SCATTERING OF SOLUTIONS WITH GROUP INVARIANCE FOR THE FOURTH-ORDER NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this paper, we consider the focusing, L^2-supercritical and $\dot{H}^2$-subcritical fourth-order nonlinear Schrödinger equations. We show the scattering of group-invariant solutions below the ground state threshold, under the hypothesis that the threshold for group-invariant solutions is less than a certain value.

1. Intrduction

We consider the focusing fourth-order nonlinear Schrödinger equation

$$i\partial_t u - \Delta^2 u + \mu\Delta u = -|u|^{p-1}u, \tag{1.1}$$

where $u = u(t,x)$ is a complex-valued unknown function of $(t,x) \in \mathbb{R}\times\mathbb{R}^d$, $\mu \geq 0$ and $p > 1$. The classical mathematical model for propagation of intense laser beams in a bulk medium with Kerr nonlinearity is given by the nonlinear Schrödinger equation. Karpman [21], Karpman–Shagalov [22] and Fibich–Ilan–Papanicolaou [14] studied (1.1) to take into account the role of small fourth-order dispersion. In Fukumoto–Moffat [16], similar equations also appear in the study of a vortex filament in an incompressible fluid.

In this article, we study the global dynamics of (1.1) in the L^2-supercritical and $\dot{H}^2$-subcritical case, where p satisfies

$$1 + \tfrac{8}{d} < p < \begin{cases} \infty & \text{if } 1 \leq d \leq 4, \\ 1 + \frac{8}{d-4} & \text{if } d \geq 5. \end{cases} \tag{1.2}$$

There is a number of results on the global dynamics of the fourth-order nonlinear Schrödinger equation. As for the defocusing equation

$$i\partial_t u - \Delta^2 u + \mu\Delta u = |u|^{p-1}u,$$

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Pausader [33] proved global well-posedness and scattering for all radially symmetric initial data in $H^2(\mathbb{R}^d)$ when $d \geq 5$ and $1 < p \leq 1 + 8/(d-4)$. Note that the radial assumption in [33] is not necessary when $p < 1 + 8/(d-4)$ and is removed later in [31, 35] in the remaining case, $p = 1 + 8/(d-4)$ when $d \geq 8$. We refer the reader to [37] for the low dimensional L^2-supercritical case: $1 \leq d \leq 4$ and $p > 1 + 8/d$. See also [36] for the high dimensional L^2-critical case, i.e., $d \geq 5$ and $p = 1 + 8/d$, with $L^2(\mathbb{R}^d)$ data.

Let us turn to the focusing case (1.1). It is widely known that ground state solutions play an important role in describing global dynamics of nonlinear dispersive equations. This is also the case for equation (1.1). Miao–Xu–Zhao [30] and Pausader [34] proved the global well-posedness and the scattering of (1.1) with $\mu = 0$ in the energy-critical case, i.e., the case where $d \geq 5$ and $p = 1 + 8/(d-4)$, for radially symmetric initial data below the ground state threshold. In [30, 34], the proofs are based on concentration compactness argument in the spirit of Kenig–Merle [24].

In the inter-critical case (1.2), Guo [17] proved scattering of radially symmetric solutions to (1.1) for $d \geq 2$ and with p satisfying (1.2), below the ground state threshold. The argument in [17] is based on the concentration compactness argument. Dinh [9] provides an alternative proof for a result similar to [17], avoiding the use of the concentration compactness argument. The proof in [9] relies on the approach introduced by Dodson–Murphy [11]. It is worth noting that in both [17, 9], the radial symmetry assumption is retained, and the one-dimensional case is excluded.

In our previous work [25], we study the scattering problem for (1.1) in the one dimensional case, which was missing in [9, 17], under the evenness assumption. As a result in this study, we proved the scattering of even solutions below the ground state threshold under the assumption that the threshold for even solutions is less than a certain constant times threshold for general solutions. The argument in [25] is not limited to the one-dimensional setting; it can be applied to a broad class of equations under evenness assumption, or more general group-invariance assumption. As a demonstration, we improve the result [9, 17] by relaxing the radial assumption. Our underlying objective is to find a reasonable extension of our argument to the multi-dimension framework.

1.1. **Sobolev space with group invariance.** Let $M_d(\mathbb{R})$ be the set of $d \times d$ matrices with real entries. Let $O(d)$ denote the set of $d \times d$ orthogonal matrices, i.e.

$$O(d) := \{\mathcal{R} \in M_d(\mathbb{R}) \mid \mathcal{R}^T\mathcal{R} = \mathcal{I}_d\},$$

where the transpose of a matrix $\mathcal{A}$ is written by $\mathcal{A}^T$ and $\mathcal{I}_d$ denotes the $d \times d$ identity matrix. Note that $O(1) = \{\pm 1\}$. We consider a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, where $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ is a group with the binary operation $(+, \cdot)$. Throughout this paper, we assume that for any $(\theta_1, \mathcal{G}_1), (\theta_2, \mathcal{G}_2) \in G$, $\mathcal{G}_1 = \mathcal{G}_2$ implies $\theta_1 = \theta_2$. The assumption reads as θ-part is given as a function of $\mathcal{G}$, and hence we can use the notation $\mathcal{G}$ without confusion to denote not only a matrix but also an element of G. In the sequel, we freely use this identification. For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$, we define the Sobolev space with G-invariance by

$$H^k_G := \{f \in H^k(\mathbb{R}^d) \mid f = \mathcal{G}f,\ \forall \mathcal{G} \in G\},$$

where $\mathcal{G}f(x) := e^{-i\theta}(f \circ \mathcal{G}^{-1})(x) = e^{-i\theta}f(\mathcal{G}^{-1}x)$ for $\mathcal{G} = (\theta, \mathcal{G}) \in G$. We remark that the above assumption of G is a natural one since one has $H^k_G = \{0\}$ if it fails. In the sequel, we also freely identify a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ with a subgroup of $O(d)$ and denote both by G in spite of the fact that there are several possibilities on the choice of the θ-part, as the above examples show. We also remark that the radial case corresponds to the case $d \geq 2$ and $G_{\mathrm{rad}} = \{0\} \times O(d)$. Other simple examples of G in $d = 1$ are $G_{\mathrm{even}} = \{(0, 1), (0, -1)\}$ and $G_{\mathrm{odd}} = \{(0, 1), (\pi, -1)\}$, which give even subspace and odd subspace of Sobolev space $H^k(\mathbb{R})$, respectively.

Note that since the Cauchy problem of (1.1) with (1.2) is well-posed in $H^2(\mathbb{R}^d)$, it is obvious that if the initial data u_0 belongs to H^2_G, then the corresponding solution u to (1.1) is also G-invariant, i.e., $u(t) = \mathcal{G}u(t)$ for all $\mathcal{G} \in G$, due to the fact that the operators Δ and Δ^2 are invariant for group actions by $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and (1.1) is gauge invariant.

1.2. **Main result.** Solutions to (1.1) preserve the mass

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 dx$$

and the energy

$$E(u(t)) := \tfrac{1}{2}\int_{\mathbb{R}^d} |\Delta u(t,x)|^2 dx + \tfrac{\mu}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2 dx - \tfrac{1}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1} dx.$$

If $\mu = 0$, the equation (1.1) is invariant under the scaling transformation

$$u_\lambda(t,x) := \lambda^{\frac{4}{p-1}} u(\lambda^4 t, \lambda x), \quad \lambda > 0.$$

By direct computation,

$$\|u_\lambda(0)\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{s - \frac{d}{2} + \frac{4}{p-1}} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)}$$

for all $s \in \mathbb{R}$. This implies that the critical exponent of homogeneous Sobolev spaces is given by

$$s_c := \tfrac{d}{2} - \tfrac{4}{p-1}.$$

We say that (1.1) is L^2-supercritical and $\dot{H}^2$-subcritical if $0 < s_c < 2$. This condition holds if and only if p satisfies (1.2).

(1.1) has standing wave solutions $u(t,x) = e^{i\omega t}\phi(x)$, where $\omega > 0$ and ϕ is a solution of the fourth-order elliptic equation

$$-\omega\phi - \Delta^2\phi + \mu\Delta\phi + |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^d. \tag{1.3}$$

For each $\omega > 0$, we define the action S_ω by

$$S_\omega(f) := E(f) + \tfrac{\omega}{2}M(f).$$

We say that Q_ω is a ground state solution to (1.3) if

$$S_\omega(Q_\omega) = \inf\{S_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ \phi \text{ solves } (1.3)\}.$$

Let K be the virial functional defined by

$$\begin{aligned}K(f) &:= \partial_{\lambda=1} S_\omega(\lambda^{\frac{d}{2}} f(\lambda x))\\ &= 2\int_{\mathbb{R}^d} |\Delta f(x)|^2 dx + \mu \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx - \tfrac{d(p-1)}{2(p+1)} \int_{\mathbb{R}^d} |f(x)|^{p+1} dx.\end{aligned}$$

Then we note that every solution ϕ of (1.3) satisfies $K(\phi) = 0$. If p satisfies (1.2), then for all $\omega > 0$,

$$S_\omega(Q_\omega) = \inf\{S_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ K(\phi) = 0\}. \tag{1.4}$$

It is known that there exists a ground state solution to (1.3) for all $p > 1$ and $\omega > 0$ (see [5, 6], for instance).

In this paper, we consider the following conjecture:

Conjecture 1.1 (Scattering below the ground state threshold)**.** *Let $d \geq 1$, $\mu \geq 0$ and p satisfy* (1.2)*. Suppose that $u_0 \in H^2(\mathbb{R}^d)$ satisfies*

$$S_\omega(u_0) < S_\omega(Q_\omega) \quad \textit{and} \quad K(u_0) > 0 \tag{1.5}$$

for some $\omega > 0$, where Q_ω is a ground state solution to (1.3)*. Then, there exists a unique global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ to* (1.1) *with initial data $u(0) = u_0$. Moreover, u scatters forward and backward in time, i.e., there exist $\phi^\pm \in H^2(\mathbb{R}^d)$ such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{-it(\Delta^2-\mu\Delta)}\phi^\pm\|_{H^2} = 0. \tag{1.6}$$

It is known that (1.1) is locally well-posed for all initial data in $H^2(\mathbb{R}^d)$. Further, if an initial data u_0 satisfies (1.5), then by the variational analysis, we see that the corresponding solution u exists globally in time and $\|u\|_{L_t^\infty H_x^2(\mathbb{R}\times\mathbb{R}^d)} < \infty$ (see Lemma 3.4 for instance). For any interval $I \subset \mathbb{R}$, we define a scattering size by

$$\|u\|_{X(I)} := \|u\|_{L_t^{q_1} L_x^r(I\times\mathbb{R}^d)},$$

where $r = p+1, q_1 = \frac{4(p-1)(p+1)}{8-(d-4)(p-1)}$. Then for all global solutions u of (1.1), the boundedness of the scattering size

$$\|u\|_{X(\mathbb{R})} < \infty$$

implies the scattering (1.6) (see Proposition 2.8). For $\omega > 0$ and for $L \geq 0$, we define

$$C_\omega(L) := \sup\{\|u\|_{X(I)} \mid S_\omega(u_0) \leq L,\ K(u_0) > 0\},$$

where the supremum is taken over all solutions u to (1.1) defined on $I \times \mathbb{R}^d$ with initial data $u_0 \in H^2(\mathbb{R}^d)$. By using Propositions 2.7 and 2.9, we see that $C_\omega(L)$ is sublinear for small $L > 0$ and that $L \mapsto C_\omega(L)$ is non-decreasing and continuous whenever $C_\omega(L)$ is finite. Let

$$L_\omega^* := \sup\{L \in [0,\infty) \mid C_\omega(L) < \infty\}.$$

Then Conjecture 1.1 is equivalent to $L_\omega^* = S_\omega(Q_\omega)$. Note that $L_\omega^* \leq S_\omega(Q_\omega)$ follows from $C(S_\omega(Q_\omega)) = \infty$, which is a consequence of the fact that $\|e^{i\omega t}Q_\omega\|_{X(\mathbb{R})} = \infty$. For $\omega > 0$ and for $L \leq 0$, we define the criterion for G-invariant solutions by

$$C_{G,\omega}(L) := \sup\{\|u\|_{X(I)} \mid S_\omega(u_0) \leq L,\ K(u_0) > 0\},$$

where the supremum is taken over all G-invariant solutions u to (1.1) defined on $I \times \mathbb{R}^d$ with initial data $u_0 \in H^2_G$. Let

$$L^*_{G,\omega} := \sup\{L \in [0,\infty) \mid C_{G,\omega}(L) < \infty\}.$$

By the definition, $C_{G,\omega}(L) \le C_\omega(L)$ holds and hence $L^*_{G,\omega} \ge L^*_\omega$. Our goal in this paper is to establish the lower bound

$$L^*_{G,\omega} \ge S_\omega(Q_\omega). \tag{1.7}$$

This implies that Conjecture 1.1 holds true for any G-invariant solutions. Note that if there exists a G-invariant ground state then one has the other inequality

$$L^*_{G,\omega} \le S_\omega(Q_\omega),$$

which shows the sharpness of the criterion (1.5). However, the symmetry of the ground states is only partially understood (see Remark 1.8, below).

For a subgroup G of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and for $\omega > 0$, we define

$$m_{G,\omega} := \inf_{|x|=1} (\#Gx) L^*_{G_x,\omega},$$

where $Gx = \{\mathcal{G}x \mid \mathcal{G} \in G\}$ and $G_x = \{\mathcal{G} \in G \mid \mathcal{G}x = x\}$ is a subgroup of G. Our main result is stated as follows.

Theorem 1.2 (Main result)**.** *Let $d \ge 1$, $\mu \ge 0$ and p satisfy* (1.2)*. Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and $\omega > 0$. Then* (1.7) *holds true under the hypothesis $L^*_{G,\omega} < m_{G,\omega}$.*

Remark 1.3. We point out that by using Proposition 2.9 in Section 2, we see that $L^*_{G,\omega} \le m_{G,\omega}$ holds in general. Hence, the hypothesis in Theorem 1.2 reads as the failure of the identity.

Remark 1.4. The hypothesis $L^*_{G,\omega} < m_{G,\omega}$ implies that $\inf_{|x|=1} \#Gx \ge 2$. Indeed, if $\inf_{|x|=1} \#Gx = 1$, then there exists $w \in \mathbb{R}^d$ such that $|w| = 1$ and $\#Gw = 1$. Since $G_w = \{\mathcal{G} \in G \mid \mathcal{G}w = w\}$ is a subgroup of G, we have $L^*_{G_w,\omega} \le L^*_{G,\omega}$. Therefore,

$$m_{G,\omega} \le (\#Gw) L^*_{G_w,\omega} \le L^*_{G,\omega},$$

which contradicts the hypothesis. Note that $\inf_{|x|=1} \#Gx \ge 2$ is equivalent to the condition that for all $x \in \mathbb{R}^d \setminus \{0\}$, there exists $\mathcal{G} \in G$ such that $\mathcal{G}x \ne x$.

Remark 1.5. If $m_{G,\omega} = \infty$, the hypothesis $L^*_{G,\omega} < m_{G,\omega}$ is not necessary in Theorem 1.2. Indeed, if $L^*_{G,\omega} = \infty$, then (1.7) is clear since the action of ground states is finite. Note that if $\inf_{|x|=1} \#Gx = \infty$, then since $L^*_{G_x,\omega} \ge L^*_\omega$ for all $x \in \mathbb{R}^d$, we have

$$m_{G,\omega} \ge \left(\inf_{|x|=1} \#Gx \right) L^*_\omega = \infty.$$

In the radial case $G_{\mathrm{rad}} = \{0\} \times O(d)$ with $d \ge 2$, one has $\inf_{|x|=1} \#G_{\mathrm{rad}}x = \infty$ As mentioned above, (1.7) was proved in this case. Theorem 1.2 is an extension of these previous works. We also remark that $\inf_{|x|=1} \#Gx = \infty$ is equivalent to the compactness of the embedding $H^1_G(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ for $2 < p < 2^* := 2d/(d-2)$ (see [2]). In our argument, the feature of a group G satisfying $\inf_{|x|=1} \#Gx = \infty$ can be characterized as the validity of the profile decomposition without spatial translations for a bounded sequence in H^2_G (see Remark 4.3).

Remark 1.6. Recall that (1.7) is still open in the case that $d = 1$ and $G = \{0\} \times \{1, -1\}$, i.e., in the case that H^2_G is a set of even functions in $H^2(\mathbb{R})$. In [25], we proved (1.7) under the assumption that $L^*_{G,\omega} < 2L^*_\omega$. Theorem 1.2 is an extension of this result. Indeed, for all $x \neq 0$, we have $\#Gx = 2$ and $G_x = \{0\} \times \{1\}$ and hence $m_{G,\omega} = 2L^*_\omega$.

Remark 1.7. Inui [19] used the group-invariance assumption to obtain scattering results for the nonlinear Schrödinger equation above the ground state threshold. In [19], it is assumed that either $\#G < \infty$ or $\inf_{|x|=1} \#Gx = \infty$ holds. Note that our result gives a partial answer in the case that $\#G = \infty$ and $\inf_{|x|=1} \#Gx < \infty$. An example in our mind is the block-radial symmetry $G = O(d_1) \times \cdots \times O(d_k)$ with $d_1 + \cdots + d_k = d$. One easily verifies that $\inf_{|x|=1} \#Gx = \infty$ if $\min_{1\le j\le k} d_j \ge 2$. On the other hand, it holds that $\inf_{|x|=1} \#Gx < \infty = \#G$ when $\min_{1\le j\le k} d_j = 1$ and $\max_{1\le j\le k} d_j \ge 2$. Indeed, when $d_1 = 1$, $d_2 = 2$, and $d = 3$, we have $\#(O(1) \times O(2)) = \infty$ and

$$\{\mathcal{G}(x_1, x_2, x_3) \mid \mathcal{G} \in O(1) \times O(2)\} = \{\pm x_1\} \times ((x_2^2 + x_3^2)^{\frac{1}{2}} S^1).$$

Hence, $\inf_{|x|=1} \#\{\mathcal{G}x \mid \mathcal{G} \in O(1) \times O(2)\} = 2$. The block-radial cases are also considered in [20].

Remark 1.8. The symmetry breaking of ground states of (1.3) occurs when $\mu < 0$. Lenzmann–Weth [28] proved that all least-action solutions to (1.3) fail to be radially symmetric when $d \ge 2$, $\mu < 0$, $1 < p < 1 + 4/(d-1)$ and ω is slightly larger than $\mu^2/4$ (see also [29]). However, these nonradial ground states can still be *even* functions. By using the Fourier symmerization methods in [8, 27], one can prove that if p is odd and $1 < p < 1 + 8/(d-4)_+$, any ground state of (1.3) for $\omega > \mu^2/4$ must be an even function up to a translation. It is expected that the symmetry breaking for ground states of (1.1) does not occur in our case, i.e., when $\mu \ge 0$. Indeed, Boulenger–Lenzmann [7] proved the existence of radially symmetric ground states for (1.3) when $\mu \ge 0$ and p is an odd integer.

Remark 1.9. It is conjectured that solutions of (1.1) with p satisfying (1.2) blows up in finite time if the initial data u_0 satisfies

$$S_\omega(u_0) < S_\omega(Q_\omega) \quad \text{and} \quad K(u_0) < 0.$$

It is still an open question. In the case $d \ge 2$, Boulenger–Lenzmann [7] proved finite time blow-up for radially symmetric solutions to (1.1) with p satisfying (1.2) and $p \le 9$. Their proofs are based on localized virial identities for (1.1), which are analogous to virial identities introduced by Ogawa–Tsutsumi [32] for the second-order nonlinear Schrödinger equations. To establish the localized virial identities, the radial Sobolev inequality was employed. Because of this, the radial assumption with $d \ge 2$ is required.

Let us briefly recall the results in [9, 17, 25].

Theorem 1.10 ([9, 17])**.** *Let $d \ge 2$, $\mu = 0$ and p satisfy* (1.2)*. Suppose that $u_0 \in H^2(\mathbb{R}^d)$ is radially symmetric and satisfies*

$$M(u_0)^{\frac{2-s_c}{s_c}} E(u_0) < M(Q)^{\frac{2-s_c}{s_c}} E(Q), \tag{1.8}$$

$$\|u_0\|_{L^2}^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_{L^2} < \|Q\|_{L^2}^{\frac{2-s_c}{s_c}} \|\Delta Q\|_{L^2}, \tag{1.9}$$

where $Q := Q_1$ is a ground state solution to (1.3) with $\omega = 1$. Then, there exists a unique global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ to (1.1) with initial data $u(0) = u_0$. Moreover, u scatters forward and backward in time, i.e., there exist $\phi^{\pm} \in H^2(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{-it\Delta^2}\phi^{\pm}\|_{H^2} = 0.$$

In the case $\mu = 0$, the condition (1.8)–(1.9) is equivalent to (1.5) for some $\omega > 0$. In [9], Theorem 1.10 is also proved in the case $\mu > 0$ under the similar conditions (1.8) and (1.9), where the ground state Q and its energy $E(Q)$ in the threshold are defined as those corresponding to the case $\mu = 0$. Note that when $\mu > 0$, our scattering condition in Theorem 1.2 improves on that in the result of [9] (see Remark 3.2).

As mentioned above, the radial assumption is used in Theorem 1.10. One motivation for the use of the radial assumption can be explained by the absence of Galilean invariance, a characteristic property of fourth-order Schrödinger equations. In [17, 25], the proofs of Theorem 1.10 are based on the argument introduced by Kenig–Merle [24] for proving global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equation (NLS). We also refer to Holmer–Roudenko [18] for the focusing 3D cubic NLS in the radial case. Roughly speaking, the argument begins reformulating the problem as a variational problem, where the failure of the theorem implies the existence of a phantom optimizer to the problem. Then, the existence of the phantom yields a contradiction, which completes the proof of the theorem. The phantom optimizer is often referred to as a *minimal blowup solution* or a *critical element*.

We recall that, in the case of usual second-order NLS, the scattering results is extended to the nonradial case (see [1, 13, 10, 12, 26]). For the second-order NLS, by using the Galilean invariance of the equation, one can control the spatial shift of a critical element, which enables the contradiction argument via the virial identity. However, in the fourth-order case, the lack of the Galilean invariance makes control of the spatial shift hard. The radial assumption immediately stems this step. In the previous study [17], which follows the argument in [18], the following radial Sobolev inequality was employed to show the spatial localization of a critical element: Let $f \in H^1(\mathbb{R}^d)$ be radially symmetric. Then for all $R > 0$,

$$\|f\|_{L^\infty(|x|>R)} \le CR^{-\frac{d-1}{2}} \|f\|_{L^2(|x|>R)}^{\frac{1}{2}} \|\nabla f\|_{L^2(|x|>R)}^{\frac{1}{2}}, \tag{1.10}$$

where C is a constant independent of f and R. When $d \ge 2$, the above estimate (1.10) implies that radially symmetric functions enjoy the stronger spatial decay as $|x| \to \infty$. This improvement makes the analysis simpler in several respects. For instance, one may deduce that no spatial translation is involved in the linear profile decomposition for sequences of radially symmetric functions. We remark that (1.10) is also used in [9] to control the error of a localized virial identity.

In one dimensional case, however, there is no advantage in view of the uniform decay property such as (1.10). In our previous work [25], we proved Theorem 1.10 for $d = 1$ under the additional assumption that $L^*_{\text{even}} < 2L^*$, where $L^*_{\text{even}} = L^*_{G,1}$ with $G = \{0\} \times \{1, -1\}$ and $L^* = L^*_1$. The main ingredient of the proof in [25] is to establish the profile decomposition for even functions in $H^2(\mathbb{R})$. It follows from the evenness assumption that if there exist profiles with an unbounded space shift, such profiles must appear as a pair.

By using this fact together with the assumption $L^*_{\mathrm{even}} < 2L^*$, we see that the action of each profile in the decomposition of the sequence optimizing L^*_{even} is less than L^* if it has an unbounded space shift. Then the solution with such a profile as initial data scatters, which leads to contradiction. Consequently, we can construct a critical element that does not shift.

In the proof of Theorem 1.2, we extend the argument in [25] to the case of group-invariance setting in general spatial dimensions. For a sequence of G-invariant functions in $H^2(\mathbb{R}^d)$, we derive the profile decomposition, in which each profile with an unbounded spatial shift is G_w-invariant and appears as a pair of $\#Gw$ profiles for some $w \in \mathbb{S}^{d-1}$. Moreover, we see that each of these pairs is G-invariant. By using this profile decomposition and applying the concentration compactness argument, we can show that a sequence optimizing $L^*_{G,\omega}$ is consist of only one pair of profiles. At this point, however, there are $\#Gw$ profiles in the pair. If $L^*_{G,\omega} < m_{G,\omega}$, then the action of each profile is less than $L^*_{G_w,\omega}$, which leads to contradiction. Therefore, we can assume that the critical element does not shift in the proof of Theorem 1.2.

The rest of the paper is organized as follows: In Section 2, we recall the Strichartz type estimates for the linear fourth-order Schrödinger equation. We also give a small data theory and prove a long-time perturbation theory. In Section 3, we study the ground state Q_ω and give some lemmas. In Section 4, we establish a linear profile decomposition for a sequence $\{f_n\}$ of G-invariant functions uniformly bounded in $H^2(\mathbb{R}^d)$. In Section 5, we construct a critical solution, which is a target counter example in contradiction argument. Finally, in Section 6, we prove Theorem 1.2 by contradiction.

2. Preliminaries

We consider the linear fourth-order Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta^2 u + \mu\Delta u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

The linear propagator associated with (2.1) is given by

$$e^{-it(\Delta^2-\mu\Delta)}f := \mathcal{F}^{-1}\left[e^{-it(|\xi|^4+\mu|\xi|^2)}\hat{f}(\xi)\right],$$

where $\mathcal{F}f = \hat{f}$ is the Fourier transform of f and $\mathcal{F}^{-1}f$ is the inverse Fourier transform of f. Since $|e^{-it(|\xi|^4+\mu|\xi|^2)}| = 1$,

$$\|e^{-it(\Delta^2-\mu\Delta)}f\|_{L^2} = \|f\|_{L^2}. \tag{2.2}$$

Ben-Artzi–Koch–Saut [3] studied the dispersive estimates for Schrödinger equations with fourth-order dispersion. Applying their result, for $\mu \ge 0$, we have

$$\|e^{-it(\Delta^2-\mu\Delta)}f\|_{L^\infty} \lesssim |t|^{-\frac{d}{4}}\|f\|_{L^1} \tag{2.3}$$

and

$$\|\partial^\alpha e^{-it(\Delta^2-\mu\Delta)}f\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}}\|f\|_{L^1}, \tag{2.4}$$

for all $\alpha \in \mathbb{Z}^d$ such that $|\alpha| = d$. Interpolating (2.2) and (2.3), for $2 \le r \le \infty$,

$$\|e^{-it(\Delta^2-\mu\Delta)} f\|_{L^r} \lesssim |t|^{-\frac{d}{4}(1-\frac{2}{r})} \|f\|_{L^{r'}}. \tag{2.5}$$

From the above dispersive estimates, we obtain the following two types of Strichartz estimate.

Lemma 2.1 (Strichartz estimates for biharmonic admissible pairs)**.** *Let $d \ge 1$ and $\mu \ge 0$. Suppose that pairs (q_1, r_1) and (q_2, r_2) satisfy*

$$2 \le q_j, r_j \le \infty, \quad (q_j, r_j, d) \ne (2, \infty, 4) \quad \textit{and} \quad \tfrac{4}{q_j} + \tfrac{d}{r_j} = \tfrac{d}{2}, \quad \textit{for } j = 1, 2. \tag{2.6}$$

Then for any $I \subset \mathbb{R}$ and $t_0 \in I$,

$$\|e^{-it(\Delta^2-\mu\Delta)} f\|_{L_t^{q_1} L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \|f\|_{L_x^2(\mathbb{R}^d)},$$

$$\left\| \int_{t_0}^t e^{-i(t-s)(\Delta^2-\mu\Delta)} F(s) ds \right\|_{L_t^{q_1} L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q_2'} L_x^{r_2'}(I\times\mathbb{R}^d)}.$$

Proof. We can prove Lemma 2.1 by using (2.5) and the theorem in Keel–Tao [23]. ☐

Lemma 2.2 (Strichartz estimates for Schrödinger admissible pairs)**.** *Let $d \ge 1$ and $\mu \ge 0$. Suppose that pairs (q_1, r_1) and (q_2, r_2) satisfy*

$$2 \le q_j, r_j \le \infty, \quad (q_j, r_j, d) \ne (2, \infty, 2) \quad \textit{and} \quad \tfrac{2}{q_j} + \tfrac{d}{r_j} = \tfrac{d}{2}, \quad \textit{for } j = 1, 2.$$

Then for any $I \subset \mathbb{R}$ and $t_0 \in I$,

$$\|e^{-it(\Delta^2-\mu\Delta)} f\|_{L_t^{q_1} L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \||\nabla|^{-\frac{2}{q_1}} f\|_{L_x^2(\mathbb{R}^d)},$$

$$\left\| \int_{t_0}^t e^{-i(t-s)(\Delta^2-\mu\Delta)} F(s) ds \right\|_{L_t^{q_1} L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \||\nabla|^{-\frac{2}{q_1}-\frac{2}{q_2}} F\|_{L_t^{q_2'} L_x^{r_2'}(I\times\mathbb{R}^d)}.$$

Proof. Lemma 2.2 follows from (2.4) and the theorem in Keel–Tao [23]. ☐

Corollary 2.3. *Let $d \ge 3$ and $\mu \ge 0$. Suppose that (q, r) satisfies* (2.6) *without indices. Then, for any $I \subset \mathbb{R}$ and $t_0 \in I$,*

$$\left\| \Delta \int_{t_0}^t e^{-it(t-s)(\Delta^2-\mu\Delta)} F(s) ds \right\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|\nabla F\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}.$$

Proof. Corollary 2.3 follows from the Sobolev inequality and Lemma 2.2. ☐

From the above Strichartz estimates, we have the following local well-posedness for H^2-initial data.

Proposition 2.4 (Local well-posedness in H^2)**.** *Let $d \ge 1$, $\mu \ge 0$. Suppose*

$$\begin{cases} p \ge 2 & \textit{if } d = 1, 2, \\ 1 + \frac{2}{d} \le p < 1 + \frac{8}{(d-4)_+} & \textit{if } d \ge 3. \end{cases}$$

Then, for any $u_0 \in H^2(\mathbb{R}^d)$, there exist $T_*, T^* \in (0,\infty)$ and a unique solution u to (1.1) with initial data $u(0) = u_0$ such that

$$u \in C((-T_*, T^*), H^2(\mathbb{R}^d)) \cap L^q_{loc}((-T_*, T^*), W^{2,r}(\mathbb{R}^d))$$

for all (q,r) satisfying (2.6). Furthermore, we can choose T_* and T^* to be non-increasing functions of $\|u_0\|_{H^2}$. Moreover, u satisfies $M(u(t)) = M(u_0)$ and $E(u(t)) = E(u_0)$ for all $t \in (-T_*, T^*)$.

Proof. See [9]. □

To prove the scattering result, we need the following Kato type inhomogeneous Strichartz estimate.

Lemma 2.5 (Kato type estimates)**.** *Let $d \geq 1$ and $\mu \geq 0$. Suppose that q_1, q_2 and r satisfy*

$$1 < q_1, q_2 < \infty, \quad 2 < r \begin{cases} \leq \infty & \text{if } 1 \leq d \leq 3, \\ < \infty & \text{if } d = 4, \\ < \frac{2d}{d-4} & \text{if } d \geq 5 \end{cases}$$

and

$$\tfrac{1}{q_1} + \tfrac{1}{q_2} = \tfrac{d}{4}\left(1 - \tfrac{2}{r}\right). \tag{2.7}$$

Then for any $I \subset \mathbb{R}$ and $t_0 \in I$,

$$\left\| \int_{t_0}^t e^{-i(t-s)(\Delta^2 - \mu\Delta)} F(s)ds \right\|_{L_t^{q_1} L_x^r(I\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q_2'} L_x^{r'}(I\times\mathbb{R}^d)}. \tag{2.8}$$

Proof. By using (2.5), we have

$$\begin{aligned} \left\| \int_{t_0}^t e^{-i(t-s)(\Delta^2-\mu\Delta)} F(s)ds \right\|_{L_x^r} &\lesssim \int_{t_0}^t \left\| e^{-i(t-s)(\Delta^2-\mu\Delta)} F(s) \right\|_{L_x^r} ds \\ &\lesssim \int_{t_0}^t |t-s|^{-\frac{d}{4}(1-\frac{2}{r})} \|F(s)\|_{L_x^{r'}} ds. \end{aligned}$$

Hence, from the Hardy–Littlewood–Sobolev inequality and (2.7), we obtain (2.8). □

Throughout this paper, for $1 + 8/d < p < 1 + 8/(d-4)_+$ we fix

$$r = p+1, \qquad q = \tfrac{8(p+1)}{d(p-1)},$$

$$q_1 = \tfrac{4(p-1)(p+1)}{8-(d-4)(p-1)}, \qquad q_2 = \tfrac{4(p-1)(p+1)}{(dp-4)(p-1)-8}.$$

Then (q,r) satisfies $\frac{4}{q} + \frac{d}{r} = \frac{d}{2}$ and (q_1, r) and (q_2, r) satisfy

$$\tfrac{4}{q_1} + \tfrac{d}{r} = \tfrac{4}{p-1} = \tfrac{d}{2} - s_c, \quad \tfrac{4}{q_2} + \tfrac{d}{r} = d - \tfrac{4}{p-1} = \tfrac{d}{2} + s_c.$$

In particular, we have $\frac{1}{q_1} + \frac{1}{q_2} = \frac{d}{4}(1 - \frac{2}{r})$. For any time interval $I \subset \mathbb{R}$, we define function space $X(I)$ with norm

$$\|u\|_{X(I)} := \|u\|_{L_t^{q_1} L_x^r(I\times\mathbb{R}^d)}.$$

We also define

$$\|F\|_{N(I)} := \|F\|_{L_t^{q_2'} L_x^{r'}(I\times\mathbb{R}^d)}.$$

Lemma 2.6 (Nonlinear estimate)**.** *Let $t_0 \in \mathbb{R}$ and let I be a time interval which contains t_0. Then*

$$\left\| \int_{t_0}^t e^{-i(t-s)(\Delta^2-\mu\Delta)}(uv)(s)ds \right\|_{X(I)} \lesssim \|u\|_{L_t^{q_1}L_x^r(I\times\mathbb{R}^d)} \|v\|_{L_t^{q_1/(p-1)}L_x^{r/(p-1)}(I\times\mathbb{R}^d)}.$$

Proof. By using Lemma 2.5,

$$\left\| \int_{t_0}^t e^{-i(t-s)(\Delta^2-\mu\Delta)}(uv)(s)ds \right\|_{X(I)} \lesssim \|uv\|_{N(I)}.$$

Since $q_2' = \frac{4(p-1)(p+1)}{p(8-(d-4)(p-1))} = q_1/p$ and $r' = \frac{p+1}{p} = r/p$, by using the Hölder inequality, we have

$$\|uv\|_{N(I)} \leq \|u\|_{L_t^{q_1}L_x^r(I\times\mathbb{R}^d)} \|v\|_{L_t^{q_1/(p-1)}L_x^{r/(p-1)}(I\times\mathbb{R}^d)}.$$

This completes the proof of Lemma 2.6. □

By using Lemma 2.6, we have the following small data theory.

Proposition 2.7 (Small data)**.** *Let $d \geq 1$, $\mu \geq 0$ and let p satisfy (1.2). There exists small $\delta_{sd} > 0$ such that the following holds. Let I be a time interval which contains t_0. If $u_0 \in H^2(\mathbb{R})$ satisfies*

$$\|e^{-it(\Delta^2-\mu\Delta)}u_0\|_{X(I)} \leq \delta_{sd},$$

then there exists a unique solution $u \in X(I)$ to (1.1) with initial data $u(t_0) = u_0$. Moreover,

$$\|u\|_{X(I)} \leq 2\|e^{-it(\Delta^2-\mu\Delta)}u_0\|_{X(I)}.$$

Proof. We can prove Proposition 2.7 by using Lemma 2.6 and applying the standard contraction mapping principle. □

In the proof of Theorem 1.2, we will use the following scattering criterion.

Proposition 2.8 (H^2-scattering)**.** *Let $d \geq 1$, $\mu \geq 0$ and let p satisfy (1.2). If $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ is a global solution to (1.1) and satisfies*

$$\|u\|_{X(\mathbb{R})} < \infty, \quad \text{and} \quad \|u\|_{L_t^\infty H_x^2(\mathbb{R}\times\mathbb{R}^d)} < \infty,$$

then u scatters forward and backward in time.

Proof. See [9] for the details. □

The following long-time perturbation theory plays crucial role in the concentration compactness argument (in Section 5).

Proposition 2.9 (Long-time perturbation theory)**.** *Let $d \geq 1$, $\mu \geq 0$ and let p satisfy* (1.2)*. For each $A \gg 1$, there exist $\delta_0 = \delta_0(A) \ll 1$ and $C = C(A) \gg 1$ such that the following holds. Let $I \subset \mathbb{R}$ be a compact time interval, and let $\tilde{u} \in C(I, H^2(\mathbb{R}^d))$ be an approximate solution of* (1.1) *in the sense that*

$$i\partial_t \tilde{u} - \Delta^2 \tilde{u} + \mu\Delta\tilde{u} = -|\tilde{u}|^{p-1}\tilde{u} + e,$$

for some $e \in N(I)$. Let $t_0 \in I$ and $u_0 \in H^2(\mathbb{R}^d)$. If

$$\|\tilde{u}\|_{X(I)} \leq A, \quad \|e\|_{N(I)} \leq \delta$$

and

$$\|e^{-i(t-t_0)(\Delta^2-\mu\Delta)}(\tilde{u}(t_0) - u_0)\|_{X(I)} \leq \delta$$

for some $\delta \in (0, \delta_0]$, then there exists a solution $u \in C(I, H^2(\mathbb{R}^d))$ to (1.1) *such that $u(t_0) = u_0$. Moreover, u satisfies*

$$\|u - \tilde{u}\|_{X(I)} \leq C\delta.$$

Proof. Put $u = \tilde{u} + w$ and let $F(z) = |z|^{p-1}z$. If w solves the equation

$$i\partial_t w - \Delta^2 w + \mu\Delta w = F(\tilde{u} + w) - F(\tilde{u}) - e \tag{2.9}$$

with initial data $w(t_0) = u_0 - \tilde{u}(t_0)$, then u solves (1.1) with initial data $u(t_0) = u_0$. Without loss of generality, we may assume $\inf I = t_0$. Since $\|\tilde{u}\|_{X(I)} \leq A$, we can partition I into $N = N(A)$ intervals $I_j = [t_j, t_{j+1}]$ such that for each j,

$$\|\tilde{u}\|_{X(I_j)} \leq \eta$$

for some $\eta > 0$ sufficiently small to be chosen later. The integral equation of (2.9) with initial time t_j is given by

$$w(t) = e^{-i(t-t_j)(\Delta^2-\mu\Delta)}w(t_j) - i\int_{t_j}^{t} e^{-i(t-s)(\Delta^2-\mu\Delta)}[F(\tilde{u} + w) - F(\tilde{u}) - e](s)ds. \tag{2.10}$$

We define

$$\begin{aligned}\Phi_j(w)(t) := {} & e^{-i(t-t_j)(\Delta^2-\mu\Delta)}w(t_j)\\ & - i\int_{t_j}^{t} e^{-i(t-s)(\Delta^2-\mu\Delta)}[F(\tilde{u} + w) - F(\tilde{u}) - e](s)ds,\end{aligned}$$

and let

$$B_j := \|e^{-i(t-t_j)(\Delta^2-\mu\Delta)}w(t_j)\|_{X(I)}.$$

Then by using Lemma 2.5, we have

$$\begin{aligned}\|\Phi_j(w)\|_{X(I_j)} &\leq \|e^{-i(t-t_j)(\Delta^2-\mu\Delta)}w(t_j)\|_{X(I_j)} + c\|F(\tilde{u} + w) - F(\tilde{u})\|_{N(I_j)}\\ &\quad + c\|e\|_{N(I_j)}\\ &\leq B_j + c\delta + c\|F(\tilde{u} + w) - F(\tilde{u})\|_{N(I_j)}, \end{aligned} \tag{2.11}$$

where c is a constant depending on $r = p+1$. By Lemma 2.6,

$$\begin{aligned}\|F(\tilde{u}+w)-F(\tilde{u})\|_{N(I_j)} &\lesssim \left(\|\tilde{u}\|_{X(I_j)}+\|w\|_{X(I_j)}\right)^{p-1}\|w\|_{X(I_j)}\\ &\lesssim \left(\eta+\|w\|_{X(I_j)}\right)^{p-1}\|w\|_{X(I_j)}. \end{aligned}\tag{2.12}$$

Let

$$X_j := \{u \in X(I_j) \mid \|u\|_{X(I_j)} \le 2(B_j + c\delta)\}.$$

If η and $B_j + c\delta$ are sufficiently small, then from (2.11) and (2.12) we have

$$\|\Phi_j(w)\|_{X(I_j)} \le 2(B_j + c\delta)$$

for all $w \in X_j$, and so $\Phi_j(w) \in X_j$. Moreover, if η and $B_j + c\delta$ are sufficiently small, for all $w_1, w_2 \in X_j$ we have

$$\|\Phi_j(w_1)-\Phi_j(w_2)\|_{X(I_j)} \le c\|F(\tilde{u}+w_1)-F(\tilde{u}+w_2)\|_{N(I_j)} \le \tfrac{1}{2}\|w_1-w_2\|_{X(I_j)}$$

and so $\Phi_j : X_j \to X_j$ is a contraction mapping. Hence, applying the fixed point theorem, there exists an unique $w \in X_j$ such that $\Phi_j(w) = w$. Now take $t = t_{j+1}$ in (2.10) and apply $e^{-i(t-t_{j+1})(\Delta^2-\mu\Delta)}$ to both side to obtain

$$\begin{aligned}e^{-i(t-t_{j+1})(\Delta^2-\mu\Delta)}w(t_{j+1}) &= e^{-i(t-t_j)(\Delta^2-\mu\Delta)}w(t_j)\\ &\quad - i\int_{t_j}^{t_{j+1}} e^{-i(t-s)(\Delta^2-\mu\Delta)}[F(\tilde{u}+w)-F(\tilde{u})-e](s)ds.\end{aligned}$$

Since the above Duhamel integral is confined to $I_j = [t_j, t_{j+1}]$, by using Lemma 2.5, we have

$$B_{j+1} \le B_j + c\delta + c\|F(\tilde{u}+w)-F(\tilde{u})\|_{N(I_j)} \le 2(B_j + c\delta),$$

if η and $B_j + c\delta$ are sufficiently small. Iterating this, we obtain

$$B_j \le 2^j B_0 + c\delta\sum_{k=1}^{j} 2^k \le 2^j\delta + 2(2^j-1)c\delta$$

for all $0 \le j \le N-1$. Thus, we can take $\delta = \delta(N) > 0$ sufficiently small so that by an inductive argument, we get $w \in X_j$ such that $\Phi_j(w) = w$ for all $0 \le j \le N-1$. Therefore, we obtain a solution w of (2.9) such that

$$\|w\|_{X(I_j)} \le 2(B_j + c\delta) \le 2^{j+1}\delta + 2(2^{j+1}-1)c\delta$$

for all $0 \le j \le N-1$. Hence,

$$\|w\|_{X(I)} \le \delta\sum_{j=0}^{N-1}(2^{j+1} + 2(2^{j+1}-1)c).$$

This completes the proof of Proposition 2.9. $\square$

3. Variational analysis

In this section, we prepare some variational lemmas.

Lemma 3.1 (Variational characterization of ground states)**.** *Let $d \geq 1$, $\mu \geq 0$ and p satisfy* (1.2)*. Let $\omega > 0$ and Q_ω be a ground state of* (1.3)*. Then,*

$$S_\omega(Q_\omega) = \inf\{\tilde{S}_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ K(\phi) \leq 0\},$$

where

$$\begin{aligned}\tilde{S}_\omega(\phi) &:= S_\omega(\phi) - \tfrac{2}{d(p-1)} K(\phi) \\ &= \tfrac{s_c}{d} \int_{\mathbb{R}^d} |\Delta\phi|^2 dx + \left(\tfrac{s_c}{d} + \tfrac{2}{d(p-1)}\right) \mu \int_{\mathbb{R}^d} |\nabla\phi|^2 dx + \tfrac{\omega}{2} \int_{\mathbb{R}^d} |\phi|^2 dx.\end{aligned}$$

Proof. By the definition of $\tilde{S}_\omega$, we see that if $K(\phi) = 0$, then $\tilde{S}_\omega(\phi) = S_\omega(\phi)$. Hence, we have

$$S_\omega(Q_\omega) \geq \inf\{\tilde{S}_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ K(\phi) \leq 0\}. \tag{3.1}$$

Let ϕ be such that $K(\phi) < 0$. For any $\lambda > 0$, we have

$$K(\lambda\phi) = \lambda^2 \left(2 \int_{\mathbb{R}^d} |\Delta\phi|^2 dx + \mu \int_{\mathbb{R}^d} |\nabla\phi|^2 dx\right) - \lambda^{p+1} \tfrac{d(p-1)}{2(p+1)} \int_{\mathbb{R}^d} |\phi|^{p+1} dx.$$

Since $K(\phi) < 0$, there exists $0 < \lambda_0 < 1$ such that $K(\lambda_0\phi) = 0$. Thus,

$$S_\omega(Q_\omega) \leq S_\omega(\lambda_0\phi) = \tilde{S}_\omega(\lambda_0\phi) = \lambda_0^2 \tilde{S}_\omega(\phi) < \tilde{S}_\omega(\phi).$$

Therefore, we obtain

$$S_\omega(Q_\omega) \leq \inf\{\tilde{S}_\omega(\phi) \mid \phi \in H^2(\mathbb{R}^d) \setminus \{0\},\ K(\phi) \leq 0\}. \tag{3.2}$$

Lemma 3.1 follows from (3.1) and (3.2). □

Remark 3.2. Suppose that $\mu \geq 0$ and $u_0 \in H^2(\mathbb{R}^d)\setminus\{0\}$ satisfies the scattering condition of the result in [9], that is,

$$M^{\frac{2-s_c}{s_c}}(u_0) E(u_0) < M^{\frac{2-s_c}{s_c}}(Q) E_0(Q), \tag{3.3}$$

$$\|u_0\|_{L^2}^{\frac{2-s_c}{s_c}} \|\Delta u_0\|_{L^2} < \|Q\|_{L^2}^{\frac{2-s_c}{s_c}} \|\Delta Q\|_{L^2}, \tag{3.4}$$

where Q is a ground state solution of (1.3) with $\mu = 0$, $\omega = 1$, and $E_0(Q) = \frac{1}{2}\int |\Delta Q|^2 - \frac{1}{p+1}\int |Q|^{p+1}$. Then, there exists $\omega > 0$ and a ground state solution Q_ω of (1.3) such that u_0 satisfies (1.5). Indeed, taking $\omega = \frac{2(2-s_c)E(u_0)}{s_c M(u_0)}$ and using (3.3), we have

$$S_\omega(u_0) = \left(\tfrac{\omega M(u_0)}{2(2-s_c)}\right)^{\frac{2-s_c}{2}} \left(\tfrac{E(u_0)}{s_c}\right)^{\frac{s_c}{2}} < \left(\tfrac{\omega M(Q)}{2(2-s_c)}\right)^{\frac{2-s_c}{2}} \left(\tfrac{E_0(Q)}{s_c}\right)^{\frac{s_c}{2}}. \tag{3.5}$$

Let $S_{0,\omega}$, $\tilde{S}_{0,\omega}$ and K_0 be functionals corresponding to the case $\mu = 0$. Let $N_{0,\omega}$ be a functional defined by

$$N_{0,\omega}(f) := \inf_{\lambda>0} S_{0,\omega}(T_\lambda f) = S_{0,\omega}(T_\lambda f)\Big|_{\lambda=\left(\frac{s_c \omega M(f)}{2(2-s_c)E_0(f)}\right)^{\frac{1}{4}}}$$
$$= \left(\tfrac{\omega M(f)}{2(2-s_c)}\right)^{\frac{2-s_c}{2}} \left(\tfrac{E_0(f)}{s_c}\right)^{\frac{s_c}{2}}, \tag{3.6}$$

where $(T_\lambda f)(x) := \lambda^{\frac{4}{p-1}} f(\lambda x)$. From (3.5) and (3.6), we have $S_\omega(u_0) < N_{0,\omega}(Q)$. Let $Q_{0,\omega}(x) := \omega^{\frac{1}{p-1}} Q(\omega^{\frac{1}{4}} x)$. Then, $Q_{0,\omega}$ is a ground state of (1.3) with $\mu = 0$ and $N_{0,\omega}(Q_{0,\omega}) = N_{0,\omega}(Q)$. Hence, we obtain

$$S_\omega(u_0) < N_{0,\omega}(Q_{0,\omega}) = S_{0,\omega}(Q_{0,\omega}).$$

Since $K_0(Q_\omega) = -\mu \int |\nabla Q_\omega|^2 \leq 0$, from Lemma 3.1 we have

$$S_{0,\omega}(Q_{0,\omega}) \leq \tilde{S}_{0,\omega}(Q_\omega) \leq \tilde{S}_\omega(Q_\omega) = S_\omega(Q_\omega).$$

Therefore, $S_\omega(u_0) < S_\omega(Q_\omega)$. Similarly, we can show $\tilde{S}_\omega(u_0) < \tilde{S}_\omega(Q_\omega)$, and so $K(u_0) > 0$ by Lemma 3.1. Note that the equality of $S_{0,\omega}(Q_{0,\omega}) \leq S_\omega(Q_\omega)$ holds if and only if $\mu = 0$. This implies that when $\mu > 0$, there exists u_0 such that (1.5) holds for some $\omega > 0$ and (3.3)–(3.4) does not hold.

From (1.4) and conservation laws, for each $\omega > 0$, the set of functions satisfying (1.5) is invariant under the flow of (1.1).

Lemma 3.3 (Invariant set). *Let $d \geq 1$, $\mu \geq 0$ and p satisfy* (1.2)*. Let $\omega > 0$ and Q_ω be a ground state of* (1.3)*. Let $u \in C(I, H^2(\mathbb{R}^d))$ be a solution to* (1.1)*. If $S_\omega(u(t_0)) < S_\omega(Q_\omega)$ and $K(u(t_0)) > 0$ for some $t_0 \in I$, then $S_\omega(u(t)) < S_\omega(Q_\omega)$ and $K(u(t)) > 0$ for all $t \in I$.*

Proof. Suppose that $S_\omega(u(t_0)) < S_\omega(Q_\omega)$ and $K(u(t_0)) > 0$ for some $t_0 \in I$. Then, by the conservation laws, we have $S_\omega(u(t)) = S_\omega(u(t_0)) < S_\omega(Q_\omega)$ for all $t \in I$. If $K(u(t_1)) \leq 0$ for some $t_1 \in I$, then by the continuity of $K(u(t))$, there exists $t_2 \in I$ such that $K(u(t_2)) = 0$. From (1.4) this implies that $S_\omega(u(t_2)) \geq S_\omega(Q_\omega)$, which is a contradiction. □

To prove rigidity for a critical element in Section 6, we use the following coercivity lemma.

Lemma 3.4 (Coercivity). *Let $d \geq 1$, $\mu \geq 0$ and p satisfy* (1.2)*. Let $\omega > 0$ and Q_ω be a ground state of* (1.3)*. Let $u \in C(I, H^2(\mathbb{R}^d))$ be a solution to* (1.1)*. If $S_\omega(u(t_0)) < S_\omega(Q_\omega)$ and $K(u(t_0)) > 0$ for some $t_0 \in I$, then u exists globally in time, i.e., $I = \mathbb{R}$. Moreover, there exists a constant $\delta > 0$ such that for all $t \in \mathbb{R}$,*

$$K(u(t)) > \delta(\|\Delta u(t)\|_{L^2}^2 + \mu \|\nabla u(t)\|_{L^2}^2). \tag{3.7}$$

Proof. From Lemma 3.3 we have $K(u(t)) > 0$ for all $t \in I$. Hence,

$$\tilde{S}_\omega(u(t)) = S_\omega(u) - \tfrac{2}{d(p-1)} K(u(t)) \leq S_\omega(u) < S_\omega(Q_\omega)$$

for all $t \in I$. This implies that $\|u(t)\|_{H^2}$ is uniformly bounded in $t \in I$, which proves the global existence. Now we will prove (6.6). Let

$$\lambda := \left(\frac{S_\omega(Q_\omega)}{S_\omega(u)}\right)^{\frac{1}{p+1}} > 1.$$

By using $K(u(t)) > 0$, we have

$$\lambda = \left(\frac{S_\omega(Q_\omega)}{\tilde{S}_\omega(u(t)) + \frac{2}{d(p-1)}K(u(t))}\right)^{\frac{1}{p+1}} < \left(\frac{S_\omega(Q_\omega)}{\tilde{S}_\omega(u(t))}\right)^{\frac{1}{p+1}}$$

for all $t \in \mathbb{R}$. Hence,

$$\tilde{S}_\omega(\lambda u(t)) \le \lambda^2 \tilde{S}_\omega(u(t)) < \lambda^{p+1}\tilde{S}_\omega(u(t)) < S_\omega(Q_\omega)$$

for all $t \in \mathbb{R}$. From Lemma 3.1, this implies that $K(\lambda u(t)) > 0$. Thus, we obtain

$$\begin{aligned} K(u(t)) &= \lambda^{-(p+1)}K(\lambda u(t)) + (1 - \lambda^{-(p-1)})(2\|\Delta u(t)\|_{L^2}^2 + \mu\|\nabla u(t)\|_{L^2}^2) \\ &> (1 - \lambda^{-(p-1)})(\|\Delta u(t)\|_{L^2}^2 + \mu\|\nabla u(t)\|_{L^2}^2) \end{aligned}$$

for all $t \in \mathbb{R}$. This completes the proof of Lemma 3.4. □

4. Profile decomposition

In this section, we derive a linear profile decomposition of a sequence of G-invariant functions for the biharmonic Schrödinger propagator.

Proposition 4.1 (Profile decomposition for a sequence of G-invariant functions)**.** *Let $d \ge 1$, $\mu \ge 0$ and p satisfy (1.2). Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$. Let $\{f_n\}$ be a sequence of G-invariant functions uniformly bounded in H^2_G. Passing to a subsequence if necessary, there exists $J^* \in \{0, 1, ...\} \cup \{\infty\}$ and for each $j \in [1, J^*]$, there exist a subgroup G^j of G, a G^j-invariant function $\phi^j \in H^2_{G^j} \setminus \{0\}$ and a sequence $(t_n^j, x_n^j) \in \mathbb{R} \times \mathbb{R}^d$ with the following properties: For each $j \in [1, J^*]$, $N^j := \#(G/G^j) < \infty$, and so, there exist $\mathcal{G}_1^j, ..., \mathcal{G}_{N^j}^j \in G$ such that $G = \bigcup_{k=1}^{N^j} \mathcal{G}_k^j G^j$ and $\mathcal{G}_k^j G^j \cap \mathcal{G}_l^j G^j = \emptyset$ for any $k \neq l$. For each $1 \le J \le J^*$, we have the decomposition*

$$f_n(x) = \sum_{j=1}^{J} e^{it_n^j(\Delta^2 - \mu\Delta)}\psi_n^j(x) + R_n^J(x), \tag{4.1}$$

where for each $j \in [1, J]$, ψ_n^j is given by

$$\psi_n^j(x) := \sum_{k=1}^{N^j} \mathcal{G}_k^j\left[\phi^j(x - x_n^j)\right], \tag{4.2}$$

and R_n^J is a remainder term. Furthermore, the following properties hold:

(i)

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|e^{-it(\Delta^2 - \mu\Delta)}R_n^J\|_{X(\mathbb{R})} = 0. \tag{4.3}$$

(ii) *For each* $j \in [1, J^*]$, *either* $x_n^j \equiv 0$ *or* $\lim_{n\to\infty} |x_n^j| = \infty$ *holds and either* $t_n^j \equiv 0$, $\lim_{n\to\infty} t_n^j = -\infty$ *or* $\lim_{n\to\infty} t_n^j = +\infty$ *holds. If* $1 \le j \ne k \le J^*$, *then for all* $\mathcal{G}, \mathcal{G}' \in G$,

$$\lim_{n\to\infty} \left\{|t_n^j - t_n^k| + |\mathcal{G}x_n^j - \mathcal{G}'x_n^k|\right\} = \infty. \tag{4.4}$$

(iii) *If* $x_n^j \equiv 0$, *then* $G^j = G$, $N^j = 1$ *and* $\psi_n^j = \phi^j$. *If* $|x_n^j| \to \infty$, *then we may assume that* $\frac{x_n^j}{|x_n^j|}$ *converges to* $w^j \in \mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid |x| = 1\}$. *In this case,* $G^j = G_{w^j} = \{\mathcal{G} \in G \mid \mathcal{G}w^j = w^j\}$ *and* $N^j = \#Gw^j$. *Moreover,* ψ_n^j *is* G*-invariant and*

$$\psi_n^j(\cdot + \mathcal{G}x_n^j) \rightharpoonup \mathcal{G}\phi^j \quad \text{weakly in } H^2(\mathbb{R}^d) \quad \text{for all } \mathcal{G} \in G, \tag{4.5}$$

$$\lim_{n\to\infty} \|\psi_n^j\|_{\dot{H}^s}^2 = (\#Gw^j)\|\phi^j\|_{\dot{H}^s}^2 \quad \text{for all } \ 0 \le s \le 2. \tag{4.6}$$

(iv) *For each* $1 \le J \le J^*$,

$$\lim_{n\to\infty} \left\{ \|f_n\|_{\dot{H}^s}^2 - \sum_{j=1}^{J} \|\psi_n^j\|_{\dot{H}^s}^2 - \|R_n^J\|_{\dot{H}^s}^2 \right\} = 0 \quad \text{for all } \ 0 \le s \le 2. \tag{4.7}$$

Remark 4.2. In [19], the similar profile decomposition for G-invariant functions for the second-order Schrödinger propagator is obtained under the assumption that $\#G < \infty$.

Remark 4.3. If $|x_n^j| \to \infty$, then $\inf_{|x|=1} \#Gx \le \#Gw^j < \infty$. Therefore, if $\inf_{|x|=1} \#Gx = \infty$, then $x_n^j \equiv 0$ for all $j \in [1, J^*]$.

To prove Proposition 4.1, we prepare some lemmas.

Lemma 4.4. *Let* $f \in H^2(\mathbb{R}^d)$, *and let* $\{(t_n, x_n)\} \subset \mathbb{R} \times \mathbb{R}^d$ *be such that* $|t_n| + |x_n| \to \infty$. *Then,*

$$e^{-it_n(\Delta^2 - \mu\Delta)} f(x + x_n) \rightharpoonup 0 \quad \text{weakly in } H^2(\mathbb{R}^d).$$

Proof. Lemma 4.4 can be proved by using a density argument and (2.5). We omit the detail. ☐

Lemma 4.5. *Let* G *be a subgroup of* $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ *and* $\{f_n\}$ *be a sequence of* G*-invariant functions uniformly bounded in* H_G^2. *Suppose that there exists a function* $\phi \in H^2(\mathbb{R}^d) \setminus \{0\}$ *and a sequence* $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d$ *such that*

$$e^{-it_n(\Delta^2 - \mu\Delta)} f_n(x + x_n) \rightharpoonup \phi \quad \text{weakly in } \ H^2(\mathbb{R}^d) \tag{4.8}$$

and $|x_n| \to \infty$ *as* $n \to \infty$. *Then passing to a subsequence, we may assume that* $\frac{x_n}{|x_n|}$ *converges to some* $w \in \mathbb{S}^{d-1}$. *Moreover,* $\#Gw < \infty$.

Proof. Since $|x_n| \to \infty$, we may assume that $x_n \ne 0$ and $\frac{x_n}{|x_n|} \in \mathbb{S}^{d-1}$. Hence, passing to a subsequence if necessary, $\frac{x_n}{|x_n|}$ converges to some $w \in \mathbb{S}^{d-1}$. Then we may assume that $x_n = |x_n|w$ for sufficiently large n.

Now we prove $\#Gw < \infty$ by contradiction argument. For $x \in \mathbb{R}^d$ and $r > 0$, we define

$$\begin{aligned} m(x, r) := \sup\{N \in \mathbb{N} \mid & \text{ there exists } \mathcal{G}_1, ..., \mathcal{G}_N \in G \\ & \text{such that } B_r(\mathcal{G}_k^{-1}x) \cap B_r(\mathcal{G}_l^{-1}x) = \emptyset, \text{ for all } k \ne l\}. \end{aligned}$$

Suppose that $\#Gw = \infty$. Then for each $r > 0$, we have

$$\lim_{n\to\infty} m(x_n, r) = \infty.$$

Hence, since f_n is G-invariant and uniformly bounded in $H^2(\mathbb{R}^d)$, if n is sufficiently large, then

$$\|f_n\|_{H^2(B_r(x_n))} \le \frac{\|f_n\|_{H^2}}{m(x_n,r)} < \varepsilon.$$

By using (4.8), for any $r > 0$ we obtain

$$\|\phi\|_{H^2(B_r(0))} \le \limsup_{n\to\infty} \|f_n\|_{H^2(B_r(x_n))} < \varepsilon.$$

On the other hand, taking $r > 0$ sufficiently large, we have

$$\|\phi\|_{H^2(\mathbb{R}^d\setminus B_r(0))} < \varepsilon.$$

Therefore, we have $\|\phi\|_{H^2} < 2\varepsilon$ for any $\varepsilon > 0$, which is a contradiction. $\square$

Lemma 4.6. *Let $d \ge 1$, $\mu \ge 0$ and p satisfy* (1.2). *Let $\{f_n\}$ be a sequence of functions in $H^2(\mathbb{R}^d)$. Suppose that*

$$A := \lim_{n\to\infty} \|f_n\|_{H^2} < \infty \quad \textit{and} \quad \varepsilon := \lim_{n\to\infty} \|e^{-it(\Delta^2-\mu\Delta)} f_n\|_{L_t^\infty L_x^{\frac{d(p-1)}{4}}} > 0. \tag{4.9}$$

Then, passing to a subsequence, there exist a function $\phi \in H^2(\mathbb{R}^d)$ and a sequence $\{(t_n, x_n)\} \subset \mathbb{R} \times \mathbb{R}^d$ such that

$$e^{-it_n(\Delta^2-\mu\Delta)} f_n(x + x_n) \rightharpoonup \phi \quad \textit{weakly in } H^2(\mathbb{R}^d),$$
$$\|\phi\|_{\dot H^{s_c}} \gtrsim A\left(\frac{\varepsilon}{A}\right)^{\frac{(d+2)(p-1)-8}{d(p-1)-8}}.$$

Proof. (4.9) implies that for sufficiently large n,

$$\|f_n\|_{H^2} \le 2A \quad \text{and} \quad \|e^{-it(\Delta^2-\mu\Delta)} f_n\|_{L_t^\infty L_x^{\frac{d(p-1)}{4}}} \ge \frac{\varepsilon}{2}. \tag{4.10}$$

Let $r \ge 1$ to be chosen later, and let $\chi(x)$ be a smooth radial function such that $\hat\chi(\xi) = 1$ for $\frac{1}{r} \le |\xi| \le r$ and $\hat\chi(\xi)$ is supported in $\frac{1}{2r} \le |\xi| \le 2r$. By the Sobolev embedding,

$$\begin{aligned}
\|e^{-it(\Delta^2-\mu\Delta)}(f_n - \chi * f_n)\|^2_{L_t^\infty L_x^{\frac{d(p-1)}{4}}} &\lesssim \int_{\mathbb{R}^d} |\xi|^{2s_c}(1-\hat\chi(\xi))^2 |\widehat{f_n}(\xi)|^2 d\xi \\
&\le \int_{|\xi|\le\frac{1}{r}} |\xi|^{2s_c}|\widehat{f_n}(\xi)|^2 d\xi + \int_{|\xi|\ge r} |\xi|^{2s_c}|\widehat{f_n}(\xi)|^2 d\xi \\
&\le r^{-2s_c}\|f_n\|^2_{L^2} + r^{-2(2-s_c)}\|f_n\|^2_{\dot H^2}.
\end{aligned}$$

Hence, from (4.10) there exists $c_1 > 0$ such that for sufficiently large n, we have

$$\|e^{-it(\Delta^2-\mu\Delta)} f_n - \chi * e^{-it(\Delta^2-\mu\Delta)} f_n\|_{L_t^\infty L_x^{\frac{d(p-1)}{4}}} \le c_1 A r^{-\min\{s_c, 2-s_c\}}.$$

Thus, taking $r = \max\{1, \left(\frac{4c_1A}{\varepsilon}\right)^{\frac{1}{\min\{s_c,2-s_c\}}}\}$ so that $c_1 A r^{-\min\{s_c,2-s_c\}} \le \frac{\varepsilon}{4}$, from (4.10) we obtain

$$\|\chi * e^{-it(\Delta^2-\mu\Delta)} f_n\|_{L_t^\infty L_x^{\frac{d(p-1)}{4}}} \ge \frac{\varepsilon}{4}.$$

On the other hand, by the Hölder inequality, we obtain

$$\begin{aligned}\|\chi * e^{-it(\Delta^2-\mu\Delta)}f_n\|_{L_t^\infty L_x^{\frac{d(p-1)}{4}}} &\le \|\chi * e^{-it(\Delta^2-\mu\Delta)}f_n\|_{L_t^\infty L_x^2}^{\frac{8}{d(p-1)}} \|\chi * e^{-it(\Delta^2-\mu\Delta)}f_n\|_{L_{t,x}^\infty}^{\frac{d(p-1)-8}{d(p-1)}} \\ &\le A^{\frac{8}{d(p-1)}} \|\chi * e^{-it(\Delta^2-\mu\Delta)}f_n\|_{L_{t,x}^\infty}^{\frac{d(p-1)-8}{d(p-1)}}.\end{aligned}$$

Therefore, we have

$$\|\chi * e^{-it(\Delta^2-\mu\Delta)}f_n\|_{L_{t,x}^\infty} \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(p-1)}{d(p-1)-8}}.$$

This implies that there exists $(t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d$ such that

$$|\chi * e^{-it_n(\Delta^2-\mu\Delta)}f_n(x_n)| \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(p-1)}{d(p-1)-8}}. \tag{4.11}$$

Let $g_n := e^{-it_n(\Delta^2-\mu\Delta)}f_n(x+x_n)$. Then $\|g_n\|_{H^2} = \|f_n\|_{H^2} \le 2A$. Hence, there exists $\phi \in H^2(\mathbb{R}^d)$ and $g_n \rightharpoonup \phi$ weakly in $H^2(\mathbb{R}^d)$. By the Hölder inequality and the Sobolev inequality and using (4.11), we have

$$\begin{aligned}\|\chi\|_{L^{\frac{p-1}{p-5}}}\|\phi\|_{\dot H^{s_c}} \gtrsim |\langle\chi,\phi\rangle_{L^2}| &= \lim_{n\to\infty}|\langle\chi, g_n\rangle_{L^2}| \\ &= \lim_{n\to\infty}\left|\int_{\mathbb{R}}\chi(x)e^{-it_n(\Delta^2-\mu\Delta)}f_n(x+x_n)dx\right| \\ &= \lim_{n\to\infty}|\chi * e^{-it_n(\Delta^2-\mu\Delta)}f_n(x_n)| \\ &\gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{d(p-1)}{d(p-1)-8}}.\end{aligned} \tag{4.12}$$

Since $\|\chi\|_{L^{\frac{p-1}{p-5}}} \lesssim r^{\frac{d(p-5)}{p-1}} \lesssim \left(\frac{A}{\varepsilon}\right)^{\frac{2(p-5)}{d(p-1)-8}}$, from (4.12) we have

$$\|\phi\|_{\dot H^{s_c}} \gtrsim A\left(\tfrac{\varepsilon}{A}\right)^{\frac{(d+2)(p-1)-8}{d(p-1)-8}}.$$

This completes the proof of Lemma 4.6. □

Proof of Proposition 4.1. For uniformly bounded sequences $\{f_n\} \subset H^2(\mathbb{R}^d)$, we define

$$\mathcal{V}(\{f_n\}) := \left\{ \phi \in H^2(\mathbb{R}^d) \;\middle|\; \begin{array}{l} \exists\{(t_n,x_n)\} \subset \mathbb{R}\times\mathbb{R}^d \text{ s.t.} \\ \text{passing to a subsequence,} \\ e^{-it_n(\Delta^2-\mu\Delta)}f_n(x+x_n) \rightharpoonup \phi \text{ weakly in } H^2(\mathbb{R}^d). \end{array} \right\}$$

and

$$\eta(\{f_n\}) := \sup_{\phi\in\mathcal{V}(\{f_n\})}\|\phi\|_{H^2}^2.$$

Then, we note that $\eta(\{f_n\}) \le \limsup_{n\to\infty}\|f_n\|_{H^2}^2$. To prove Proposition 4.1, we first show the decomposition (4.1) with the property (i) replaced by

$$\lim_{J\to J^*}\eta(\{R_n^J\}) = 0, \tag{4.13}$$

and with properties (ii), (iii) and (iv). We will extract ϕ^j, t_n^j, x_n^j by an inductive process.

Set $R_n^0 := f_n$. If $\eta(\{f_n\}) = 0$, then the proof is complete with $J^* = 0$. If not, by the definition of $\eta(\{f_n\})$, there exists $\phi^1 \in \mathcal{V}(\{f_n\})$ such that

$$\|\phi^1\|_{H^2} \geq \tfrac{1}{2}\eta(\{f_n\}) > 0.$$

In particular, $\phi^1 \neq 0$. Moreover, by the definition of $\mathcal{V}(\{f_n\})$, there exists $\{(t_n^1, x_n^1)\} \subset \mathbb{R} \times \mathbb{R}^d$ such that

$$e^{-it_n^1(\Delta^2-\mu\Delta)} f_n(x + x_n^1) \rightharpoonup \phi^1 \quad \text{weakly in } H^2(\mathbb{R}^d). \tag{4.14}$$

Passing to a subsequence, one may suppose that either $\lim n \to \infty |x_n^1| = \infty$ or $x_n^1 \to x^1 \in \mathbb{R}^d$ as $n \to \infty$. We claim that one can choose $x_n^1 \equiv 0$ in the latter case. This is because

$$\begin{aligned}\langle e^{-it_n^1(\Delta^2-\mu\Delta)} f_n, \psi \rangle_{H^2} &= \langle e^{-it_n^1(\Delta^2-\mu\Delta)} f_n(\cdot + x_n^1), \psi(\cdot + x_n^1) \rangle_{H^2} \\ &\to \langle \phi^1, \psi(\cdot + x^1) \rangle_{H^2} \\ &= \langle \phi^1(\cdot + x^1), \psi \rangle_{H^2}\end{aligned}$$

as $n \to \infty$, for all $\psi \in C_c^\infty(\mathbb{R}^d)$. This implies that (4.14) is valid with the choice $x_n^1 \equiv 0$ with the modification $\phi^1 \mapsto \phi^1(\cdot - x^1)$. The claim is proved. Similarly, one verifies that either $\lim_{n\to\infty} |t_n^1| = \infty$ or $t_n^1 \equiv 0$ holds, by passing to a subsequence if necessary.

Since f_n is G-invariant, from (4.14) we have

$$e^{-it_n^1(\Delta^2-\mu\Delta)} (\mathcal{G}^{-1} f_n)(x + x_n^1) \rightharpoonup \phi^1 \quad \text{weakly in } H^2(\mathbb{R}^d) \tag{4.15}$$

for all $\mathcal{G} \in G$. If $x_n^1 \equiv 0$, then (4.15) implies that ϕ^1 is G-invariant. In this case, we set $G^1 = G$, $N^1 = 1$ and $\psi_n^1 = \phi^1$. If $|x_n^1| \to \infty$, then passing to a subsequence, we may assume that $\frac{x_n^1}{|x_n^1|}$ converges to some $w^1 \in \mathbb{S}^{d-1}$ and $x_n^1 = |x_n^1| w^1$. In this case, we let G^1 be a subgroup of G defined by $G^1 = G_{w^1} = \{\mathcal{G} \in G \mid \mathcal{G} w^1 = w^1\}$ and $N^1 := \#(G/G^1)$. Then from (4.15), we see that ϕ^1 is G^1-invariant. From Lemma 4.5, we have $N^1 = \#(G/G^1) = \# G w^1 < \infty$, and hence, there exist $\mathcal{G}_1^1, ..., \mathcal{G}_{N^1}^1 \in G$ such that $G = \bigcup_{k=1}^{N^1} \mathcal{G}_k^1 G^1$ and $\mathcal{G}_k^1 G^1 \cap \mathcal{G}_l^1 G^1 = \emptyset$ for any $k \neq l$. We define

$$\psi_n^1(x) = \sum_{k=1}^{N^1} \mathcal{G}_k^1[\phi^1(x - x_n^1)].$$

Then ψ_n^1 is G-invariant. Indeed,

$$\mathcal{G}\psi_n^1 = \sum_{k=1}^{N^1} \mathcal{G}\mathcal{G}_k^1[\phi^1(x - x_n^1)] = \sum_{k=1}^{N^1} (\mathcal{G}\mathcal{G}_k^1 \phi^1)(x - \mathcal{G}\mathcal{G}_k^1 x_n^1).$$

Since $G = \bigcup_{k=1}^{N^1} \mathcal{G}_k^1 G^1$, for all $1 \leq k \leq N^1$, there exist $\mathcal{G}' \in G^1$ and $\mathcal{G}_{l_k}^1$ such that $\mathcal{G}\mathcal{G}_k^1 = \mathcal{G}_{l_k}^1 \mathcal{G}'$. Hence, by using $\mathcal{G}' x_n^1 = |x_n^1| \mathcal{G}' w^1 = x_n^1$ and $\mathcal{G}'\phi^1 = \phi^1$, we have

$$\mathcal{G}\psi_n^1 = \sum_{k=1}^{N^1} (\mathcal{G}_{l_k}^1 \phi^1)(x - \mathcal{G}_{l_k}^1 x_n^1) = \sum_{k=1}^{N^1} \mathcal{G}_{l_k}^1[\phi(x - x_n^1)] = \psi_n^1,$$

where the last equality follows from the fact that $k \mapsto \mathcal{G}^1_{l_k}$ is injective. Further, for each $\mathcal{G} \in G$, there exists $\mathcal{G}^1_l$ such that $\mathcal{G}x^1_n = \mathcal{G}^1_l x^1_n$. Since $|\mathcal{G}^1_k x^1_n - \mathcal{G}^1_l x^1_n| \to \infty$ for any $k \neq l$, we have

$$\psi^1_n(x + \mathcal{G}x^1_n) = \sum_{k=1}^{N^1} (\mathcal{G}^1_k \phi^1)(x + \mathcal{G}^1_l x^1_n - \mathcal{G}^1_k x^1_n) \rightharpoonup \mathcal{G}^1_l \phi^1 = \mathcal{G}\phi^1,$$

where the last equality follows from the fact that $\mathcal{G}^{-1}\mathcal{G}^1_l \in G^1$ and ϕ^1 is G^1-invariant. Furthermore, since $|\mathcal{G}^1_k x^1_n - \mathcal{G}^1_l x^1_n| \to \infty$ for any $1 \leq k \neq l \leq N^1$, we have

$$\begin{aligned}\lim_{n\to\infty} \|\psi^1_n\|^2_{\dot{H}^s} &= N^1 \|\phi^1\|^2_{\dot{H}^s} \\ &\quad + \lim_{n\to\infty} 2 \sum_{1\leq k<l\leq N^1} \left\langle \mathcal{G}^1_k \phi^1, \mathcal{G}^1_l \phi^1(x - (\mathcal{G}^1_l x^1_n - \mathcal{G}^1_k x^1_n)) \right\rangle_{\dot{H}^s} \\ &= (\#Gw^1)\|\phi^1\|^2_{\dot{H}^s}.\end{aligned}$$

Set $R^1_n := f_n - e^{it^1_n(\Delta^2 - \mu\Delta)}\psi^1_n$. Then R^1_n is G-invariant and we have

$$\begin{aligned}\|f_n\|^2_{\dot{H}^s} - \|R^1_n\|^2_{\dot{H}^s} &= 2\langle f_n, e^{it^1_n(\Delta^2-\mu\Delta)}\psi^1_n\rangle_{\dot{H}^s} - \|e^{it^1_n(\Delta^2-\mu\Delta)}\psi^1_n\|^2_{\dot{H}^s} \\ &= 2\sum_{k=1}^{N^1} \langle e^{-it^1_n(\Delta^2-\mu\Delta)}((\mathcal{G}^1_k)^{-1} f_n)(x + x_n), \phi^1\rangle_{\dot{H}^s} - \|\psi^1_n\|^2_{\dot{H}^s} \\ &\to (\#Gw^1)\|\phi^1\|^2_{\dot{H}^s}.\end{aligned}$$

Hence, the properties (iii) and (iv) in Proposition 4.1 hold for $J = 1$.

Let $J_0 \geq 1$ be fixed. We suppose that for all $1 \leq J \leq J_0$ we have the decomposition (4.1) with the properties (iii) and (iv), and also suppose that (4.4) holds for any $1 \leq j \neq k \leq J_0$. If $\eta(\{R^{J_0}_n\}) = 0$, then we stop the induction and set $J^* = J_0$. If $\eta(\{R^{J_0}_n\}) > 0$, then by the definition of $\eta(\{R^{J_0}_n\})$, we find $\phi^{J_0+1} \in \mathcal{V}(\{R^{J_0}_n\})$ such that

$$\|\phi^{J_0+1}\|_{H^2} \geq \tfrac{1}{2}\eta(\{R^{J_0}_n\}) > 0. \tag{4.16}$$

Moreover, there exist $\{(t^{J_0+1}_n, x^{J_0+1}_n)\} \subset \mathbb{R} \times \mathbb{R}^d$ such that

$$e^{-it^{J_0+1}_n(\Delta^2-\mu\Delta)} R^{J_0}_n(x + x^{J_0+1}_n) \rightharpoonup \phi^{J_0+1} \quad \text{weakly in } H^2(\mathbb{R}^d).$$

Arguing as above, passing to a subsequence if necessary, we see that $\{t^{J_0+1}_n\}$ and $\{x^{J_0+1}_n\}$ satisfy the alternative in the property (ii).

Since $R^{J_0}_n$ is G-invariant, we have

$$e^{-it^{J_0+1}_n(\Delta^2-\mu\Delta)}(\mathcal{G}^{-1}R^{J_0}_n)(x + x^{J_0+1}_n) \rightharpoonup \phi^{J_0+1} \quad \text{wealy in } H^2(\mathbb{R}^d) \tag{4.17}$$

for all $\mathcal{G} \in G$. If $x^{J_0+1}_n \equiv 0$, then (4.17) implies that ϕ^{J_0+1} is G-invariant. In this case, we set $G^{J_0+1} = G$, $N^{J_0+1} = 1$ and $\psi^{J_0+1}_n = \phi^{J_0+1}$. If $|x^{J_0+1}_n| \to \infty$, then passing to a subsequence, we may assume that $\frac{x^{J_0+1}_n}{|x^{J_0+1}_n|}$ converges to some $w^{J_0+1} \in \mathbb{S}^{d-1}$ and $x^{J_0+1}_n = |x^{J_0+1}_n| w^{J_0+1}$. In this case, we let G^{J_0+1} be a subgroup of G defined by $G^{J_0+1} = G_{w^{J_0+1}} = \{\mathcal{G} \in G \mid \mathcal{G}w^{J_0+1} = w^{J_0+1}\}$ and $N^{J_0+1} := \#(G/G^{J_0+1})$. Then from (4.17), we see that ϕ^{J_0+1} is G^{J_0+1}-invariant. From Lemma 4.5, we have $N^{J_0+1} = \#(G/G^{J_0+1}) = \#Gw^{J_0+1} <$

∞, and hence, there exist $\mathcal{G}_1^{J_0+1}, ..., \mathcal{G}_{N^{J_0+1}}^{J_0+1} \in G$ such that $G = \bigcup_{k=1}^{N^{J_0+1}} \mathcal{G}_k^{J_0+1} G^{J_0+1}$ and $\mathcal{G}_k^{J_0+1} G^{J_0+1} \cap \mathcal{G}_l^{J_0+1} G^{J_0+1} = \emptyset$ for any $k \neq l$. We define

$$\psi_n^{J_0+1}(x) = \sum_{k=1}^{N^{J_0+1}} \mathcal{G}_k^{J_0+1}[\phi^{J_0+1}(x - x_n^{J_0+1})].$$

Then $\psi_n^{J_0+1}$ is G-invariant and

$$\psi_n^{J_0+1}(x + \mathcal{G} x_n^{J_0+1}) \rightharpoonup \mathcal{G}\phi^{J_0+1} \quad \text{for all } \mathcal{G} \in G,$$
$$\lim_{n\to\infty} \|\psi_n^{J_0+1}\|_{\dot{H}^s}^2 = (\#Gw^{J_0+1})\|\phi^{J_0+1}\|_{\dot{H}^s}^2.$$

We set $R_n^{J_0+1} := R_n^{J_0} - e^{it_n^{J_0+1}(\Delta^2-\mu\Delta)}\psi_n^{J_0+1}(x)$. Then $R_n^{J_0+1}$ is G-invariant and we have

$$\|R_n^{J_0}\|_{\dot{H}^s}^2 - \|R_n^{J_0+1}\|_{\dot{H}^s}^2 \to (\#Gw^{J_0+1})\|\phi^{J_0+1}\|_{\dot{H}^s}^2.$$

Iterating this, we obtain (4.7) for $J = J_0 + 1$. Hence, we have the decomposition (4.1) with properties (iii) and (iv) for $J = J_0 + 1$.

To prove (4.4) for all $1 \le j \neq k \le J_0 + 1$, it suffices to claim that for all $1 \le j \le J_0$ and for all $\mathcal{G}, \mathcal{G}' \in G$,

$$|t_n^j - t_n^{J_0+1}| + |\mathcal{G}x_n^j - \mathcal{G}'x_n^{J_0+1}| \to \infty. \tag{4.18}$$

Suppose that there exists $1 \le j_0 \le J_0$ and $\mathcal{G}, \mathcal{G}' \in G$ such that $|t_n^{j_0} - t_n^{J_0+1}|$ and $|\mathcal{G}x_n^{j_0} - \mathcal{G}'x_n^{J_0+1}|$ are finite. Then passing to a subsequence, we may assume that

$$t_n^{j_0} - t_n^{J_0+1} \to t_0 \quad \text{and} \quad \mathcal{G}x_n^{j_0} - \mathcal{G}'x_n^{J_0+1} \to x_0. \tag{4.19}$$

By using $\psi(x + x_n^{J_0}) \rightharpoonup \phi^{J_0}$, we have $e^{-it_n^{j_0}(\Delta^2-\mu\Delta)}R_n^{j_0}(x + x_n^{j_0}) \rightharpoonup 0$. Hence, from (4.19) we have

$$\begin{aligned}
&\operatorname*{w-lim}_{n\to\infty} e^{-it_n^{J_0+1}(\Delta^2-\mu\Delta)}R_n^{j_0}(x + \mathcal{G}'x_n^{J_0+1}) \\
&= \operatorname*{w-lim}_{n\to\infty} e^{i(t_n^{j_0}-t_n^{J_0+1})(\Delta^2-\mu\Delta)} \left[e^{-it_n^{j_0}(\Delta^2-\mu\Delta)}\mathcal{G}[R_n^{j_0}(\cdot + x_n^{j_0})]\right](x - (\mathcal{G}x_n^{j_0} - \mathcal{G}'x_n^{J_0+1})) \\
&= 0.
\end{aligned}$$

Thus, if $j_0 = J_0$, then $\mathcal{G}'\phi^{J_0+1} = 0$. If $J_0 \ge 2$ and $1 \le j_0 \le J_0 - 1$, using the inductive relation $R_n^{J_0} = R_n^{j_0} - \sum_{j=j_0+1}^{J_0} e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j$, we have

$$\begin{aligned}
\mathcal{G}'\phi^{J_0+1} &= \operatorname*{w-lim}_{n\to\infty} e^{-it_n^{J_0+1}(\Delta^2-\mu\Delta)}R_n^{J_0}(x + \mathcal{G}'x_n^{J_0+1}) \\
&= \operatorname*{w-lim}_{n\to\infty} e^{-it_n^{J_0+1}(\Delta^2-\mu\Delta)}R_n^{j_0}(x + \mathcal{G}'x_n^{J_0+1}) \\
&\quad - \sum_{j=j_0+1}^{J_0} \operatorname*{w-lim}_{n\to\infty} e^{i(t_n^j-t_n^{J_0+1})(\Delta^2-\mu\Delta)}\psi_n^j(x + \mathcal{G}'x_n^{J_0+1}) \\
&= -\sum_{j=j_0+1}^{J_0} \operatorname*{w-lim}_{n\to\infty} e^{i(t_n^j-t_n^{J_0+1})(\Delta^2-\mu\Delta)}\psi_n^j(x + \mathcal{G}'x_n^{J_0+1}).
\end{aligned}$$

Since $|x_n^j - \mathcal{G}x_n^{j_0}| + |t_n^j - t_n^{j_0}| \to \infty$ for all $j_0+1 \le j \le J_0$, from (4.19) and Lemma 4.4,

$$\begin{aligned}
&\operatorname*{w-lim}_{n\to\infty} e^{i(t_n^j - t_n^{J_0+1})(\Delta^2-\mu\Delta)}\psi_n^j(x+\mathcal{G}'x_n^{J_0+1})\\
=\ &\operatorname*{w-lim}_{n\to\infty} e^{i(t_n^j - t_n^{J_0+1})(\Delta^2-\mu\Delta)}\psi_n^j(x+x_n^j-(x_n^j-\mathcal{G}'x_n^{J_0+1}))\\
=\ &\operatorname*{w-lim}_{n\to\infty} e^{i(t_n^{j_0} - t_n^{J_0+1})(\Delta^2-\mu\Delta)}\left[e^{i(t_n^j - t_n^{j_0})(\Delta^2-\mu\Delta)}\phi^j(\cdot-(x_n^j-\mathcal{G}x_n^{j_0}))\right](x-(\mathcal{G}x_n^{j_0}-\mathcal{G}'x_n^{J_0+1}))\\
=\ &0
\end{aligned}$$

for all $j_0+1 \le j \le J_0$. Hence, $\mathcal{G}'\phi^{J_0+1} = 0$. This contradicts (4.16) and so we have proved that (4.4) holds for all $1 \le j \ne k \le J_0+1$.

If $\eta(\{R_n^{J_0+1}\}) = 0$, we stop the induction and set $J^* = J_0+1$. In this case, (4.13) holds automatically. If $\eta(\{R_n^{J_0+1}\}) > 0$, we continue the induction. If the inductive process does not terminate in finitely many steps, we set $J^* = \infty$. In this case, from (4.7) we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j=1}^{J}\|\psi_n^j\|_{H^2}^2 \le \limsup_{n\to\infty}\|f_n\|_{H^2}^2 < \infty.$$

This together with (4.6) implies that $(\#Gw^j)\|\phi^j\|_{H^2}^2 \to 0$ as $j\to\infty$. Since $\#Gw^j \ge 1$ for all $j \ge 1$, we have $\|\phi^j\|_{H^2} \to 0$ as $j \to \infty$. Thus, by using $\|\phi^j\|_{H^2} \ge \frac{1}{2}\eta(\{R_n^j\})$, we obtain (4.13).

Now we will prove the property (i) in Proposition 4.1. Note by the interpolation, the Strichartz estimate (Lemma 2.1) and the Sobolev inequality, it suffices to show

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|e^{-it(\Delta^2-\mu\Delta)}R_n^J\|_{L_t^\infty L_x^{\frac{d(p-1)}{4}}(\mathbb{R}\times\mathbb{R}^d)} = 0. \tag{4.20}$$

For each $0 \le J \le J^*$, let

$$A_J := \limsup_{n\to\infty}\|R_n^J\|_{H^2} \quad\text{and}\quad \varepsilon_J := \limsup_{n\to\infty}\|e^{-it(\Delta^2-\mu\Delta)}R_n^J\|_{L_t^\infty L_x^{\frac{p-1}{4}}(\mathbb{R}\times\mathbb{R}^d)}.$$

Applying Lemma 4.6, there exist $\tilde\phi^{J+1} \in H^2(\mathbb{R}^d)$ and $\{(\tilde t_n^{J+1}, \tilde x_n^{J+1})\} \subset \mathbb{R}\times\mathbb{R}^d$ such that

$$e^{-i\tilde t_n^{J+1}(\Delta^2-\mu\Delta)}R_n^J(x+\tilde x_n^{J+1}) \rightharpoonup \tilde\phi^{J+1} \quad \text{weakly in } H^2(\mathbb{R}^d)$$

and

$$\|\tilde\phi^{J+1}\|_{\dot H^{s_c}} \gtrsim A_J\left(\frac{\varepsilon_J}{A_J}\right)^{\frac{(d+2)(p-1)-8}{d(p-1)-8}} = A_J^{-\frac{2(p-1)}{d(p-1)-8}}\varepsilon_J^{\frac{(d+2)(p-1)-8}{d(p-1)-8}}. \tag{4.21}$$

Since $\tilde\phi^{J+1} \in \mathcal{V}(\{R_n^J\})$, by using (4.13),

$$\|\tilde\phi^{J+1}\|_{\dot H^{s_c}} \lesssim \|\tilde\phi^{J+1}\|_{H^2} \le \eta(\{R_n^J\}) \to 0 \quad \text{as } J \to J^*. \tag{4.22}$$

Since $\|R_n^{J+1}\|_{H^2}^2 = \|R_n^J\|_{H^2}^2 - \|\psi_n^{J+1}\|_{H^2}^2 + o(1)$, we see $A_{J+1} \le A_J$. In particular, $A_J \le A_0 = \limsup_{n\to\infty}\|f_n\|_{H^2} < \infty$ for all $0 \le J \le J^*$. Hence, (4.21) and (4.22) imply $\varepsilon_J \to 0$ as $J \to J^*$ and so (4.20) follows. ☐

Corollary 4.7. *In the situation of Proposition 4.1, for each $1 \le J \le J^*$ we have*

$$\lim_{n\to\infty}\left\{\|f_n\|_{L^{p+1}}^{p+1} - \sum_{j=1}^{J}\|e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j\|_{L^{p+1}}^{p+1} - \|R_n^J\|_{L^{p+1}}^{p+1}\right\} = 0,$$

where for all $j \in [1, J^*]$,

$$\lim_{n\to\infty} \|e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j\|_{L^{p+1}}^{p+1} = (\#Gw^j)\|e^{it_n^j(\Delta^2-\mu\Delta)}\phi^j\|_{L^{p+1}}^{p+1}.$$

Proof. Corollary 4.7 can be proved by using (4.4), (4.6) and (4.7) and by applying the dispersive estimate (2.5). □

5. Concentration compactness

In this section, we construct a critical G-invariant solution u of (1.1) that satisfies $S_\omega(u) = L^*_{G,\omega}$, under the assumption that $L^*_{G,\omega} < S_\omega(Q_\omega)$. To do this, we need the additional hypothesis $L^*_{G,\omega} < m_{G,\omega}$.

Lemma 5.1 (Concentration compactness)**.** *Let $d \geq 1$, $\mu \geq 0$ and p satisfy* (1.2)*. Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and $\omega > 0$. Suppose $L^*_{G,\omega} < S_\omega(Q_\omega)$, and let $\{u_n\}$ be a sequence of global G-invariant solutions to* (1.1) *with initial data $u_{n,0} \in H^2_G$ such that*

$$S_\omega(u_n) \to L^*_{G,\omega}, \quad K(u_{n,0}) > 0$$

and there exists $\{t_n\} \subset \mathbb{R}$ such that

$$\lim_{n\to\infty} \|u_n\|_{X((-\infty,t_n])} = \lim_{n\to\infty} \|u_n\|_{X([t_n,+\infty))} = \infty.$$

If $L^*_{G,\omega} < m_{G,\omega}$, then there exists a G-invariant function $\phi \in H^2_G$ such that passing to a subsequence, $u_n(t_n)$ converges to ϕ strongly in $H^2(\mathbb{R}^d)$.

Proof. Since $L^*_{G,\omega} < S_\omega(Q_\omega)$, from Lemma 3.3 we have $K(u_n(t)) > 0$ for all $t \in \mathbb{R}$. Thus, by using time translation symmetry, we may assume $t_n \equiv 0$ without loss of generality. Then

$$\lim_{n\to\infty} \|u_n\|_{X((-\infty,0])} = \lim_{n\to\infty} \|u_n\|_{X([0,+\infty))} = \infty. \tag{5.1}$$

Applying Proposition 4.1 and Corollary 4.7 to $u_n(0)$, passing to a subsequence if necessary, for each $1 \leq J \leq J^*$ we have a decomposition

$$u_n(0) = \sum_{j=1}^{J} e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j + R_n^J,$$

with

$$M(u_n) = \sum_{j=1}^{J} M(\psi_n^j) + M(R_n^J) + o(1), \tag{5.2}$$

$$E(u_n) = \sum_{j=1}^{J} E(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) + E(R_n^J) + o(1), \tag{5.3}$$

$$\|u_n(0)\|_{\dot H^s}^2 = \sum_{j=1}^{J} \|\psi_n^j\|_{\dot H^s}^2 + \|R_n^J\|_{\dot H^s}^2 + o(1) \quad \text{for } 0 \leq s \leq 2, \tag{5.4}$$

where $o(1) \to 0$ as $n \to \infty$ for each $1 \le J \le J^*$. Note that for all $j \in [1, J^*]$, ψ_n^j and R_n^j are G-invariant. From (5.2) and (5.3), we have

$$\lim_{n\to\infty} \sum_{j=1}^{J} S_\omega(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) + \lim_{n\to\infty} S_\omega(R_n^J) = \lim_{n\to\infty} S_\omega(u_n) = L_{G,\omega}^* < S_\omega(Q_\omega). \tag{5.5}$$

Since $K(u_n(0)) > 0$, we have

$$\lim_{n\to\infty} \tilde{S}_\omega(u_n(0)) \le \lim_{n\to\infty} S_\omega(u_n(0)) = L_{G,\omega}^* < S_\omega(Q_\omega) = \tilde{S}_\omega(Q_\omega).$$

Hence, from (5.4), we have

$$\lim_{n\to\infty} \sum_{j=1}^{J} \tilde{S}_\omega(\psi_n^j) + \lim_{n\to\infty} \tilde{S}_\omega(R_n^J) = \lim_{n\to\infty} \tilde{S}_\omega(u_n(0)) < \tilde{S}_\omega(Q_\omega).$$

Thus, for n sufficiently large, $\tilde{S}_\omega(\psi_n^j) < \tilde{S}_\omega(Q_\omega)$ for each $j \in [1, J^*]$, and so, by Lemma 3.1, we have $K(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) > 0$ for each $j \in [1, J^*]$. Therefore, we see that for each $j \in [1, J^*]$,

$$\limsup_{n\to\infty} S_\omega(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) \ge \tilde{S}_\omega(\psi_n^j) > 0. \tag{5.6}$$

Similarly, we have

$$\limsup_{n\to\infty} S_\omega(R_n^J) \ge 0.$$

Now we consider the following three cases: $J^* = 0$, $J^* = 1$ and $J^* \ge 2$. We will show that the first case and the third case lead to contradiction. In the second case, we will get desired $\phi \in H_G^2$.

Case 1: $J^* = 0$. In this case, by (4.3), we have

$$\limsup_{n\to\infty} \|e^{-it(\Delta^2-\mu\Delta)}u_n(0)\|_{X(\mathbb{R})} = \limsup_{n\to\infty} \|e^{-it(\Delta^2-\mu\Delta)}R_n^0\|_{X(\mathbb{R})} = 0.$$

Hence, for n sufficiently large, we obtain

$$\|e^{-it(\Delta^2-\mu\Delta)}u_n(0)\|_{X(\mathbb{R})} \le \delta_{sd}.$$

Thus, by using Proposition 2.7, we have

$$\|u_n\|_{X(\mathbb{R})} \le 2\|e^{-it(\Delta^2-\mu\Delta)}u_n(0)\|_{X(\mathbb{R})} \le 2\delta_{sd}.$$

This contradicts (5.1) and so $J^* = 0$ does not occur.

Case 2: $J^* = 1$. In this case, we can write

$$u_n(0) = e^{it_n(\Delta^2-\mu\Delta)}\psi_n + R_n.$$

From the property (ii) in Proposition 4.1, either $t_n \equiv 0$, $\lim_{n\to\infty} t_n = -\infty$ or $\lim_{n\to\infty} t_n = +\infty$ holds. If $t_n \to -\infty$, then by the Strichartz estimate and the monotone convergence theorem,

$$\|e^{-i(t-t_n)(\Delta^2-\mu\Delta)}\psi_n\|_{X([0,+\infty))} = \|e^{-it(\Delta^2-\mu\Delta)}\psi_n\|_{X([-t_n,+\infty))} \to 0$$

as $n \to \infty$. Hence, from (4.3), we have

$$\limsup_{n\to\infty} \|e^{-it(\Delta^2-\mu\Delta)}u_n(0)\|_{X([0,+\infty))} = 0.$$

Thus, by Proposition 2.7, for n sufficiently large, we obtain

$$\|u_n\|_{X([0,+\infty))} \leq 2\delta_{sd}.$$

This contradicts (5.1) and so $t_n \to -\infty$ does not occur. By the similar argument, $t_n \to +\infty$ does not occur. Therefore, we may assume that $t_n \equiv 0$. Thus, we have

$$u_n(0) = \psi_n + R_n.$$

We claim that $\limsup_{n\to\infty} S_\omega(\psi_n) = L^*_{G,\omega}$. To prove this, suppose that

$$\limsup_{n\to\infty} S_\omega(\psi_n) < L^*_{G,\omega}$$

and let v_n be a solution to (1.1) such that $v_n(0) = \psi_n$. Since ψ_n is G-invariant, we see that v_n is also G-invariant. Then by the definition of $L^*_{G,\omega}$, we have $\limsup_{n\to\infty} \|v_n\|_{X(\mathbb{R})} < \infty$. Let

$$\tilde{u}_n(t) := v_n(t) + e^{-it(\Delta^2-\mu\Delta)}R_n$$

and

$$\begin{aligned} e_n &:= i\partial_t\tilde{u}_n - \Delta^2\tilde{u}_n + \mu\Delta\tilde{u}_n + |\tilde{u}_n|^{p-1}\tilde{u}_n \\ &= |\tilde{u}_n|^{p-1}\tilde{u}_n - |v_n|^{p-1}v_n. \end{aligned}$$

Then $\tilde{u}_n(0) = u_n(0)$ and by Lemma 2.6, we have

$$\|e_n\|_{N(\mathbb{R})} \lesssim \left(\|v_n\|_{X(\mathbb{R})} + \|e^{-it(\Delta^2-\mu\Delta)}R_n\|_{X(\mathbb{R})}\right)^{p-1} \|e^{-it(\Delta^2-\mu\Delta)}R_n\|_{X(\mathbb{R})}.$$

Hence, by using $\limsup_{n\to\infty} \|v_n\|_{X(\mathbb{R})} < \infty$ and $\limsup_{n\to\infty} \|e^{-it(\Delta^2-\mu\Delta)}R_n\|_{X(\mathbb{R})} = 0$, we obtain

$$\limsup_{n\to\infty} \|e_n\|_{N(\mathbb{R})} = 0.$$

Thus, by using Proposition 2.9,

$$\|u_n\|_{X(\mathbb{R})} \leq \|\tilde{u}_n\|_{X(\mathbb{R})} + o(1) \tag{5.7}$$

as $n \to \infty$. Since $\|\tilde{u}_n\|_{X(\mathbb{R})} \leq \|v_n\|_{X(\mathbb{R})} + \|e^{-it(\Delta^2-\mu\Delta)}R_n\|_{X(\mathbb{R})} < \infty$ for sufficiently large n, (5.7) contradicts (5.1), and so, $\limsup_{n\to\infty} S_\omega(\psi_n) = L^*_{G,\omega}$. Therefore, we see that $\limsup_{n\to\infty} S_\omega(R_n) = 0$, and thus, $u_n(0) = \psi_n + o(1)$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$.

From Proposition 4.1, there exist $\phi \in H^2(\mathbb{R}^d)$, $x_n \in \mathbb{R}^d$ and $\mathcal{G}_1, ..., \mathcal{G}_N \in G$ such that

$$\psi_n = \sum_{k=1}^{N} \mathcal{G}_k[\phi(x - x_n)]$$

and the following properties hold: Either $x_n \equiv 0$ or $|x_n| \to \infty$. If $x_n \equiv 0$, then $N = 1$ and $\psi_n = \phi$. If $|x_n| \to \infty$, then we may assume that $\frac{x_n}{|x_n|} \to w \in \mathbb{S}^{d-1}$ and $N = \#Gw < \infty$.

Moreover, $|\mathcal{G}_k x_n - \mathcal{G}_l x_n| \to 0$ for any $1 \le k \ne l \le N$. Now we claim that $x_n \equiv 0$. If $|x_n| \to \infty$, then from Proposition 4.1 and Corollary 4.7, we have

$$(\#Gw)S_\omega(\phi) = \limsup_{n\to\infty} S_\omega(\psi_n) = L^*_{G,\omega} < m_{G,\omega} \le (\#Gw)L^*_{G_w,\omega}. \tag{5.8}$$

This implies that $S_\omega(\phi) < L^*_{G_w,\omega}$. Similarly, we have $K(\phi) > 0$. Let v be a solution to (1.1) such that $v(0) = \phi$. Then by the definition of $L^*_{G_w,\omega}$, we have $\|v\|_{X(\mathbb{R})} < \infty$. We define

$$\tilde{u}_n(t) := \sum_{k=1}^{N} \mathcal{G}_k[v(t, x - x_n)]$$

and

$$\begin{aligned} e_n &:= i\partial_t \tilde{u}_n - \Delta^2 \tilde{u}_n + \mu\Delta\tilde{u}_n + |\tilde{u}_n|^{p-1}\tilde{u}_n \\ &= |\tilde{u}_n|^{p-1}\tilde{u}_n - \sum_{k=1}^{N} |\mathcal{G}_k v(t, x - \mathcal{G}_k x_n)|^{p-1}\mathcal{G}_k v(t, x - \mathcal{G}_k x_n). \end{aligned}$$

Then, by using $\|v\|_{X(\mathbb{R})} < \infty$ and $N < \infty$, we obtain

$$\|\tilde{u}_n\|_{X(\mathbb{R})} \le N\|v\|_{X(\mathbb{R})} < \infty.$$

Since $|\mathcal{G}_k x_n - \mathcal{G}_l x_n| \to 0$ for any $1 \le k \ne l \le N$, we have

$$\begin{aligned} \|e_n\|_{N(\mathbb{R})} &\lesssim \sum_{1\le k\ne l\le N} \||\mathcal{G}_k v(t, x - \mathcal{G}_k x_n)||\mathcal{G}_l v(t, x - \mathcal{G}_l x_n)|^{p-1}\|_{N(\mathbb{R})} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Thus, by Proposition 2.9, we have $\|u_n\|_{X(\mathbb{R})} < \infty$ for sufficiently large n, which contradicts (5.1). Therefore, $x_n \equiv 0$, and so, $N = 1$ and $\psi_n = \phi$. Consequently, $u_n(0)$ converges to a function ϕ strongly in $H^2(\mathbb{R}^d)$. In particular, ϕ is G-invariant. The proof of Lemma 5.1 is complete if we have proved that $J^* \ge 2$ does not occur.

Case 3: $J^* \ge 2$. In this case, there exist more than one profiles, and from (5.5) and (5.6) there exists $\varepsilon > 0$ such that

$$0 < \limsup_{n\to\infty} S_\omega(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) \le L^*_{G,\omega} - \varepsilon \tag{5.9}$$

for all $1 \le j \le J^*$. From the property (ii) in Proposition 4.1, for each $1 \le j \le J^*$, t_n^j has a limit $t_0^j \in \{-\infty, 0, \infty\}$ as $n \to \infty$. By using Lemma 2.7, we can define an unique solution w_n^j of (1.1) such that

$$w_n^j(0) = e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j. \tag{5.10}$$

From (5.9) and (5.10) we have

$$\limsup_{n\to\infty} S_\omega(w_n^j) \le L^*_{G,\omega} - \varepsilon < S_\omega(Q_\omega).$$

For n sufficiently large, by using $K(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) > 0$ and (5.10), we have

$$K(w_n^j(0)) > 0.$$

Hence, by Lemma 3.4, we see that w_n^j exists globally in time if n is sufficiently large. From (5.5) and $\limsup_{n\to\infty} S_\omega(R_n^J) \geq 0$, we have

$$\limsup_{n\to\infty} \sum_{j=1}^{J} S_\omega(w_n^j) = \limsup_{n\to\infty} \sum_{j=1}^{J} S_\omega(e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j) \leq L_{G,\omega}^*.$$

Since $C_{G,\omega}$ is sublinear around 0 and bounded on $[0, L_{G,\omega}^*]$, we have

$$\begin{aligned}\limsup_{n\to\infty} \sum_{j=1}^{J} \|w_n^j\|_{X(\mathbb{R})} \leq & \limsup_{n\to\infty} \sum_{j=1}^{J} C_{G,\omega}\left(S_\omega(w_n^j)\right) \\ \lesssim_{L_{G,\omega}^*,\varepsilon} & \limsup_{n\to\infty} \sum_{j=1}^{J} S_\omega(w_n^j) \\ \lesssim_{L_{G,\omega}^*,\varepsilon} & \ 1. \end{aligned} \tag{5.11}$$

From Proposition 4.1, for each $j \in [1, J^*]$, there exist $\phi^j \in H^2(\mathbb{R}^d)$, $x_n^j \in \mathbb{R}^d$ and $\mathcal{G}_1^j, ..., \mathcal{G}_{N^j}^j \in G$ such that

$$\psi_n^j = \sum_{k=1}^{N^j} \mathcal{G}_k^j[\phi^j(x - x_n^j)]$$

and the following properties hold: For each $j \in [1, J^*]$, either $x_n^j \equiv 0$ or $|x_n^j| \to \infty$. If $x_n^j \equiv 0$, then $N^j = 1$ and $\psi_n^j = \phi^j$. If $|x_n^j| \to \infty$, then we may assume that $\frac{x_n^j}{|x_n^j|} \to w^j \in \mathbb{S}^{d-1}$ and $N^j = \#Gw^j$. Moreover, $|\mathcal{G}_k^j x_n^j - \mathcal{G}_l^j x_n^j| \to 0$ for any $1 \leq k \neq l \leq N^j$. By using Proposition 2.7, we find an unique solution v^j of (1.1) defined on a neighborhood of $-t_0^j$ such that

$$\|v^j(-t_n^j) - e^{it_n^j(\Delta^2-\mu\Delta)}\phi^j\|_{H^2} \to 0 \quad \text{as } n \to \infty. \tag{5.12}$$

Note that for each $j \in [1, J^*]$,

$$\begin{aligned}\limsup_{n\to\infty} \|\psi_n^j\|_{\dot{H}^s}^2 &= N^j \|\phi^j\|_{\dot{H}^s}^2, \\ \limsup_{n\to\infty} \|e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j\|_{L^{p+1}}^{p+1} &= N^j \|e^{it_n^j(\Delta^2-\mu\Delta)}\phi^j\|_{L^{p+1}}^{p+1}.\end{aligned}$$

Hence, $S_\omega(v^j) < S_\omega(Q_\omega)$ and $K(v^j(-t_n^j)) > 0$. Thus, from Lemma 3.4, v^j exists globally in time. Let

$$v_n^j := \sum_{k=1}^{N^j} \mathcal{G}_k^j[v^j(t - t_n^j, x - x_n^j)]$$

and let

$$\begin{aligned} e_n^j &:= i\partial_t v_n^j - \Delta^2 v_n^j + \mu\Delta v_n^j + |v_n^j|^{p-1} v_n^j \\ &= F(v_n^j) - \sum_{k=1}^{N^j} F\left(\mathcal{G}_k^j[v^j(t - t_n^j, x - x_n^j)]\right), \end{aligned}$$

where $F(z) = |z|^{p-1}z$ for $z \in \mathbb{C}$. Then by (5.12), we have

$$\|v_n^j(0) - w_n^j(0)\|_{H^2} = \|v_n^j(0) - e^{it_n^j(\Delta^2-\mu\Delta)}\psi_n^j\|_{H^2} \to 0 \quad \text{as } n \to \infty \tag{5.13}$$

and by using the property that $|\mathcal{G}_k^j x_n^j - \mathcal{G}_l^j x_n^j| \to 0$ for any $1 \le k \ne l \le N^j$, we have

$$\|e_n^j\|_{N(\mathbb{R})} \to 0 \quad \text{as } n \to \infty.$$

Hence, by Proposition 2.9, for each $1 \le J \le J^*$, we can take n sufficiently large so that

$$\|v_n^j - w_n^j\|_{X(\mathbb{R})} \le \frac{1}{J}$$

for all $1 \le j \le J$. Thus, by using (5.11), we have

$$\begin{aligned}\limsup_{n\to\infty} \sum_{j=1}^J \|v_n^j\|_{X(\mathbb{R})} &\lesssim \limsup_{n\to\infty} \sum_{j=1}^J \left(\|w_n^j\|_{X(\mathbb{R})} + \frac{1}{J} \right) \\ &\lesssim_{L^*_{G,\omega},\varepsilon} 1. \end{aligned} \tag{5.14}$$

For each $1 \le J \le J^*$ we define

$$\tilde{u}_n^J := \sum_{j=1}^J v_n^j + e^{-it(\Delta^2-\mu\Delta)} R_n^J.$$

Then

$$\|\tilde{u}_n^J\|_{X(\mathbb{R})} \le \sum_{j=1}^J \|v_n^j\|_{X(\mathbb{R})} + \|e^{-it(\Delta^2-\mu\Delta)} R_n^J\|_{X(\mathbb{R})}. \tag{5.15}$$

Therefore, from (4.3) and (5.14), we obtain

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|\tilde{u}_n^J\|_{X(\mathbb{R})} \lesssim_{L^*_{G,\omega},\varepsilon} 1. \tag{5.16}$$

For each $1 \le J \le J^*$, we define

$$\begin{aligned} e_n^J &:= i\partial_t \tilde{u}_n^J - \Delta^2 \tilde{u}_n^J + \mu\Delta\tilde{u}_n^J + |\tilde{u}_n^J|^{p-1}\tilde{u}_n^J \\ &= F\left(\sum_{j=1}^J v_n^j + e^{-it(\Delta^2-\mu\Delta)} R_n^J \right) - \sum_{j=1}^J F(v_n^j), \end{aligned}$$

where $F(z) = |z|^{p-1}z$ for $z \in \mathbb{C}$. Since

$$\begin{aligned} &\left\| F\left(\sum_{j=1}^J v_n^j + e^{-it(\Delta^2-\mu\Delta)} R_n^J \right) - F\left(\sum_{j=1}^J v_n^j \right) \right\|_{N(\mathbb{R})} \\ \lesssim &\left\| \left(\left| \sum_{j=1}^J v_n^j \right|^{p-1} + |e^{-it(\Delta^2-\mu\Delta)} R_n^J|^{p-1} \right) |e^{-it(\Delta^2-\mu\Delta)} R_n^J| \right\|_{N(\mathbb{R})} \\ \lesssim &\left(\sum_{j=1}^J \|v_n^j\|_{X(\mathbb{R})}^{p-1} + \|e^{-it(\Delta^2-\mu\Delta)} R_n^J\|_{X(\mathbb{R})}^{p-1} \right) \|e^{-it(\Delta^2-\mu\Delta)} R_n^J\|_{X(\mathbb{R})}, \end{aligned}$$

from (4.3) and (5.14), we have

$$\lim_{J\to J^*}\limsup_{n\to\infty}\left\|F\left(\sum_{j=1}^{J}v_n^j+e^{-it(\Delta^2-\mu\Delta)}R_n^J\right)-F\left(\sum_{j=1}^{J}v_n^j\right)\right\|_{N(\mathbb{R})}=0.$$

For each $1\le J\le J^*$, using (4.4), we can take n sufficiently large so that

$$\sum_{1\le j\ne k\le J}\||v_n^j||v_n^k|^{r-1}\|_{L_t^{q_1/r}L_x^1(\mathbb{R}\times\mathbb{R}^d)}\le\tfrac{1}{C_J J^2}.$$

Hence, we obtain

$$\left\|F\left(\sum_{j=1}^{J}v_n^j\right)-\sum_{j=1}^{J}F(v_n^j)\right\|_{N(\mathbb{R})}\lesssim C_J\sum_{1\le j\ne k\le J}\||v_n^j||v_n^k|^{p-1}\|_{N(\mathbb{R})}=o_J(1).$$

Therefore, we get

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|e_n^J\|_{N(\mathbb{R})}=0. \tag{5.17}$$

From (5.13) we have

$$\tilde{u}_n^J(0)=\sum_{j=1}^{J}v_n^j(0)+R_n^J=u_n(0)+o_J(1) \tag{5.18}$$

as $n\to\infty$ in $H^2(\mathbb{R}^d)$. From (5.16), (5.17) and (5.18), using Proposition 2.9, we have

$$\limsup_{n\to\infty}\|u_n\|_{X(\mathbb{R})}\le\lim_{J\to J^*}\limsup_{n\to\infty}\|\tilde{u}_n^J\|_{X(\mathbb{R})}\lesssim_{L^*_{G,\omega},\varepsilon}1.$$

This contradicts (5.1) and so $J^*\ge 2$ does not occur. □

Proposition 5.2 (Existence and precompactness of critical solution)**.** *Let $d\ge 1$, $\mu\ge 0$ and p satisfy* (1.2)*. Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z}\times O(d)$ and $\omega>0$. Suppose $L^*_{G,\omega}<S_\omega(Q_\omega)$. If $L^*_{G,\omega}<m_{G,\omega}$, then there exists $\phi\in H^2_G$ such that*

$$S_\omega(\phi)=L^*_{G,\omega}\quad\textit{and}\quad K(\phi)>0. \tag{5.19}$$

Let u be a solution to (1.1) *with initial data ϕ. Then*

$$\|u\|_{X((-\infty,0])}=\|u\|_{X([0,+\infty))}=\infty. \tag{5.20}$$

Moreover, $\{u(t)\mid t\in\mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$.

Proof. Since $C_{G,\omega}(L):[0,L^*_{G,\omega})\to[0,\infty)$ is continuous, non-decreasing and $C_{G,\omega}(L^*_{G,\omega})=\infty$, we can take L_n such that $C_{G\omega}(L_n)=n$ and $L_n\nearrow L^*_{G,\omega}$ as $n\to\infty$. Then by the definition of $C_{G,\omega}(L_n)$, there exists a sequence of G-invariant solutions u_n to (1.1) with a initial data $u_{n,0}\in H^2_G$ such that

$$L_{n-1}\le S_\omega(u_n)<L_n,\quad K(u_{n,0})>0 \tag{5.21}$$

and $\|u_n\|_{X(\mathbb{R})} \geq n - \frac{1}{2}$. Let $f(t) := \|u_n\|_{X((-\infty,t])}$. Then $f(-\infty) = 0$ and $f(+\infty) \geq n - \frac{1}{2}$. Since $f(t)$ is continuous, we find $t_n \in \mathbb{R}$ such that $\frac{1}{4}(n - \frac{1}{2}) \leq f(t_n) \leq \frac{1}{2}(n - \frac{1}{2})$. Then

$$\|u_n\|_{X((-\infty,t_n])} = f(t_n) \geq \tfrac{1}{4}(n - \tfrac{1}{2}) \to \infty,$$
$$\|u_n\|_{X([t_n,+\infty)} = \|u_n\|_{X(\mathbb{R})} - f(t_n) \geq \tfrac{1}{2}(n - \tfrac{1}{2}) \to \infty.$$

Moreover, by Lemma 3.3,

$$K(u_n(t_n)) > 0. \tag{5.22}$$

Applying Lemma 5.1, there exists $\phi \in H^2_G$ such that $u_n(t_n) \to \phi$ in $H^2(\mathbb{R}^d)$ as $n \to \infty$. From this strong convergence, (5.21) and (5.22), we have (5.19). Let u be a solution to (1.1) with initial data ϕ. Then by using Propsition 2.9, we obtain (5.20).

Finally, we prove that $\{u(t) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$. Take a sequence $\{t'_n\} \subset \mathbb{R}$. Then from (5.20) we have

$$\|u\|_{X((-\infty,t'_n])} = \|u\|_{X([t'_n,+\infty))} = \infty.$$

Applying Lemma 5.1 again, there exists $\phi' \in H^2(\mathbb{R}^d)$ such that $u(t'_n)$ has a subsequence that converges ϕ' in $H^2(\mathbb{R}^d)$. This completes the proof of Proposition 5.2. □

6. Rigidity

In this section, we will complete the proof of Theorem 1.2. We first prepare the following Lemma, in which precompactness in $H^2(\mathbb{R}^d)$ implies tightness in $H^2(\mathbb{R}^d)$.

Lemma 6.1 (Tightness in H^2)**.** *Let u be a global solution to* (1.1) *such that $\{u(t) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$. Then for all $\eta > 0$, there exists $R_0 > 0$ such that for all $R \geq R_0$ and for all $t \in \mathbb{R}$,*

$$\int_{|x|\geq R} |u(t,x)|^2 + |\nabla u(t,x)|^2 + |\Delta u(t,x)|^2 dx < \eta.$$

Proof. Suppose that Lemma 6.1 does not hold. Then there exist $\eta_0 > 0$, $R_n > 0$ and $t_n \in \mathbb{R}$ such that $R_n \to \infty$ and

$$\int_{|x|\geq R_n} |u(t_n,x)|^2 + |\nabla u(t,x)|^2 + |\Delta u(t_n,x)|^2 dx \geq \eta_0 \tag{6.1}$$

for all n. Since $\{u(t) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$, there exists $\phi \in H^2(\mathbb{R}^d)$ such that $u(t_n)$ has a subsequence that converges ϕ in $H^2(\mathbb{R}^d)$. In particular, there exists n_1 such that

$$\|u(t_n) - \phi\|^2_{H^2} < \tfrac{\eta_0}{4} \tag{6.2}$$

for all $n \geq n_1$. On the other hand, since $R_n \to \infty$, there exists n_2 such that

$$\int_{|x|\geq R_n} |\phi(x)|^2 + |\nabla \phi(x)|^2 + |\Delta \phi(x)|^2 dx < \tfrac{\eta_0}{4} \tag{6.3}$$

for all $n \geq n_2$. By the triangle inequality, (6.2) and (6.3) contradict (6.1) when $n \geq \max\{n_1, n_2\}$. □

Next, we show that there is no nontrivial solution which has a precompcat orbit in $H^2(\mathbb{R}^d)$ and satisfies (1.5). To prove this, we use the localized virial identity associated with fourth-order nonlinear Schrödinger equation (1.1) (see Lemma A.1).

Proposition 6.2 (Rigidity)**.** *Let $d \geq 1$, $\mu \geq 0$ and p satisfy* (1.2)*. Let u be a solution to* (1.1) *with initial data u_0 such that*

$$S_\omega(u) < S_\omega(Q_\omega) \quad \text{and} \quad K(u_0) > 0 \tag{6.4}$$

and $\{u(t) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$. Then $u_0 = 0$.

Proof. Suppose $u_0 \neq 0$. From (6.4) and Lemma 3.4, u exists globally in time. In particular, $K(u(t)) > 0$ for all $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^2}^2 < \infty. \tag{6.5}$$

Moreover, from Lemma 3.1, there exists $\delta > 0$ such that for all $t \in \mathbb{R}$,

$$K(u(t)) > \delta(\|\Delta u(t)\|_{L^2}^2 + \mu\|\nabla u(t)\|_{L^2}^2) \geq 2\delta E(u). \tag{6.6}$$

Since $K(u(t)) > 0$ for all $t \in \mathbb{R}$, we have

$$\begin{aligned} E(u(t)) &= \tfrac{1}{4}K(u(t)) + \tfrac{\mu}{4}\|\nabla u(t)\|_{L^2}^2 + \tfrac{s_c(p-1)}{4(p+1)}\|u(t)\|_{L^{p+1}}^{p+1} \\ &\geq \tfrac{s_c(p-1)}{4(p+1)}\|u(t)\|_{L^{p+1}}^{p+1} \end{aligned} \tag{6.7}$$

for all $t \in \mathbb{R}$. In particular, $E(u) = E(u_0) > 0$. Let $\psi \in C_0^\infty([0,\infty))$ be a non-increasing function and such that $\psi(s) = 1$ for $0 \leq s \leq 1$ and $\psi(s) = 0$ for $s \geq 2$. For each $R > 0$, we define

$$\Psi_R(x) := R^2 \int_0^{\frac{|x|}{R}} \int_0^r \psi(s) ds dr$$

and

$$M_R(t) := 2\operatorname{Im} \int_{\mathbb{R}^d} \overline{u}(t,x)\nabla\Psi_R(x) \cdot \nabla u(t,x) dx.$$

Then by the Hölder inequality,

$$|M_R(t)| \lesssim R\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \leq R\|u(t)\|_{H^2}^2 \tag{6.8}$$

for all $t \in \mathbb{R}$. From Lemma A.1, we obtain

$$\tfrac{d}{dt}M_R(t) = 4K(u(t)) + A_R(t) + B_R(t), \tag{6.9}$$

where

$$\begin{aligned} |A_R(t)| \lesssim & \int_{|x|\geq R} |\Delta u(t,x)|^2 + \mu|\nabla u(t,x)|^2 dx \\ & + R^{-2}\|\nabla u(t)\|_{L^2}^2 + (\mu R^{-2} + R^{-4})\|u(t)\|_{L^2}^2 \end{aligned}$$

and

$$B_R(t) \geq 0$$

for all $t \in \mathbb{R}$. Since $\{u(t) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R}^d)$, from Lemma 6.1 and (6.5) there exists $R_0 > 0$ such that for all $t \in \mathbb{R}$,

$$|A_{R_0}(t)| \leq 4\delta E(u).$$

Hence, from (6.6), (6.7) and (6.9), we obtain

$$\tfrac{d}{dt}M_{R_0}(t) \geq 4\delta E(u) \geq \tfrac{\delta s_c(p-1)}{p+1}\|u_0\|_{L^{p+1}}^{p+1}$$

for all $t \in \mathbb{R}$. By using (6.8), for any time interval $I \subset \mathbb{R}$ we have

$$|I|\|u_0\|_{L^{p+1}}^{p+1} \lesssim \sup_{t\in I}|M_{R_0}(t)| \lesssim R_0 \sup_{t\in\mathbb{R}}\|u(t)\|_{H^2}^2.$$

From (6.5), this contradicts $u_0 \neq 0$ for sufficiently large $|I|$. □

Proof of Theorem 1.2. Let G be a subgroup of $\mathbb{R}/2\pi\mathbb{Z} \times O(d)$ and $\omega > 0$. Assume $L^*_{G,\omega} < m_{G,\omega}$. To prove Theorem 1.2 by contradiction argument, suppose that $L^*_{G,\omega} < S_\omega(Q_\omega)$. Then from Proposition 5.2, there exists a global G-invariant solution u of (1.1) with an initial data u_0 such that

$$S_\omega(u) < S_\omega(Q_\omega), \quad K(u_0) > 0$$

and

$$\|u\|_{X((-\infty,0])} = \|u\|_{X([0,+\infty))} = \infty. \tag{6.10}$$

Moreover, $\{u(t) \mid t \in \mathbb{R}\}$ is precompact in $H^2(\mathbb{R})$. By using Proposition 6.2, $u_0 = 0$. Hence, $u \equiv 0$ which contradicts (6.10). Thus, $L^*_{G,\omega} \geq S_\omega(Q_\omega)$. □

Appendix A. Localized virial identity

In this appendix, we recall the localized virial identity for (1.1). In [7], the localized virial identity associated with (1.1) was established to prove blow-up results for (1.1). We remark that in [7], only radially symmetric solutions of (1.1) were treated.

Let ψ be a smooth function defined on $[0, \infty)$ such that $\psi(s) = 1$ for $0 \leq s \leq 1$, $\psi(s) = 0$ for $s \geq 2$ and $\psi'(s) \leq 0$ for $s \geq 0$. For $R > 0$, we define

$$\Psi_R(x) := R^2 \int_0^{\frac{|x|}{R}} \int_0^r \psi(s)dsdr.$$

Then,

$$\|\nabla^j \Psi_R\|_{L^\infty} \lesssim R^{2-j} \tag{A.1}$$

for all $j \geq 1$. Let we denote radial derivative by $\partial_r = \frac{\partial}{\partial r}$ with $r = |x|$. Since $\psi(s) = 1$ for $0 \leq s \leq 1$, we see that

$$\tfrac{\partial_r \Psi_R}{|x|} - 1 = \tfrac{R}{|x|}\int_0^{\frac{|x|}{R}} \psi(s)ds - 1 = 0, \tag{A.2}$$

$$\partial_r^2 \Psi_R - \tfrac{\partial_r \Psi_R}{|x|} = \left\{\psi\left(\tfrac{|x|}{R}\right) - 1\right\} + \left\{1 - \tfrac{\partial_r \Psi_R}{|x|}\right\} = 0 \tag{A.3}$$

for $|x| \le R$. Moreover, since ψ' is a non-increasing function, we see that

$$\partial_r^2 \Psi_R - \tfrac{\partial_r \Psi_R}{|x|} = \tfrac{R}{|x|} \int_0^{\frac{|x|}{R}} \left[\psi\left(\tfrac{|x|}{R}\right) - \psi(s) \right] ds \le 0 \tag{A.4}$$

for all x. For $u \in C(I, H^2(\mathbb{R}^d))$ and $R > 0$, we define the localized virial quantity $M_R(t)$ by

$$M_R(t) := 2 \operatorname{Im} \int_{\mathbb{R}^d} \overline{u}(t,x) \nabla \Psi_R(x) \cdot \nabla u(t,x) dx.$$

Lemma A.1. *Let $d \ge 1$, $\mu \ge 0$. Suppose that $u \in C(I, H^2(\mathbb{R}^d))$ is a solution to* (1.1). *Then for $R > 0$ and for all $t \in I$,*

$$\tfrac{d}{dt} M_R(t) = 4K(u(t)) + A_R(t) + B_R(t),$$

where

$$\begin{aligned} |A_R(t)| \lesssim & \int_{|x| \ge R} |\Delta u(t,x)|^2 + |\nabla u(t,x)|^2 dx \\ & + R^{-2} \|\nabla u(t)\|_{L^2}^2 + (R^{-4} + \mu R^{-2}) \|u(t)\|_{L^2}^2, \\ B_R(t) \ge & \ 0. \end{aligned}$$

Proof. Define an operator Z by $Z := (\nabla \Psi_R) \cdot \nabla + \nabla \cdot (\nabla \Psi_R)$. By using (1.1),

$$\tfrac{d}{dt} M_R(t) = \langle u | [\Delta^2, Z] u \rangle + \langle u | [-\mu \Delta, Z] u \rangle + \langle u | [-|u|^{p-1}, Z] u \rangle,$$

where $\langle \cdot | \cdot \rangle$ denote the real part of L^2-inner product and $[A, B] = AB - BA$. To estimate $\langle u | [\Delta^2, Z] u \rangle$ and $\langle u | [-\mu \Delta, Z] u \rangle$, we use two commutator identities. First, we have

$$[\Delta, Z] = 4 \partial_k (\partial_{kl} \Psi_R) \partial_l + (\Delta^2 \Psi_R). \tag{A.5}$$

Second, by using the fact that $\Delta A + A \Delta = \partial_k A \partial_k + [\partial_k, [\partial_k, A]]$ for an operator A, we have

$$\begin{aligned} [\Delta^2, Z] &= \Delta [\Delta, Z] + [\Delta, Z] \Delta \\ &= 2 \partial_k [\Delta, Z] \partial_k + [\partial_k, [\partial_k, [\Delta, Z]]] \\ &= 8 \partial_{kl} (\partial_{lm} \Psi_R) \partial_{mk} + 4 \partial_k (\partial_{kl} \Delta \Psi) \partial_l + 2 \partial_k (\Delta^2 \Psi_R) \partial_k + (\Delta^3 \Psi_R). \end{aligned} \tag{A.6}$$

By using (A.6), we integrate by parts to obtain

$$\begin{aligned} \langle u | [\Delta^2, Z] u \rangle = & \ 8 \operatorname{Re} \int \partial_{kl} \overline{u} (\partial_{lm} \Psi_R) \partial_{mk} u - 4 \operatorname{Re} \int \partial_k \overline{u} (\partial_{kl} \Delta \Psi_R) \partial_l u \\ & - 2 \operatorname{Re} \int \partial_k \overline{u} (\Delta^2 \Psi_R) \partial_k u + \int (\Delta^3 \Psi_R) |u|^2. \end{aligned}$$

Since

$$\partial_{kl} \Psi_R = \left(\delta_{kl} - \tfrac{x_k x_l}{|x|^2} \right) \tfrac{\partial_r \Psi_R}{|x|} + \tfrac{x_k x_l}{|x|^2} \partial_r^2 \Psi_R, \tag{A.7}$$

we have

$$\begin{aligned}
&8\,\mathrm{Re}\int \partial_{kl}\overline{u}(\partial_{lm}\Psi_R)\partial_{mk}u \\
&= \; 8\int |\Delta u|^2 + \left(\tfrac{\partial_r\Psi_R}{|x|} - 1\right)|\Delta u|^2 + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right)|\tfrac{x}{|x|}\cdot\nabla\partial_k u|^2.
\end{aligned}$$

From (A.1), we have

$$\begin{aligned}
\left|\int \partial_k\overline{u}(\partial_{kl}\Delta\Psi_R)\partial_l u\right| &\lesssim \|\partial_{kl}\Delta\Psi_R\|_{L^\infty}\|\nabla u(t)\|_{L^2}^2 \lesssim R^{-2}\|\nabla u(t)\|_{L^2}^2, \\
\left|\int \partial_k\overline{u}(\Delta^2\Psi_R)\partial_k u\right| &\lesssim \|\Delta^2\Psi_R\|_{L^\infty}\|\nabla u(t)\|_{L^2}^2 \lesssim R^{-2}\|\nabla u(t)\|_{L^2}^2, \\
\left|\int (\Delta^3\Psi_R)|u|^2\right| &\lesssim \|\Delta^3\Psi_R\|_{L^\infty}\|u(t)\|_{L^2}^2 \lesssim R^{-4}\|u(t)\|_{L^2}^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\langle u|[\Delta^2, Z]u\rangle \\
&= \; 8\int |\Delta u|^2 + \left(\tfrac{\partial_r\Psi_R}{|x|} - 1\right)|\Delta u|^2 + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right)|\tfrac{x}{|x|}\cdot\nabla\partial_k u|^2 \\
&\quad + O(R^{-2}\|\nabla u(t)\|_{L^2}^2 + R^{-4}\|u(t)\|_{L^2}^2). \qquad \text{(A.8)}
\end{aligned}$$

By using (A.5), we integrate by parts to obtain

$$\langle u|[-\mu\Delta, Z]u\rangle = 4\mu\,\mathrm{Re}\int \partial_k\overline{u}(\partial_{kl}\Psi_R)\partial_l u + \mu\int(\Delta^2\Psi_R)|u|^2.$$

By using (A.7) again, we have

$$\begin{aligned}
&4\mu\,\mathrm{Re}\int \partial_k\overline{u}(\partial_{kl}\Psi_R)\partial_l u \\
&= \; 4\mu\int |\nabla u|^2 + \left(\tfrac{\partial_r\Psi_R}{|x|} - 1\right)|\nabla u|^2 + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right)|\tfrac{x}{|x|}\cdot\nabla u|^2.
\end{aligned}$$

From (A.1), we have

$$\left|\int(\Delta^2\Psi_R)|u|^2\right| \lesssim \|\Delta^2\Psi_R\|_{L^\infty}\|u(t)\|_{L^2}^2 \lesssim R^{-2}\|u(t)\|_{L^2}^2.$$

Thus, we have

$$\begin{aligned}
&\langle u|[-\mu\Delta, Z]u\rangle \\
&= \; 4\mu\int |\nabla u|^2 + \left(\tfrac{\partial_r\Psi_R}{|x|} - 1\right)|\nabla u|^2 + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right)|\tfrac{x}{|x|}\cdot\nabla u|^2 \\
&\quad + O(\mu R^{-2}\|u(t)\|_{L^2}^2). \qquad \text{(A.9)}
\end{aligned}$$

We integrate by parts to get

$$\begin{aligned}\langle u|[-|u|^{p-1}, Z]u\rangle &= 2\int |u|^2\nabla\Psi_R\cdot\nabla(|u|^{p-1})\\ &= -\tfrac{2(p-1)}{p+1}\int(\Delta\Psi_R)|u|^{p+1}\\ &= -\tfrac{2(p-1)}{p+1}\int d|u|^{p+1} + (\Delta\Psi_R - d)|u|^{p+1}. \end{aligned}\tag{A.10}$$

Therefore, from (A.8), (A.9) and (A.10), we obtain

$$\begin{aligned}\tfrac{d}{dt}M_R(t) &= 8\int|\Delta u|^2 + 4\mu\int|\nabla u|^2 - \tfrac{2d(p-1)}{p+1}\int|u|^{p+1} + A_R(t) + B_R(t)\\ &= 4K(u(t)) + A_R(t) + B_R(t),\end{aligned}\tag{A.11}$$

where

$$\begin{aligned}A_R(t) = 8\int &\left(\tfrac{\partial_r\Psi_R}{|x|}-1\right)|\Delta u|^2 + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right)|\tfrac{x}{|x|}\cdot\nabla\partial_k u|^2\\ &+ 4\mu\int\left(\tfrac{\partial_r\Psi_R}{|x|}-1\right)|\nabla u|^2 + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right)|\tfrac{x}{|x|}\cdot\nabla u|^2\\ &+ O(R^{-2}\|\nabla u(t)\|_{L^2}^2 + (\mu R^{-2}+R^{-4})\|u(t)\|_{L^2}^2)\end{aligned}$$

and

$$B_R(t) = -\tfrac{2(p-1)}{p+1}\int(\Delta\Psi_R - d)|u|^{p+1}.$$

From (A.2) and (A.3), we have

$$\begin{aligned}|A_R(t)| \lesssim &\int_{|x|\geq R}|\Delta u(t,x)|^2 + |\nabla u(t,x)|^2dx\\ &+ R^{-2}\|\nabla u(t)\|_{L^2}^2 + (\mu R^{-2}+R^{-4})\|u(t)\|_{L^2}^2.\end{aligned}\tag{A.12}$$

By using (A.7) with $k = l$ and (A.4), we have

$$\Delta\Psi_R - d = d\left(\tfrac{\partial_r\Psi_R}{|x|}-1\right) + \left(\partial_r^2\Psi_R - \tfrac{\partial_r\Psi_R}{|x|}\right) \leq 0.$$

Hence,

$$B_R(t) \geq 0.\tag{A.13}$$

Lemma A.1 follows from (A.11), (A.12) and (A.13). □

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