ON THE STATISTICAL LIE GROUPS OF NORMAL DISTRIBUTIONS

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Abstract. The α-connections (other than the Levi-Civita connection of the Fisher metric) on the statistical Lie group of the normal distributions can not be the Levi-Civita connection of any left invariant semi-Riemannian metrics.

1. Introduction

The set $\mathcal{N}$ of all normal distributions is identified with the upper half plane (as a smooth manifold). The Fisher metric g^{F} is regarded as a Riemannian metric of constant curvature $-1/2$ on the upper half plane $\mathcal{N}$. Moreover there exists a one-parameter family $\{\nabla^{(\alpha)}\}_{\alpha\in\mathbb{R}}$ of torsion free linear connections on $\mathcal{N}$. In Information Geometry, the linear connection $\nabla^{(\alpha)}$ is referred as to the *Amari-Chentosov* α-connection. The connection $\nabla^{\mathsf{e}} = \nabla^{(1)}$ and $\nabla^{\mathsf{m}} = \nabla^{(-1)}$ are called the exponential connection and mixture connection, respectively. The connection $\nabla^{(0)}$ coincides with the Levi-Civita connection of the Fisher metric g^{F}. The triplets $(\mathcal{N}, g^{\mathsf{F}}, \nabla^{(\alpha)})$ are statistical manifolds. Here recall the notion of statistical manifold.

A smooth manifold M equipped with a pair (g, ∇) consisting of a Riemannian metric g and a torsion free linear connection ∇ is said to be a *statistical manifold* if $C = \nabla g$ is totally symmetric. In particular a statistical manifold $M = (M, g, \nabla)$ with flat ∇ is called a *Hessian manifold*. More generally a statistical manifold M is said to be *conjugate symmetric* if the curvature tensor field R^* of the conjugate connection ∇^* coincides with the curvature tensor field R of ∇.

A statistical manifold (M, g, ∇) is said to be *homogeneous* if there exits a Lie group G acting transitively, isometrically with respect to g and affinely with respect to ∇ on M [1, 2].

One can see that $(\mathcal{N}, g^{\mathsf{F}}, \nabla^{(\alpha)})$ is statistical for any α. In particular $\nabla^{(1)}$ and $\nabla^{(-1)}$ are flat. Thus $(\mathcal{N}, g^{\mathsf{F}}, \nabla^{(\pm 1)})$ are Hessian. In our previous work [1], we proved that $(\mathcal{N}, g^{\mathsf{F}}, \nabla^{(\alpha)})$ is a homogeneous statistical manifold for any α. Moreover we showed that $(\mathcal{N}, g^{\mathsf{F}}, \nabla^{(\alpha)})$ is identified with the orientation preserving affine transformation group $\mathrm{GA}^+(1)$ of the real line $\mathbb{R}$ equipped with a one-parameter family of left invariant statistical structures. It should be emphasized that $\nabla^{(\alpha)}$ is flat when and only when $\alpha = \pm 1$.

On the other hand, Wolf [6] proved the existence of flat left invariant Lorentz metrics on $\mathrm{GA}^+(1)$. Nomizu [4] proved that every left invariant Lorentz metric on $\mathrm{GA}^+(1)$ is of constant curvature. From these facts one may expect that the connections $\nabla^{(\pm 1)}$ are Levi-Civita

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connections of some left invariant Lorentz metrics on $\mathrm{GA}^+(1)$. The purpose of this letter is to give a negative answer to this question.

Theorem 1. *The α-connection of the statistical manifold $\mathcal{N}$ of normal distributions can be the Levi-Civita connection of some left invariant semi-Riemannian metrics on* $\mathrm{GA}^+(1)$ *if and only if $\alpha = 0$ and the metric is homothetic to the Fisher metric.*

Corollary 1. *The* e*-connection and* m*-connection of the statistical manifold of normal distributions can not be the Levi-Civita connection of any left invariant semi-Riemannian metrics.*

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2. Preliminaries

The set $\mathcal{N} = \{N(x, y^2) \mid x, y \in \mathbb{R},\ y > 0\}$ of all normal distributions is identified with the orientation preserving affine transformation group

$$\mathrm{GA}^+(1) = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \middle| \ x, y \in \mathbb{R},\ \ y > 0 \right\}$$

of the real line $\mathbb{R}$. The Lie algebra $\mathfrak{ga}(1)$ of $\mathrm{GA}^+(1)$ is given by

$$\left\{ \begin{pmatrix} v & u \\ 0 & 0 \end{pmatrix} \middle| \ u, v \in \mathbb{R} \right\}.$$

Take a basis

$$E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

of $\mathfrak{ga}(1)$. Then, by performing left translations to E_1 and E_2, we obtain left invariant vector fields

$$E_1 = y\frac{\partial}{\partial x}, \quad E_2 = y\frac{\partial}{\partial y}$$

on $\mathrm{GA}^+(1)$. For any positive constant λ, we set

$$e_1 = E_1, \quad e_2 = \frac{1}{\lambda}E_2.$$

Then we obtain a left invariant Riemannian metric

$$g_\lambda = \frac{dx^2 + \lambda^2 dy^2}{y^2}.$$

The left invariant frame field $\{e_1, e_2\}$ is orthonormal with respect to g_λ. The Levi-Civita connection ${}^\lambda\nabla$ of g_λ is described as

$${}^\lambda\nabla_{e_1}e_1 = \frac{1}{\lambda}e_2, \quad {}^\lambda\nabla_{e_1}e_2 = -\frac{1}{\lambda}e_1, \quad {}^\lambda\nabla_{e_2}e_1 = {}^\lambda\nabla_{e_2}e_2 = 0.$$

The metric $g^{\mathsf{F}} := g_{\sqrt{2}}$ is the blue*Fisher metric* of the normal distribution. It is of constant curvature $-1/2$. The Amari-Chentsov α-connection of $\mathcal{N}$ is extended to the following one-parameter family of left invariant connections on $\mathrm{GA}^+(1)$ (see [1]):

$$\nabla^{(\alpha)}_{e_1} e_1 = \frac{1-\alpha}{\lambda} e_2, \quad \nabla^{(\alpha)}_{e_1} e_2 = -\frac{1+\alpha}{\lambda} e_1, \quad \nabla^{(\alpha)}_{e_2} e_1 = -\frac{\alpha}{\lambda} e_1, \quad \nabla^{(\alpha)}_{e_2} e_2 = -\frac{2\alpha}{\lambda} e_2.$$

Obviously $\nabla^{(0)} = {}^{\lambda}\nabla$. When $\lambda = \sqrt{2}$, $\nabla^{\mathsf{e}} = \nabla^{(1)}$ and $\nabla^{\mathsf{m}} = \nabla^{(-1)}$ are called the e-connection (exponential connection) and m-connection (mixture connection), respectively.

3. Left invariant metrics

Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathfrak{ga}(1)$. We denote by g the left invariant semi-Riemannian metric on $\mathrm{GA}^+(1)$ induced from $\langle \cdot, \cdot \rangle$. We set

$$g_{ij} = \langle e_i, e_j \rangle, \quad i, j = 1, 2.$$

By using the Koszul formula and $[e_1, e_2] = -e_1/\lambda$, the Levi-Civita connection ∇^g of g is computed as

$$\begin{aligned}
&\langle \nabla^g_{e_1} e_1, e_1 \rangle = 0, \quad \langle \nabla^g_{e_1} e_1, e_2 \rangle = g_{11}/\lambda, \\
&\langle \nabla^g_{e_1} e_2, e_1 \rangle = -g_{11}/\lambda, \quad \langle \nabla^g_{e_1} e_2, e_2 \rangle = 0, \\
&\langle \nabla^g_{e_2} e_1, e_1 \rangle = 0, \quad \langle \nabla^g_{e_2} e_1, e_2 \rangle = g_{12}/\lambda, \\
&\langle \nabla^g_{e_2} e_2, e_1 \rangle = -g_{12}/\lambda, \quad \langle \nabla^g_{e_2} e_2, e_2 \rangle = 0.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\nabla^g_{e_1} e_1 &= \frac{g_{11}}{\lambda \det(g_{ij})}(-g_{12} e_1 + g_{11} e_2), \\
\nabla^g_{e_1} e_2 &= -\frac{g_{11}}{\lambda \det(g_{ij})}(g_{22} e_1 - g_{12} e_2), \\
\nabla^g_{e_2} e_1 &= \frac{g_{12}}{\lambda \det(g_{ij})}(-g_{12} e_1 + g_{11} e_2), \\
\nabla^g_{e_2} e_2 &= -\frac{g_{12}}{\lambda \det(g_{ij})}(g_{22} e_1 - g_{12} e_2).
\end{aligned}$$

Example 1. Let us take a scalar product $g_{11} = \varepsilon_1 = \pm 1$, $g_{12} = 0$ and $g_{22} = \varepsilon_2 = \pm 1$. Then we have

$$\nabla^g_{e_1} e_1 = \frac{\varepsilon_1 \varepsilon_2}{\lambda} e_2, \quad \nabla^g_{e_1} e_2 = -\frac{1}{\lambda} e_1, \quad \nabla^g_{e_2} e_1 = \nabla^g_{e_2} e_2 = 0.$$

Then the Riemannian curvature R^g of g is given by

$$R^g(e_1, e_2) e_2 = -\frac{1}{\lambda^2}\, e_1.$$

Hence the sectional curvature K of g is computed as

$$K = \frac{\langle R^g(e_1, e_2) e_2, e_1 \rangle}{\det(g_{ij})} = -\frac{\varepsilon_2}{\lambda^2}.$$

Thus we obtain

- Riemannian metric $(dx^2 + \lambda^2 dy^2)/y^2$: $\quad K = -1/\lambda^2$.
- Lorentz metric $(-dx^2 + \lambda^2 dy^2)/y^2$: $\quad K = -1/\lambda^2$.

- Lorentz metric $(dx^2 - \lambda^2 dy^2)/y^2$: $\quad K = 1/\lambda^2$.

Nomizu [4] proved that every left invariant Lorentz metric on $\mathrm{GA}^+(1)$ is of constant curvature.

Example 2 (Wolf's example). Let us take a scalar product $g_{11} = g_{22} = 0$, $g_{12} = \varepsilon \neq 0$. Then we have

$$\nabla^g_{e_1} e_1 = \nabla^g_{e_1} e_2 = 0, \quad \nabla^g_{e_2} e_1 = \frac{1}{\lambda} e_1, \quad \nabla^g_{e_2} e_2 = -\frac{1}{\lambda} e_2.$$

The Riemannian curvature R^g of g is given by

$$R^g(e_1, e_2)e_2 = 0.$$

Hence the the metric

$$g = \frac{2\lambda\varepsilon}{y^2}\, dx\, dy$$

is flat. The Lorentz metric g with $\varepsilon = \lambda = 1$ is due to Wolf [6, p. 58, Corollary].

4. Proof of Theorem 1

Let us consider $(\mathrm{GA}^+(1), g_\lambda, \nabla^{(\alpha)})$. Assume that $\nabla^{(\alpha)} = \nabla^g$ for some left invariant semi-Riemannian metric g on $\mathrm{GA}^+(1)$. Then we obtain the system:

$$\begin{aligned}
\nabla^{(\alpha)}_{e_1} e_1 = \nabla^g_{e_1} e_1 &\Longleftrightarrow g_{11} g_{12} = 0, \quad \frac{(g_{11})^2}{\det(g_{ij})} = 1 - \alpha, \\
\nabla^{(\alpha)}_{e_1} e_2 = \nabla^g_{e_1} e_2 &\Longleftrightarrow g_{11} g_{12} = 0, \quad \frac{g_{11} g_{22}}{\det(g_{ij})} = 1 + \alpha, \\
\nabla^{(\alpha)}_{e_2} e_1 = \nabla^g_{e_2} e_1 &\Longleftrightarrow g_{11} g_{12} = 0, \quad \frac{(g_{12})^2}{\det(g_{ij})} = \alpha, \\
\nabla^{(\alpha)}_{e_2} e_2 = \nabla^g_{e_2} e_2 &\Longleftrightarrow g_{12} g_{22} = 0, \quad \frac{(g_{12})^2}{\det(g_{ij})} = -2\alpha.
\end{aligned}$$

From this system we deduce that $\nabla^{(\alpha)} = \nabla^g$ holds when and only when

$$\alpha = 0, \quad g_{11} = g_{22} = k, \quad g_{12} = 0,$$

for some non-zero constant k. Hence the metric g is $k g_\lambda$. Thus in case $\lambda = \sqrt{2}$, $g = k g^{\mathsf{F}}$ for some non-zero constant k. Note that $\nabla^g = \nabla^{k g^{\mathsf{F}}} = \nabla^{g^{\mathsf{F}}}$. $\square$

5. Concluding remarks

5.1. **Flat metrics.** It should be emphasized that the left invariance of the assumption in Theorem 1 can not be removed. Indeed, there exists a Riemannian metric g^{e} on $\mathcal{N}$ whose Levi-Civita connection is ∇^{e} but *not* left invariant.

Let us take the so-called *natural coordinate system* (see *e.g.*, [5, Example 6.1]):

$$\theta_1 = \frac{1}{2y^2}, \quad \theta_2 = \frac{x}{y^2}$$

of $\mathcal{N}$. The coordinate system (θ_1, θ_2) defined on the right half plane

$$\{(\theta_1, \theta_2) \in \mathbb{R}^2 \mid \theta_1 > 0\}.$$

is an affine coordinate system with respect to e-connection. With respect to the natural coordinate system (θ_1, θ_2), the probability density function

$$p(t; x, y) = \frac{1}{\sqrt{2\pi y^2}} \exp\left(-\frac{(t-x)^2}{2y^2}\right)$$

of the normal distribution $N(x, y^2)$ is rewritten as

$$p(t; \theta_1, \theta_2) = \exp\left(F_1(t)\theta_1 + F_2(t)\theta_2 - \psi(\theta_1, \theta_2)\right), \tag{5.1}$$

where

$$F_1(t) = -t^2, \quad F_2(t) = t, \quad \psi(\theta_1, \theta_2) = \frac{\theta_2^2}{4\theta_1} + \frac{1}{2}\log\frac{\pi}{\theta_1}.$$

The formula (5.1) implies that $\mathcal{N}$ is an exponential family. The Fisher metric is rewritten as

$$g^{\mathsf{F}} = \sum_{i,j=1}^{2} \frac{\partial^2 \psi}{\partial\theta_i \partial\theta_j}\, d\theta_i\, d\theta_j = \frac{1}{2\theta_1}\left\{\left(\left(\frac{\theta_2}{\theta_1}\right)^2 + \frac{1}{\theta_1}\right) d\theta_1^2 - \frac{2\theta_2}{\theta_1}\, d\theta_1\, d\theta_2 + d\theta_2^2\right\}.$$

The global frame field

$$\frac{\partial}{\partial\theta_1} = -2xy^2\frac{\partial}{\partial x} - y^3\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial\theta_2} = y^2\frac{\partial}{\partial x}$$

is *not* left invariant under the solvable Lie group structure. Let us introduce a flat Riemannian metric

$$g^{\mathsf{e}} = d\theta_1^2 + d\theta_2^2 = \left(-\frac{dy}{y^3}\right)^2 + \left(\frac{ydx - 2xdy}{y^3}\right)^2.$$

Then the Levi-Civita connection of g^{e} coincides with the e-connection. Although the e-connection is left invariant, but the metric g^{e} is *not*.

5.2. **Moduli problem.** Kito [3] studied the moduli problems of Hessian structures on the Euclidean n-space $\mathbb{E}^n$ and the hyperbolic n-space $\mathbb{H}^n$ of constant curvature -1 with $n > 1$. More precisely he studied the set

$$\mathcal{H}(M, g) = \{C \in \varGamma(T^*M \odot T^*M \odot T^*M) \mid (M, g, C) \text{ is Hessian }\}$$

for $M = \mathbb{E}^n$ and $M = \mathbb{H}^n$. Here we interpret a Hessian structure on a manifold M as a pair (g, C) consisting of a Riemannian metric g and a symmetric covariant tensor field C of degree 3. By suitable modification, Kito's result is rephrased for $\mathcal{N}$ as follows:

Theorem 2. *The set $\mathcal{H}(\mathcal{N}, g^{\mathsf{F}})$ has at least the freedom of one functions on $\mathbb{R}$.*

On the other hand, in our previous work [1] we studied left invariant statistical structures on $\mathcal{N}$. We proved the following characterization of α-connections.

Theorem 3. *The only left invariant linear connections on $\mathcal{N}$ compatible to the Fisher metric g^{F} which are conjugate symmetric are Amari-Chentosov α-connections.*

Motivated by Theorem 2 and Theorem 3, here we propose the following problem:

Problem 1. Classify all the left invariant linear connections on the Lie group of normal distributions which is compatible to the Fisher metric g^{F}, especially flat ones.

By virtue of Ambrose-Singer theorem [7] and Sekigawa-Kirichenko's theorem [8, 9], the local statistical homogeneity of a statistical manifold (M, g, ∇) is characterized as follows.

Theorem 4. *A statistical manifold (M, g, ∇) is locally homogeneous if and only if there exists a pair (S, C) where S is a tensor field of type $(1, 2)$ and C is a section of $\Gamma(T^*M \odot T^*M \odot T^*M)$ satisfying the following conditions*:

(1) $g(S(X)Y, Z) + g(Y, S(X)Z) = 0$,
(2) $(\nabla^g_X R)(Y, Z) = -[S(X), R^g(Y, Z)] + R(S(X)Y, Z) + R(Y, S(X)Z)$,
(3) $(\nabla^g_X S)(Y) = [S(X), S(Y)] - S(S(X))Y$ *and*
(4)

$$\begin{aligned}(\nabla^g_X C)(Y, Z, W) = \quad & -S(X)C(Y, Z, W) + C(S(X)Y, Z, W) \\ & + C(Y, S(X)Z, W) + C(Y, Z, S(X)W),\end{aligned}$$

where ∇^g and R^g are the Levi-Civita connection and the Riemannian curvature of g, respectively.

Such a pair (S, C) is referred as to a *homogeneous statistical structure* [2]. We close this report with the following problem:

Problem 2. Classify all the homogeneous statistical structure on $(\mathcal{N}, g^{\mathsf{F}})$.

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