ON BOUNDEDNESS OF CHARACTERISTIC CLASSES VIA QUASI-MORPHISMS

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Abstract. In this paper, we characterize the second bounded characteristic classes of foliated bundles in terms of the non-descendible quasi-morphisms on the universal covering of the structure group. As an application, we study the boundedness of obstruction classes for (contact) Hamiltonian fibrations and show the non-existence of foliated structures on some Hamiltonian fibrations.

1. Introduction

Let G be a connected topological group which admits the universal covering $\pi\colon \tilde{G} \to G$ and G^{δ} denote the group G with the discrete topology. Cohomology classes of the classifying spaces BG and BG^{δ} are considered universal characteristic classes of principal G-bundles and foliated G-bundles (or flat G-bundles), respectively. In this paper, we concentrate our interest on the characteristic classes of degree two. The identity homomorphism $\iota\colon G^{\delta} \to G$ induces a continuous map $B\iota\colon BG^{\delta} \to BG$ and a homomorphism

$$B\iota^*\colon H^2(BG;\mathbb{R}) \to H^2(BG^{\delta};\mathbb{R}).$$

In this article, an element of $\mathrm{Im}(B\iota^*)$ is simply called a *characteristic class of foliated G-bundles*. Hence, in our terminology, if a characteristic class is non-zero for a foliated G-bundle E, the bundle E is non-trivial not only as a foliated G-bundle but also as a G-bundle.

Let $H^2_{\mathrm{grp}}(G;\mathbb{R})$ and $H^2_b(G;\mathbb{R})$ be the second group cohomology and second bounded cohomology of G, respectively. Then there exists a canonical map

$$c_G\colon H^2_b(G;\mathbb{R}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$$

called the comparison map. A group cohomology class $\alpha \in H^2_{\mathrm{grp}}(G;\mathbb{R})$ is said to be *bounded* if it is in the image of c_G.

Since the cohomology group $H^2(BG^{\delta};\mathbb{R})$ is canonically isomorphic to $H^2_{\mathrm{grp}}(G;\mathbb{R})$, we can consider the intersection

$$\mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*)$$

as a subspace of $H^2_{\mathrm{grp}}(G;\mathbb{R})$. This intersection $\mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*)$ is the vector space of bounded characteristic classes of foliated G-bundles.

Our main theorem stated below characterizes the space $\mathrm{Im}(c_G)\cap\mathrm{Im}(B\iota^*)$ in terms of the homogeneous quasi-morphisms on the universal covering $\tilde{G}$ of G. Let $Q(\tilde{G})$ and $Q(G)$ be the vector spaces of all homogeneous quasi-morphisms on $\tilde{G}$ and on G, respectively (see Subsection 3.1 for the definition). Then let $\pi^*\colon Q(G) \to Q(\tilde{G})$ be the pullback induced by $\pi\colon \tilde{G} \to G$.

Theorem 1.1 (Theorem 6.4)**.** *There exists an isomorphism*

$$Q(\tilde{G})/\big(H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R}) + \pi^*Q(G)\big) \xrightarrow{\cong} \mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*).$$

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The following corollary, which is mainly used in applications, immediately follows from Theorem 1.1.

Corollary 1.2 (Corollary 6.5). *Let G be a topological group whose universal covering $\tilde{G}$ satisfies $H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R}) = 0$. Then there exists an isomorphism*

$$Q(\tilde{G})/\pi^*Q(G) \xrightarrow{\cong} \mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*) \ \left(\subset H^2_{\mathrm{grp}}(G;\mathbb{R})\right).$$

In particular, if $\mu \in Q(\tilde{G})$ does not descend to G, i.e., $\mu \notin \pi^*Q(G)$, then μ gives rise to a non-trivial element of $H^2_{\mathrm{grp}}(G;\mathbb{R})$.

In the present paper, we apply the above results to the group of (contact) Hamiltonian diffeomorphisms. As an important topic of symplectic and contact topology, many researchers have studied quasi-morphisms on these groups (see, for example, [EP03], [GG04], [Py06b], [FOOO19], [BZ15], and [FPR18]). By combining these outcomes with our results (Theorem 1.1 and Corollary 1.2), we obtain some results on the *ordinary* group cohomology of these groups. For example, see Corollaries 2.1 and 2.2, Examples 2.4, 2.6, 2.5, and 2.7, Corollary 2.9, and Propositions A.2 and A.3.

Remark 1.3. Let $G = \mathrm{Homeo}_+(S^1)$ be the group of orientation preserving homeomorphisms of the circle. From a theorem of Thurston [Thu74], we have $H^2_{\mathrm{grp}}(G;\mathbb{R}) = \mathrm{Im}(B\iota^*) \cong \mathbb{R}\cdot e$, where e is the Euler class of $\mathrm{Homeo}_+(S^1)$. It is known that the space $Q(\tilde{G})$ is spanned by Poincaré's rotation number $\mathrm{rot}\colon \tilde{G} \to \mathbb{R}$, that is, $Q(\tilde{G}) \cong \mathbb{R}\cdot \mathrm{rot}$ (see [Ghy01]). Therefore, we have $Q(\tilde{G}) \cong H^2_{\mathrm{grp}}(G;\mathbb{R})$. Note that the cohomology $H^2_{\mathrm{grp}}(G;\mathbb{R})$ is equal to $\mathrm{Im}(B\iota^*) \cap \mathrm{Im}(c_G)$, since the Euler class is bounded. Moreover, the space $Q(\tilde{G})$ is equal to $Q(\tilde{G})/\left(H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R}) + \pi^*Q(G)\right)$, since G is a uniformly perfect group and $\tilde{G}$ is a perfect group. Thus, Theorem 1.1 can be seen as a generalization of this isomorphism to an arbitrary topological group.

2. Applications to symplectic and contact geometry

We apply Corollary 1.2 to symplectic and contact geometry. A symplectic manifold (M,ω) has the natural transformation group $\mathrm{Ham}(M,\omega)$ called the *group of Hamiltonian diffeomorphisms* [Ban97, Definition 4.2.4.], [PR14, Subsection 1.2]. A contact manifold (M,ξ) also has the natural transformation group $\mathrm{Cont}_0(M,\xi)$ called the *group of contact Hamiltonian diffeomorphisms* [Gei08].

2.1. **Boundedness of characteristic classes.**

It is an interesting and difficult problem to determine whether a given characteristic class is bounded. The Milnor-Wood inequality ([Mil58], [Woo71]) implies that the Euler class of foliated $SL(2,\mathbb{R})$-bundles (and foliated $\mathrm{Homeo}_+(S^1)$-bundles) is bounded. It has been shown that every element of $\mathrm{Im}(B\iota^*)$ is bounded for every real algebraic subgroups of $GL(n,\mathbb{R})$ ([Gro82]) and for every virtually connected Lie group with linear radical ([CMPSC11]).

As far as the authors know, for homeomorphism groups and diffeomorphism groups, in contrast, the boundedness of characteristic classes is known only as in the following specific examples.

- The Euler class of $\mathrm{Homeo}_+(S^1)$ is bounded [Woo71].
- The Godbillon-Vey class integrated along the fiber, which is a second cohomology class of $\mathrm{Diff}_+(S^1)$, is unbounded [Thu72].
- Every non-zero cohomology class of $\mathrm{Homeo}_0(\mathbb{R}^2)$ is unbounded [Cal04].
- Every non-zero cohomology class of $\mathrm{Homeo}_0(T^2)$ is unbounded, where T^2 is the two-dimensional torus [MR18].

- Some cohomology classes of $\mathrm{Homeo}_0(M)$ are unbounded, where M is a closed Seifert-fibered 3-manifold such that the inclusion $SO(2) \to \mathrm{Homeo}_0(M)$ induces an inclusion of $\pi_1(SO(2))$ as a direct factor in $\pi_1(\mathrm{Homeo}_0(M))$ [Man20] (see also [MN21]).

Using Corollary 1.2, we will show the boundedness and unboundedness of characteristic classes on (contact) Hamiltonian diffeomorphism groups. Let us consider the symplectic manifold $(S^2 \times S^2, \omega_\lambda)$ and the contact manifold (S^3, ξ). The symplectic form ω_λ is defined by $\omega_\lambda = \mathrm{pr}_1^*\omega_0 + \lambda \cdot \mathrm{pr}_2^*\omega_0$, where ω_0 is the area form on S^2 and $\mathrm{pr}_j \colon S^2 \times S^2 \to S^2$ is the j-th projection. The contact structure ξ is the standard one on S^3.

To simplify the notation, we set $G_\lambda = \mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$ and $H = \mathrm{Cont}_0(S^3, \xi)$. For $1 < \lambda \leq 2$, we have $H^2(BG_\lambda; \mathbb{Z}) \cong \mathbb{Z}$ and $H^2(BH; \mathbb{Z}) \cong \mathbb{Z}$ (see Section 6). Let $\mathfrak{o}_{G_\lambda} \in H^2(BG_\lambda; \mathbb{Z})$ and $\mathfrak{o}_H \in H^2(BH; \mathbb{Z})$ be the generators (or the "primary obstruction classes with coefficients in $\mathbb{Z}$" of G_λ-bundles and H-bundles, respectively).

Using Corollary 1.2, we can verify the difference between these classes in terms of boundedness. For every $c \in H^2_{\mathrm{grp}}(G; \mathbb{Z})$, let $c_\mathbb{R} \in H^2_{\mathrm{grp}}(G; \mathbb{R})$ denote the corresponding cohomology class with coefficients in $\mathbb{R}$.

Corollary 2.1. *The following properties hold.*

(1) *The cohomology class*
$$B\iota^*(\mathfrak{o}_{G_\lambda})_\mathbb{R} \in H^2_{\mathrm{grp}}(G_\lambda; \mathbb{R})$$
is bounded.

(2) *The cohomology class*
$$B\iota^*(\mathfrak{o}_H)_\mathbb{R} \in H^2_{\mathrm{grp}}(H; \mathbb{R})$$
is unbounded.

We will prove Corollary 2.1 in Section 6. In order to prove Corollary 2.1, we use Ostrover's Calabi quasi-morphism, which is a Hamiltonian Floer theoretic invariant.

Moreover, we will show the Milnor-Wood type inequality in Section 7 (Theorem 7.1). Applying it to the obstruction class $(\mathfrak{o}_{G_\lambda})_\mathbb{R}$, we obtain the following corollary.

Corollary 2.2. *Let Σ_h be a closed orientable surface of genus $h \geq 1$. Then there exist infinitely many isomorphism classes of Hamiltonian fibrations over Σ_h with the structure group $G_\lambda = \mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$ which do not admit foliated G_λ-bundle structures.*

Remark 2.3. The boundedness of c is equivalent to the one of $c_\mathbb{R}$; that is, the integer cohomology class c is bounded if and only if the real cohomology class $c_\mathbb{R}$ is bounded (this can be shown by the same arguments as in [CMPSC11, Lemma 29]). Hence, the statement in Corollary 2.1 holds for the integer coefficients cohomology classes $B\iota^*(\mathfrak{o}_{G_\lambda})$ and $B\iota^*(\mathfrak{o}_H)$.

Corollaries 2.1 and 2.2 will be restated in more general forms (see Corollary 6.6 and Theorem 7.2, respectively).

2.2. **Cohomology of (contact) Hamiltonian diffeomorphism group.** Many researchers have constructed non-trivial cohomology classes of $\mathrm{Ham}(M, \omega)$ and $\mathrm{Cont}_0(M, \xi)$ as characteristic classes of certain (contact) Hamiltonian fibrations. For example, see [Rez97], [JK02], [GKT11], [McD04], [SS20], [Mar20], and [CS16]. Here, $\mathrm{Ham}(M, \omega)$ and $\mathrm{Cont}_0(M, \xi)$ are associated with the discrete topology or the C^∞-topology. Many non-trivial homogeneous quasi-morphisms on $\mathrm{Ham}(M, \omega)$ also have been constructed in several papers (see, for example, [BG92], [EP03], [GG04], [Py06b], [Py06a], [McD10], [FOOO19, Theorem 1.10 (1)], [Ish14], and [Bra15]).

From these homogeneous quasi-morphisms on $\mathrm{Ham}(M,\omega)$, we can construct non-trivial elements of $H_b^2(\mathrm{Ham}(M,\omega);\mathbb{R})$ under the canonical map

$$\mathbf{d}\colon Q(\mathrm{Ham}(M,\omega)) \to H_b^2(\mathrm{Ham}(M,\omega);\mathbb{R})$$

(for the map $\mathbf{d}$, see Section 3). Note that the classes obtained by this map are trivial as ordinary group cohomology classes in $H^2_{\mathrm{grp}}(\mathrm{Ham}(M,\omega);\mathbb{R})$.

On the other hand, using Corollary 1.2, we can construct the non-trivial second bounded cohomology classes of $\mathrm{Ham}(M,\omega)$ and $\mathrm{Cont}_0(M,\xi)$, which are also non-trivial as *ordinary* cohomology classes. Note that the universal coverings $\widetilde{\mathrm{Ham}}(M,\omega)$ and $\widetilde{\mathrm{Cont}}_0(M,\xi)$ are perfect groups for closed symplectic and contact manifolds ([Ban78], [Ryb10]). Therefore, these groups satisfy the assumption in Corollary 1.2.

In the following cases, Corollary 1.2 provides non-trivial cohomology classes in $H^2_{\mathrm{grp}}(\mathrm{Ham}(M,\omega);\mathbb{R})$ and $H^2_{\mathrm{grp}}(\mathrm{Cont}_0(M,\xi);\mathbb{R})$.

Example 2.4. Ostrover [Ost06] constructed quasi-morphism μ^λ on $\widetilde{\mathrm{Ham}}(S^2 \times S^2,\omega_\lambda)$ for $\lambda > 1$. The homogeneous quasi-morphism μ^λ does not descend to $\mathrm{Ham}(S^2 \times S^2,\omega_\lambda)$ (Proposition 3.2). Hence, we obtain a non-trivial cohomology class of $\mathrm{Ham}(S^2 \times S^2,\omega_\lambda)$ from μ^λ.

Example 2.5. Ostrover and Tyomkin [OT09] constructed two homogeneous quasi-morphisms $\mu^1,\mu^2\colon \widetilde{\mathrm{Ham}}(M,\omega)\to\mathbb{R}$, where (M,ω) is the 1-point blow-up of $\mathbb{C}P^2$ with some toric Fano symplectic form. The restrictions of μ_1,μ_2 to $\pi_1(\mathrm{Ham}(M,\omega))$ are linearly independent. Hence, Corollary 1.2 implies the dimension of $H^2_{\mathrm{grp}}(\mathrm{Ham}(M,\omega);\mathbb{R})$ is greater than one.

Example 2.6. Fukaya, Oh, Ohta, and Ono [FOOO19, Theorem 1.10 (3)] constructed quasi-morphisms on $\widetilde{\mathrm{Ham}}(M,\omega)$, where (M,ω) is the k-points blow-up of $\mathbb{C}P^2$ with some toric symplectic form, where $k \geq 2$. Their quasi-morphisms do not descend to $\mathrm{Ham}(M,\omega)$ [FOOO19, Theorem 30.13]. Hence, we can construct a non-trivial element of $H^2_{\mathrm{grp}}(\mathrm{Ham}(M,\omega);\mathbb{R})$ from their quasi-morphisms.

Example 2.7. Givental [Giv90] constructed a homogeneous quasi-morphism μ on $\widetilde{\mathrm{Cont}}_0(\mathbb{R}P^{2n+1},\xi)$ that is called the *non-linear Maslov index* (see also [Sim07], [BZ15]). This quasi-morphism μ does not descend to $\mathrm{Cont}_0(\mathbb{R}P^{2n+1},\xi)$. Hence, we can obtain a non-trivial element of $H^2_{\mathrm{grp}}(\mathrm{Cont}_0(\mathbb{R}P^{2n+1},\xi);\mathbb{R})$.

In Section 6, we also prove the following corollary.

Corollary 2.8. *Let (M,ω) be a closed symplectic manifold. Then there exists an injective homomorphism*

$$\mathfrak{d}_b\colon Q(\widetilde{\mathrm{Ham}}(M,\omega)) \to H_b^2(\mathrm{Ham}(M,\omega);\mathbb{R}).$$

In [She14], for every closed symplectic manifold (M,ω), Shelukhin constructed a non-trivial homogeneous quasi-morphism $\mu_S\colon \widetilde{\mathrm{Ham}}(M,\omega)\to\mathbb{R}$. Therefore, the following corollary follows from Corollary 2.8.

Corollary 2.9. *For every closed symplectic manifold (M,ω), the bounded cohomology group $H_b^2(\mathrm{Ham}(M,\omega);\mathbb{R})$ is non-zero.*

Remark 2.10. Quasi-morphisms in [EP03], [Ost06], [OT09], [McD10], [FOOO19], [Ush11], [Bor13], [Cas17], and [Via18] are constructed via the Hamiltonian Floer theory. As good textbooks on this topic, we refer the reader to [PR14] and [FOOO19].

Disclaimer 2.11. Throughout the present paper, we tacitly assume that topological group G is path-connected, locally path-connected, and semilocally simply connected. In particular, every topological group G in the present paper admits the universal covering $\pi\colon \tilde{G} \to G$.

2.3. **Organization of the paper.** Section 3 collects preliminary facts. Section 4 and Section 5 are devoted to showing an isomorphism theorem (Theorem 5.4) for an arbitrary group extension. In Section 6, we prove Theorem 1.1 by applying the isomorphism theorem to a topological group and its universal covering. We provide applications in Section 7 and Section 8. In Section 7, we prove a Milnor-Wood type inequality and show the non-existence of foliated structures on Hamiltonian fibrations. In Section 8, we consider an extension problem of homomorphisms on $\pi_1(G)$ to $\tilde{G}$. In Appendix A, we provide examples of non-trivial (contact) Hamiltonian fibrations.

3. Preliminaries

3.1. **(Bounded) group cohomology and quasi-morphism.** We briefly review the (bounded) cohomology of (discrete) group and the quasi-morphism. Let G be a group and A be an abelian group. Further, let $C^n_{\mathrm{grp}}(G;A)$ denote the set of n-cochains $c\colon G^n \to A$. The coboundary map $\delta\colon C^n_{\mathrm{grp}}(G;A) \to C^{n+1}_{\mathrm{grp}}(G;A)$ is defined by

$$\begin{aligned}&\delta c(g_1, \cdots, g_{n+1})\\ &= c(g_2, \cdots, g_{n+1}) + \sum_{i=1}^{n}(-1)^i c(g_1, \cdots, g_i g_{i+1}, \cdots, g_{n+1}) + (-1)^{n+1}c(g_1, \cdots, g_n),\end{aligned}$$

where $c \in C^n_{\mathrm{grp}}(G;A)$ and $g_1, \cdots, g_{n+1} \in G$. The cohomology $H^\bullet_{\mathrm{grp}}(G;A)$ of the cochain complex $(C^\bullet_{\mathrm{grp}}(G;A), \delta)$ is called the (*ordinary*) *group cohomology of* G.

It is known that the cohomology of group G is canonically isomorphic to the cohomology of classifying space BG^δ of discrete group G^δ. This isomorphism is given by an isomorphism of cochains (see, for example, [Dup78]). Under this isomorphism, we identify $H^\bullet(BG^\delta;A)$ with $H^\bullet_{\mathrm{grp}}(G;A)$.

Let $A = \mathbb{Z}$ or $\mathbb{R}$, and let $C^n_b(G;A)$ denote the set of bounded n-cochains, i.e., $c \in C^n_{\mathrm{grp}}(G;A)$ such that

$$\|c\|_\infty = \sup_{g_1,\ldots,g_n \in G} |c(g_1, \ldots, g_n)| < +\infty.$$

The cohomology $H^\bullet_b(G,A)$ of the cochain complex $(C^\bullet_b(G;A), \delta)$ is called the *bounded cohomology of* G. The inclusion map from $C^\bullet_b(G;A)$ to $C^\bullet_{\mathrm{grp}}(G;A)$ induces the homomorphism $c_G\colon H^\bullet_b(G;A) \to H^\bullet_{\mathrm{grp}}(G;A)$, which is called the *comparison map*.

Definition 3.1. A real-valued function μ on a group G is called a *quasi-morphism* if

$$D(\mu) = \sup_{g,h \in G} |\mu(gh) - \mu(g) - \mu(h)|$$

is finite. The value $D(\mu)$ is called the *defect* of μ. A quasi-morphism μ on G is called *homogeneous* if $\mu(g^n) = n\mu(g)$ for all $g \in G$ and $n \in \mathbb{Z}$. Let $Q(G)$ denote the real vector space of homogeneous quasi-morphisms on G.

It is known that every homogeneous quasi-morphism is conjugation-invariant; that is, $\mu \in Q(G)$ satisfies

$$\mu(ghg^{-1}) = \mu(h) \tag{3.1}$$

for every $g, h \in G$ (see, for example, [Cal09, Section 2.2.3]).

By definition, the coboundary $\delta\mu$ of a homogeneous quasi-morphism $\mu \in Q(G)$ defines a bounded two-cocycle on G. This induces the following exact sequence:

$$0 \to H^1_{\mathrm{grp}}(G;\mathbb{R}) \to Q(G) \xrightarrow{\mathbf{d}} H^2_b(G;\mathbb{R}) \xrightarrow{c_G} H^2_{\mathrm{grp}}(G;\mathbb{R}) \tag{3.2}$$

(see, for example, [Cal09, Theorem 2.50]).

The following property of homogeneous quasi-morphisms is important in the present paper:

$$\mu(fg) = \mu(f) + \mu(g) = \mu(gf) \text{ for every } f, g \in G \text{ with } fg = gf \tag{3.3}$$

(see, for example, [PR14, Proposition 3.1.4]).

In the present paper, we often use Ostrover's Calabi quasi-morphism, so we explain it here. Let $(S^2 \times S^2, \omega_\lambda)$ be the symplectic manifold defined in Subsection 2.1. Entov and Polterovich [EP03] constructed a homogeneous quasi-morphism μ^1 on $\widetilde{\mathrm{Ham}}(S^2 \times S^2, \omega_1)$ using the Hamiltonian Floer theory. More precisely, μ^1 was constructed as the homogenization of Oh-Schwarz's spectral invariants ([Sch00], [Oh05]), which is a Hamiltonian Floer theoretic invariant. (See also Remark 2.10.) They also proved that μ^1 descends to $\mathrm{Ham}(S^2 \times S^2, \omega_1)$.

After their work, Ostrover [Ost06] applied Entov and Polterovich's idea to $\widetilde{\mathrm{Ham}}(S^2 \times S^2, \omega_\lambda)$ for $\lambda > 1$ and studied a quasi-morphism $\mu^\lambda \colon \widetilde{\mathrm{Ham}}(S^2 \times S^2, \omega_\lambda) \to \mathbb{R}$. In contrast to Entov-Polterovich quasi-morphisms, Ostrover's Calabi quasi-morphism μ^λ does not descend to $\mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$.

Proposition 3.2 ([Ost06])**.** *For $\lambda > 1$, there exists $\tilde{g} \in \pi_1(\mathrm{Ham}(S^2 \times S^2, \omega_\lambda))$ such that $\mu(\tilde{g}) \neq 0$. In particular, μ^λ does not descend to* $\mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$.

3.2. **Characteristic classes.** For a fibration, the *primary obstruction class* is defined as an obstruction to the construction of a cross-section. We briefly recall the definition of the obstruction class via the Serre spectral sequence (see [Whi78] for details). Let $F \to E \to B$ be a fibration. For simplicity, we suppose the following: the base space B is one-connected, the fiber F is path-connected, and the fundamental group $\pi_1(F)$ is abelian. Further, let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence with coefficients in $\pi_1(F)$. Since B is one-connected, every local coefficient system on B is simple, and therefore we have

$$E_2^{p,q} \cong H^p(B; H^q(F; \pi_1(F))).$$

Hence, we obtain $E_2^{2,0} \cong H^2(B;\pi_1(F))$ and $E_2^{0,1} \cong H^1(F;\pi_1(F))$. Since the cohomology group $H^1(F;\pi_1(F))$ is isomorphic to $\mathrm{Hom}(\pi_1(F);\pi_1(F))$, the derivation map $d_2^{0,1} \colon E_2^{0,1} \to E_2^{2,0}$ defines a map

$$d_2^{0,1} \colon \mathrm{Hom}(\pi_1(F), \pi_1(F)) \to H^2(B; \pi_1(F)),$$

where here we abuse the symbol $d_2^{0,1}$.

We are now ready to state the definition of the primary obstruction class of fibrations.

Definition 3.3. Let $F \to E \to B$ be a fibration such that B is one-connected, F is path-connected, and $\pi_1(F)$ is abelian. Further, let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of this fibration. The cohomology class $\mathfrak{o}(E) = -d_2^{0,1}(\mathrm{id}_{\pi_1(F)}) \in H^2(B;\pi_1(F))$ is called the *primary obstruction class* of E, where $\mathrm{id}_{\pi_1(F)} \in \mathrm{Hom}(\pi_1(F), \pi_1(F))$ is the identity homomorphism.

Remark 3.4. It is known that the above definition is equivalent to the classical definition of the obstruction class for the construction of a cross-section (see, for example, [Whi78, (6.10) Corollary in Chapter VI and (7.9*) Theorem in Chapter XIII]).

By the naturality of the spectral sequence, the primary obstruction class is a characteristic class. Its universal element $\mathfrak{o}$ is given as the primary obstruction class of the principal universal bundle $G \to EG \to BG$. Note that the classifying space BG is one-connected and $\pi_1(G)$ is abelian.

Remark 3.5. The class $\mathfrak{o}$ is also obtained as follows. By taking classifying spaces of the exact sequence $0 \to \pi_1(G) \to \tilde{G} \to G \to 1$, we obtain the following fibration:

$$B\pi_1(G) \to B\tilde{G} \to BG. \tag{3.4}$$

Note that the fundamental group of $B\pi_1(G)$ is isomorphic to $\pi_1(G)$, which is abelian. Then the primary obstruction class of fibration (3.4) is the class $\mathfrak{o} \in H^2(BG; \pi_1(G))$.

Let $f\colon \pi_1(G) \to \mathbb{R}$ be a homomorphism and

$$f_*\colon H^\bullet(-; \pi_1(G)) \to H^\bullet(-; \mathbb{R})$$

denote the change of coefficients homomorphism. Further, let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of (3.4) with coefficients in $\mathbb{R}$. Since $E_2^{0,1} \cong H^1(B\pi_1(G); \mathbb{R}) \cong \mathrm{Hom}(\pi_1(G); \mathbb{R})$ and $E_2^{2,0} \cong H^2(BG; \mathbb{R})$, the derivation $d_2^{0,1}\colon E_2^{0,1} \to E_2^{2,0}$ defines a homomorphism

$$d_2^{0,1}\colon \mathrm{Hom}(\pi_1(G), \mathbb{R}) \to H^2(BG; \mathbb{R}).$$

Proposition 3.6. *Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of* (3.4) *with coefficients in $\mathbb{R}$. For a homomorphism $f\colon \pi_1(G) \to \mathbb{R}$, the equality*

$$-d_2^{0,1}(f) = f_*\mathfrak{o} \in H^2(BG; \mathbb{R})$$

holds.

Proof. Let $(E_r'^{p,q}, d_r'^{p,q})$ be the Serre spectral sequence of (3.4) with coefficients in $\pi_1(G)$. Then the equality $-d_2'^{0,1}(\mathrm{id}_{\pi_1(G)}) = \mathfrak{o}$ holds. Since the derivation maps in the Serre spectral sequence are compatible with the change of coefficients homomorphisms, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(\pi_1(G), \pi_1(G)) \cong E_2'^{0,1} \arrow[r, "d_2'^{0,1}"] \arrow[d, "f_*"] & E_2'^{2,0} \cong H^2(BG; \pi_1(G)) \arrow[d, "f_*"] \\
\mathrm{Hom}(\pi_1(G), \mathbb{R}) \cong E_2^{0,1} \arrow[r, "d_2^{0,1}"] & E_2^{2,0} \cong H^2(BG; \mathbb{R}).
\end{tikzcd}$$

Since $f = f_*(\mathrm{id}_{\pi_1(G)})$, we obtain

$$-d_2^{0,1}(f) = -d_2^{0,1}(f_*(\mathrm{id}_{\pi_1(G)})) = f_*(-d_2'^{0,1}(\mathrm{id}_{\pi_1(G)})) = f_*\mathfrak{o}$$

and the proposition follows. □

Remark 3.7. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence of (3.4) with coefficients in $\mathbb{R}$. Then the map $d_2^{0,1}$ is an isomorphism. Indeed, the E_2-page of the spectral sequence induces an exact sequence

$$0 \to H^1(BG; \mathbb{R}) \to H^1(B\tilde{G}; \mathbb{R}) \to H^1(B\pi_1(G); \mathbb{R}) \xrightarrow{d_2^{0,1}} H^2(BG; \mathbb{R}) \to H^2(B\tilde{G}; \mathbb{R}).$$

Since $\tilde{G}$ is one-connected, the classifying space $B\tilde{G}$ is two-connected. Hence, the cohomology groups $H^1(B\tilde{G}; \mathbb{R})$ and $H^2(B\tilde{G}; \mathbb{R})$ are trivial, which implies that the derivation map $d_2^{0,1}$ is an isomorphism. In particular, the class $f_*\mathfrak{o} = -d_2^{0,1}(f)$ is non-zero if and only if the homomorphism f is non-zero.

4. Construction of group cohomology classes

Let us consider an exact sequence

$$1 \to K \xrightarrow{i} \Gamma \xrightarrow{\pi} G \to 1 \tag{4.1}$$

of discrete groups.

Definition 4.1. A subspace $\mathcal{C}(\Gamma)$ of $C^1(\Gamma; A)$ is defined by

$$\begin{aligned} \mathcal{C}(\Gamma) = \{F \in & C^1_{\mathrm{grp}}(\Gamma; A) \\ & \mid F(k\gamma) = F(\gamma k) = F(\gamma) + F(k) \text{ for every } \gamma \in \Gamma, k \in K\}. \end{aligned} \tag{4.2}$$

We define a map $\mathfrak{D} \colon \mathcal{C}(\Gamma) \to C^2_{\mathrm{grp}}(G; A)$ by setting

$$\mathfrak{D}(F)(g_1, g_2) = F(\gamma_2) - F(\gamma_1\gamma_2) + F(\gamma_1),$$

where γ_j is an element of Γ satisfying $\pi(\gamma_j) = g_j$.

Lemma 4.2. *The map $\mathfrak{D} \colon \mathcal{C}(\Gamma) \to C^2_{\mathrm{grp}}(G; A)$ is well-defined.*

Proof. Let γ'_j be another element of Γ satisfying $\pi(\gamma'_j) = g_j$. Then there exist $k_1, k_2 \in K$ satisfying $\gamma'_1 = k_1\gamma_1$ and $\gamma'_2 = \gamma_2 k_2$. By the definition of $\mathcal{C}(\Gamma)$, we have

$$\begin{aligned} & F(\gamma'_2) - F(\gamma'_1\gamma'_2) + F(\gamma'_1) \\ & = F(\gamma_2 k_2) - F(k_1\gamma_1\gamma_2 k_2) + F(k_1\gamma_1) \\ & = (F(\gamma_2) + F(k_2)) - (F(k_1) + F(\gamma_1\gamma_2) + F(k_2)) + (F(k_1) + F(\gamma_1)) \\ & = F(\gamma_2) - F(\gamma_1\gamma_2) + F(\gamma_1). \end{aligned}$$

This implies the well-definedness of the map $\mathfrak{D}$. □

Lemma 4.3. *For every $F \in \mathcal{C}(\Gamma)$, the cochain $\mathfrak{D}(F)$ is a cocycle.*

Proof. Since $\pi^*\mathfrak{D}(F) = -\delta F$ by the definition of $\mathfrak{D}(f)$, we have

$$\pi^*(\delta\mathfrak{D}(F)) = -\delta\delta F = 0.$$

From the surjectivity of $\pi \colon \Gamma \to G$, we have $\delta\mathfrak{D}(F) = 0$. □

Definition 4.4. A homomorphism $\mathfrak{d} \colon \mathcal{C}(\Gamma) \to H^2_{\mathrm{grp}}(G; A)$ is defined by

$$\mathfrak{d}(F) = [\mathfrak{D}(F)] \in H^2_{\mathrm{grp}}(G; A).$$

For an element F of $\mathcal{C}(\Gamma)$, the restriction $F|_K = i^*F$ to K is a homomorphism. Moreover, $F|_K$ is Γ-invariant, since

$$F(\gamma^{-1}k\gamma) = F(\gamma \cdot \gamma^{-1}k\gamma) - F(\gamma) = F(k\gamma) - F(\gamma) = F(k).$$

Let $H^1_{\mathrm{grp}}(K; A)^\Gamma$ denote the space of Γ-invariant homomorphisms from K to A. Then the restriction to K defines a homomorphism $i^* \colon \mathcal{C}(\Gamma) \to H^1_{\mathrm{grp}}(K; A)^\Gamma$.

Lemma 4.5. *The homomorphism $i^* \colon \mathcal{C}(\Gamma) \to H^1_{\mathrm{grp}}(K; A)^\Gamma$ is surjective.*

Proof. Let $s \colon G \to \Gamma$ be a section of $p \colon \Gamma \to G$ satisfying $s(1_G) = 1_\Gamma$, where $1_G \in G$ and $1_\Gamma \in \Gamma$ are the unit elements of G and Γ, respectively. Since $\gamma \cdot s(\pi(\gamma))^{-1}$ is in $\mathrm{Ker}(\pi \colon \Gamma \to G)$, we regard $\gamma \cdot s(\pi(\gamma))^{-1}$ as an element of K under the injection $i \colon K \to \Gamma$. For an element f of $H^1_{\mathrm{grp}}(K; A)^\Gamma$, define $f_s \colon \Gamma \to A$ by

$$f_s(\gamma) = f(\gamma \cdot s(\pi(\gamma))^{-1}).$$

Note that the restriction of f_s to K is equal to f. Moreover, the equalities

$$\begin{aligned} f_s(k\gamma) &= f(k\gamma \cdot s(\pi(k\gamma))^{-1}) = f(k\gamma \cdot s(\pi(\gamma))^{-1}) \\ &= f_s(k) + f_s(\gamma \cdot s(\pi(k\gamma))^{-1}) = f_s(k) + f_s(\gamma) \end{aligned}$$

and

$$\begin{aligned}
f_s(\gamma k) &= f(\gamma k \cdot s(\pi(\gamma k))^{-1}) = f(\gamma k \cdot s(\pi(\gamma))^{-1}) \\
&= f(\gamma k \gamma^{-1}) + f_s(\gamma \cdot s(\pi(k\gamma))^{-1}) \\
&= f(k) + f_s(\gamma \cdot s(\pi(k\gamma))^{-1}) \\
&= f_s(k) + f_s(\gamma \cdot s(\pi(k\gamma))^{-1}) = f_s(k) + f_s(\gamma)
\end{aligned}$$

hold, where we use the Γ-invariance of f in the second equalities. Hence, f_s belongs to $\mathcal{C}(\Gamma)$ and the surjectivity follows. $\square$

For sequence (4.1), there is an exact sequence

$$\begin{aligned}
0 \to H^1_{\mathrm{grp}}(G;\mathbb{R}) &\xrightarrow{\pi^*} H^1_{\mathrm{grp}}(\Gamma;\mathbb{R}) \xrightarrow{i^*} H^1_{\mathrm{grp}}(K;\mathbb{R})^{\Gamma} \\
&\xrightarrow{\tau} H^2_{\mathrm{grp}}(G;\mathbb{R}) \xrightarrow{\pi^*} H^2_{\mathrm{grp}}(\Gamma;\mathbb{R})
\end{aligned} \tag{4.3}$$

called the *five-term exact sequence*. This five-term exact sequence is obtained from the Hochschild-Serre spectral sequence $(E_r^{p,q}, d_r^{p,q})$ of (4.1), and the map τ is the derivation $d_2^{0,1} \colon E_2^{0,1} = H^1_{\mathrm{grp}}(K;\mathbb{R})^{\Gamma} \to E_2^{2,0} = H^2_{\mathrm{grp}}(G;\mathbb{R})$.

Lemma 4.6. *The diagram*

$$\begin{tikzcd}
\mathcal{C}(\Gamma) \arrow[d, "i^*"'] \arrow[dr, "\mathfrak{d}"] & \\
H^1_{\mathrm{grp}}(K;A)^{\Gamma} \arrow[r, "\tau"'] & H^2_{\mathrm{grp}}(G;A).
\end{tikzcd}$$

commutes.

Proof. By Definition 4.4 and Proposition 4.7 below, the commutativity follows. $\square$

Proposition 4.7 ([NSW08, (1.6.6) Proposition])**.** *For every Γ-invariant homomorphism $f \in H^1_{\mathrm{grp}}(K;A)^{\Gamma}$, there exist a one-cochain $F \colon \Gamma \to A$ such that $i^*F = f$ and a cocycle $c \in C^2_{\mathrm{grp}}(G;A)$ satisfying $c(\pi(\gamma_1), \pi(\gamma_2)) = \delta F(\gamma_1, \gamma_2)$ for every $\gamma_1, \gamma_2 \in \Gamma$. Moreover, the class $\tau(f)$ is equal to $[c] \in H^2_{\mathrm{grp}}(G;A)$.*

5. A diagram via bounded cohomology and quasi-morphism

From this section, we mainly consider cohomology with coefficients in $\mathbb{R}$. In this section, we refine the commutative diagram in view of bounded cohomology and homogeneous quasi-morphism. Recall that a cohomology class $\alpha \in H^2_{\mathrm{grp}}(G;\mathbb{R})$ is called *bounded* if α is in the image of the comparison map $c_G \colon H^2_b(G;\mathbb{R}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$.

Proposition 5.1. *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{C}(\Gamma) \cap Q(\Gamma) \arrow[r, "\mathfrak{d}_b"] \arrow[d, "i^*"'] \arrow[dr, "\mathfrak{d}"] & H^2_b(G;\mathbb{R}) \arrow[d, "c_G"] \\
H^1_{\mathrm{grp}}(K;\mathbb{R})^{G} \arrow[r, "\tau"'] & H^2_{\mathrm{grp}}(G;\mathbb{R}).
\end{tikzcd}$$

Proof. Let F be an element of $\mathcal{C}(\Gamma) \cap Q(\Gamma)$. Then the cocycle $\mathfrak{D}(F)$ is bounded, since F is a quasi-morphism and $\mathfrak{D}(F)(g_1, g_2) = F(\gamma_2) - F(\gamma_1\gamma_2) + F(\gamma_1)$ for every $g_1, g_2 \in G$ and their lifts $\gamma_1, \gamma_2 \in \Gamma$. Hence, the homomorphism $\mathfrak{D} \colon \mathcal{C}(\Gamma) \to C^2_{\mathrm{grp}}(G;\mathbb{R})$ induces a homomorphism

$$\mathfrak{d}_b \colon \mathcal{C}(\Gamma) \cap Q(\Gamma) \to H^2_b(G;\mathbb{R}).$$

By the definition of the comparison map c_G, we have $\mathfrak{d} = c_G \circ \mathfrak{d}_b$. $\square$

Remark 5.2. An exact sequence

$$0 \to A \xrightarrow{i} \Gamma \xrightarrow{\pi} G \to 1$$

(or simply Γ) is called a *central extension* if the image $i(A)$ is contained in the center of Γ. For a central extension Γ, the space $Q(\Gamma)$ is contained in $\mathcal{C}(\Gamma)$. Indeed, by definition, we have $a\gamma = \gamma a$ for every $a \in A$ and $\gamma \in \Gamma$. Hence, by (3.3), every homogeneous quasi-morphism $\mu \in Q(\Gamma)$ satisfies

$$\mu(a\gamma) = \mu(\gamma a) = \mu(a) + \mu(\gamma).$$

This implies that $Q(\Gamma) \subset \mathcal{C}(\Gamma)$. Moreover, every homomorphism $f\colon A \to \mathbb{R}$ is Γ-invariant, since $\gamma^{-1}a\gamma = a\gamma^{-1}\gamma = a$ for every $\gamma \in \Gamma$ and every $a \in A$. From the above together with Proposition 5.1, we obtain the following commutative diagram:

$$\begin{tikzcd} Q(\Gamma) \arrow[r, "\mathfrak{d}_b"] \arrow[d, "i^*"'] \arrow[dr, "\mathfrak{d}"] & H_b^2(G;\mathbb{R}) \arrow[d, "c_G"] \\ H^1_{\mathrm{grp}}(A;\mathbb{R}) \arrow[r, "\tau"'] & H^2_{\mathrm{grp}}(G;\mathbb{R}) \end{tikzcd}$$

for a central extension $0 \to A \to \Gamma \to G \to 1$.

Lemma 5.3. *Let μ be a homogeneous quasi-morphism on Γ whose restriction to K is a homomorphism. Then μ is contained in $\mathcal{C}(\Gamma)$.*

Proof. For every $\gamma \in \Gamma$, $k \in K$, and $n \in \mathbb{N}$, the equalities

$$(k\gamma)^n = k \cdot \gamma k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \cdots \cdot \gamma^{n-1} k \gamma^{-(n-1)} \cdot \gamma^n$$

and

$$(\gamma k)^n = \gamma^n \cdot \gamma^{-(n-1)} k \gamma^{n-1} \cdot \cdots \cdot \gamma^{-2} k \gamma^2 \cdot \gamma^{-1} k \gamma \cdot k$$

hold. By (3.1), the restriction $\mu|_K$ is Γ-invariant. Hence, we have

$$\mu(k \cdot \gamma k \gamma^{-1} \cdot \gamma^2 k \gamma^{-2} \cdot \cdots \cdot \gamma^{n-1} k \gamma^{-(n-1)}) = \mu(k^n)$$

and

$$\mu(\gamma^{-(n-1)} k \gamma^{n-1} \cdot \cdots \cdot \gamma^{-2} k \gamma^2 \cdot \gamma^{-1} k \gamma \cdot k) = \mu(k^n).$$

These equalities imply that

$$n \cdot |\mu(k\gamma) - \mu(k) - \mu(\gamma)| = |\mu((k\gamma)^n) - \mu(k^n) - \mu(\gamma^n)| < D(\mu)$$

and

$$n \cdot |\mu(\gamma k) - \mu(\gamma) - \mu(k)| = |\mu((\gamma k)^n) - \mu(\gamma^n) - \mu(k^n)| < D(\mu).$$

From these, we obtain $\mu(k\gamma) = \mu(k) + \mu(\gamma)$ and $\mu(\gamma k) = \mu(\gamma) + \mu(k)$. $\square$

Theorem 5.4. *The homomorphism $\mathfrak{d}\colon \mathcal{C}(\Gamma) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$ induces an isomorphism*

$$(\mathcal{C}(\Gamma) \cap Q(\Gamma))/(H^1_{\mathrm{grp}}(\Gamma;\mathbb{R}) + \pi^* Q(G)) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G).$$

Proof. Let us consider the following commutative diagram whose rows and columns are exact:

$$\begin{tikzcd}
H^1(K;\mathbb{R})^\Gamma \arrow[r] & Q(K)^\Gamma \arrow[r] & H_b^2(K;\mathbb{R})^\Gamma & \\
H^1(\Gamma;\mathbb{R}) \arrow[r] \arrow[u] & Q(\Gamma) \arrow[r, "\mathbf{d}"] \arrow[u] & H_b^2(\Gamma;\mathbb{R}) \arrow[r] \arrow[u] & H^2(\Gamma;\mathbb{R}) \\
H^1(G;\mathbb{R}) \arrow[r] \arrow[u] & Q(G) \arrow[r] \arrow[u, "\pi^*"] & H_b^2(G;\mathbb{R}) \arrow[r, "c_G"] \arrow[u, "\pi^*"] & H^2(G;\mathbb{R}) \arrow[u, "\pi^*"] \\
 & & 0 \arrow[u] & H^1(K;\mathbb{R})^\Gamma, \arrow[u, "\tau"]
\end{tikzcd}$$

where the exactness of the third column was shown in [Bou95]. By the definition of $\mathfrak{d}_b$, we have $\pi^*\mathfrak{d}_b(\mu) = \mathbf{d}(\mu)$ for $\mu \in \mathcal{C}(\Gamma) \cap Q(\Gamma)$. Hence, the map $\pi^*\colon H_b^2(G;\mathbb{R}) \to H_b^2(\Gamma;\mathbb{R})$ gives an isomorphism

$$\pi^*\colon H_b^2(G;\mathbb{R}) \xrightarrow{\cong} \mathbf{d}(\mathcal{C}(\Gamma) \cap Q(\Gamma)).$$

Then, in this diagram, the map $\mathfrak{d}$ is given as the composite

$$c_G \circ (\pi^*)^{-1} \circ \mathbf{d}\colon \mathcal{C}(\Gamma) \cap Q(\Gamma) \to H^2_{\mathrm{grp}}(G;\mathbb{R}).$$

The equality $\mathrm{Ker}(\mathfrak{d}) = H^1_{\mathrm{grp}}(\Gamma;\mathbb{R}) + \pi^*Q(G)$ is verified by a diagram chasing argument. By Lemma 5.3 and a diagram chasing argument, the surjectivity of the map $\mathfrak{d}\colon \mathcal{C}(\Gamma) \cap Q(\Gamma) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G)$ also follows. □

Remark 5.5. For a central extension Γ of G, the homomorphism $\mathfrak{d}\colon \mathcal{C}(\Gamma) \to H^2(G;\mathbb{R})$ induces an isomorphism

$$Q(\Gamma)/(H^1_{\mathrm{grp}}(\Gamma;\mathbb{R}) + \pi^*Q(G)) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G),$$

since $\mathcal{C}(\Gamma) \cap Q(\Gamma) = Q(\Gamma)$ (see Remark 5.2).

6. On topological groups

6.1. **General topological groups.** Let G be a topological group and $\pi\colon \tilde{G} \to G$ be its universal covering. Since the exact sequence

$$0 \to \pi_1(G) \xrightarrow{i} \tilde{G} \xrightarrow{\pi} G \to 1 \tag{6.1}$$

is a central extension ([Pon86, Theorem 15])), Remark 5.2 gives the commutative diagram

$$\begin{tikzcd}
Q(\tilde{G}) \arrow[r, "\mathfrak{d}_b"] \arrow[d, "i^*"] \arrow[dr, "\mathfrak{d}"] & H_b^2(G;\mathbb{R}) \arrow[d, "c_G"] \\
H^1_{\mathrm{grp}}(\pi_1(G);\mathbb{R}) \arrow[r, "\tau"'] & H^2_{\mathrm{grp}}(G;\mathbb{R})
\end{tikzcd} \tag{6.2}$$

Moreover, from Remark 5.5, the homomorphism $\mathfrak{d}\colon Q(\tilde{G}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$ induces an isomorphism

$$Q(\tilde{G})/(H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R}) + \pi^*Q(G)) \to \mathrm{Im}(\tau) \cap \mathrm{Im}(c_G). \tag{6.3}$$

In this section, we clarify the relation between the class $\mathfrak{d}(\mu) \in H^2_{\mathrm{grp}}(G;\mathbb{R})$ and the primary obstruction class $\mathfrak{o} \in H^2(BG;\pi_1(G))$.

By taking the classifying spaces of (6.1), we obtain a commutative diagram of fibrations

$$\begin{tikzcd} B\pi_1(G) \arrow[r] \arrow[d, equal] & B\tilde{G}^\delta \arrow[r] \arrow[d] & BG^\delta \arrow[d, "B\iota"] \\ B\pi_1(G) \arrow[r] & B\tilde{G} \arrow[r] & BG. \end{tikzcd}$$

In what follows, we regard the pullback $B\iota^* \colon H^\bullet(BG;\mathbb{R}) \to H^\bullet(BG^\delta;\mathbb{R})$ as a homomorphism

$$B\iota^* \colon H^\bullet(BG;\mathbb{R}) \to H^\bullet_{\mathrm{grp}}(G;\mathbb{R})$$

under the isomorphism $H^\bullet(BG^\delta;\mathbb{R}) \cong H^\bullet_{\mathrm{grp}}(G;\mathbb{R})$.

Lemma 6.1. *Let $(E_r^{p,q}, d_r^{p,q})$ be the $\mathbb{R}$-coefficients cohomology Serre spectral sequence of the fibration $B\pi_1(G) \to B\tilde{G} \to BG$. Then the equality*

$$B\iota^* \circ d_2^{0,1} = \tau \colon H^1_{\mathrm{grp}}(\pi_1(G);\mathbb{R}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$$

holds, where we identify $E_2^{0,1} = H^1(B\pi_1(G);\mathbb{R})$ with $H^1_{\mathrm{grp}}(\pi_1(G);\mathbb{R})$.

Proof. Let $({}^\delta E_r^{p,q}, {}^\delta d_r^{p,q})$ be the Hochschild-Serre spectral sequence of central extension (6.1). Note that the spectral sequence $({}^\delta E_r^{p,q}, {}^\delta d_r^{p,q})$ is isomorphic to the Serre spectral sequence of the fibration $B\pi_1(G) \to B\tilde{G}^\delta \to BG^\delta$ (see, for example, [Ben91]). Since the map τ is equal to the derivation map ${}^\delta d_2^{0,1}$ by definition, the naturality of the Serre spectral sequence implies that

$$B\iota^* \circ d_2^{0,1} = {}^\delta d_2^{0,1} = \tau,$$

and the lemma follows. □

Corollary 6.2. *Let $\mathfrak{o} \in H^2(BG;\mathbb{R})$ be the primary obstruction class for G-bundles. Then, for every homogeneous quasi-morphism $\mu \in Q(\tilde{G})$, the equality*

$$\mathfrak{D}(\mu) = -B\iota^*((\mu|_{\pi_1(G)})_*\mathfrak{o})$$

holds.

Proof. Let $(E_r^{p,q}, d_r^{p,q})$ be the Serre spectral sequence as in Lemma 6.1. Using Proposition 3.6, we obtain

$$B\iota^* \circ d_2^{0,1}(\mu|_{\pi_1(G)}) = -B\iota^*((\mu|_{\pi_1(G)})_*\mathfrak{o}).$$

On the other hand, using Lemma 6.1 and commutative diagram (6.2), we obtain

$$B\iota^* \circ d_2^{0,1}(\mu|_{\pi_1(G)}) = \tau(\mu|_{\pi_1(G)}) = \tau(i^*(\mu)) = \mathfrak{D}(\mu).$$

Hence, the equality $\mathfrak{D}(\mu) = -B\iota^*((\mu|_{\pi_1(G)})_*\mathfrak{o})$ holds. □

Corollary 6.3. *If $H^1(\tilde{G};\mathbb{R})$ is trivial, then the homomorphism*

$$B\iota^* \colon H^2(BG;\mathbb{R}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$$

is injective.

Proof. From the five-term exact sequence

$$\begin{aligned} 0 \to H^1_{\mathrm{grp}}(G;\mathbb{R}) \to H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R}) &\to H^1_{\mathrm{grp}}(\pi_1(G);\mathbb{R}) \\ &\xrightarrow{\tau} H^2_{\mathrm{grp}}(G;\mathbb{R}) \to H^2_{\mathrm{grp}}(\tilde{G};\mathbb{R}), \end{aligned}$$

the triviality of $H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R})$ implies the injectivity of the map τ. Hence, Lemma 6.1 and Remark 3.7 imply that the map $B\iota^*$ is injective. □

Theorem 6.4. *The homomorphism $\mathfrak{d}\colon Q(\tilde{G}) \to H^2_{\mathrm{grp}}(G;\mathbb{R})$ induces an isomorphism*

$$Q(\tilde{G})/(H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R}) + \pi^* Q(G)) \to \mathrm{Im}(B\iota^*) \cap \mathrm{Im}(c_G).$$

Proof. The equality

$$\mathrm{Im}(B\iota^*) = \mathrm{Im}(\tau)$$

holds by Lemma 6.1 and Remark 3.7. Hence, isomorphism (6.3) implies the theorem. □

The following corollary immediately follows from Theorem 6.4.

Corollary 6.5. *If the first cohomology $H^1_{\mathrm{grp}}(\tilde{G};\mathbb{R})$ is trivial, then the homomorphism $\mathfrak{d}$ induces an isomorphism*

$$Q(\tilde{G})/\pi^* Q(G) \to \mathrm{Im}(B\iota^*) \cap \mathrm{Im}(c_G).$$

In particular, if $\mu \in Q(\tilde{G})$ does not descend to G, then the class $\mathfrak{d}(\mu) \in H^2_{\mathrm{grp}}(G;\mathbb{R})$ is non-zero.

By using Corollaries 6.2 and 6.3, and Theorem 6.4, we can obtain the following corollary.

Corollary 6.6. *Let G be a topological group and $\tilde{G}$ be its universal covering.*

(1) *Suppose $\mu\colon \tilde{G} \to \mathbb{R}$ is a homogeneous quasi-morphism which does not descend to G and denote the primary obstruction class of G by $\mathfrak{o} \in H^2(BG;\pi_1(G))$. Then the cohomology class*

$$B\iota^*(((\mu|_{\pi_1(G)})_* \mathfrak{o})_{\mathbb{R}}) \in H^2_{\mathrm{grp}}(G;\mathbb{R})$$

is bounded. Here, $(\mu|_{\pi_1(G)})_*\colon H^2(BG;\pi_1(G)) \to H^2(BG;\mathbb{R})$ is the change of coefficients homomorphism induced by $\mu|_{\pi_1(G)}\colon \pi_1(G) \to \mathbb{R}$.

(2) *Assume that the space $Q(\tilde{G})$ is trivial. Then, for every non-zero element $\mathfrak{c}$ of $H^2(BG;\mathbb{R})$, the cohomology class*

$$(B\iota)^*(\mathfrak{c}) \in H^2_{\mathrm{grp}}(G;\mathbb{R})$$

is unbounded.

6.2. **Hamiltonian and contact Hamiltonian diffeomorphism groups.** We set $G_\lambda = \mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$ and $H = \mathrm{Cont}_0(S^3, \xi)$. For $1 < \lambda \leq 2$, it is known that $\pi_1(G_\lambda) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ([Anj02]) and $\pi_1(H) \cong \mathbb{Z}$ ([Eli92]). From Remark 3.7, we have

$$H^2(BG_\lambda;\mathbb{Z}) \cong H^1(B\pi_1(G_\lambda);\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(G_\lambda),\mathbb{Z}) \cong \mathbb{Z}$$

and

$$H^2(BH;\mathbb{Z}) \cong H^1(B\pi_1(H);\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(H),\mathbb{Z}) \cong \mathbb{Z}.$$

Let $\mathfrak{o}_H$ be the primary obstruction class of H-bundles, which is a generator of $H^2(BH;\mathbb{Z})$. The primary obstruction class $\mathfrak{o}$ of G_λ-bundles is defined as the identity homomorphism in $H^2(BG_\lambda;\pi_1(G_\lambda)) \cong \mathrm{Hom}(\pi_1(G_\lambda),\pi_1(G_\lambda))$. Let

$$\phi\colon \pi_1(G_\lambda) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \tag{6.4}$$

be the homomorphism sending $(n,a,b) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ to $n \in \mathbb{Z}$. We set

$$\mathfrak{o}_{G_\lambda} = \phi_* \mathfrak{o} \in H^2(BG_\lambda;\mathbb{Z}).$$

Then the class $\mathfrak{o}_{G_\lambda}$ is a generator of $H^2(BG_\lambda;\mathbb{Z})$.

Proof of Corollary 2.1. First, we prove (1). Recall that the restriction $\mu^{\lambda}|_{\pi_1(G_\lambda)}$ of Ostrover's Calabi quasi-morphism is a non-trivial homomorphism to $\mathbb{R}$ (Proposition 3.2). Hence, there exists a non-zero constant a such that

$$\phi = a\mu^{\lambda}|_{\pi_1(G_\lambda)} \colon \pi_1(G_\lambda) \to \mathbb{R},$$

where ϕ is the homomorphism given by (6.4). Therefore, we have

$$B\iota^*(\mathfrak{o}_{G_\lambda})_{\mathbb{R}} = a \cdot B\iota^*((\mu|_{\pi_1(G_\lambda)})_*\mathfrak{o})_{\mathbb{R}}.$$

Since Ostrover's Calabi quasi-morphism does not descend to G_λ, Corollary 6.6 (1) gives that the class $B\iota^*(\mathfrak{o}_{G_\lambda})_{\mathbb{R}}$ is bounded.

Next, we prove (2). Because the universal covering group $\widetilde{H} = \widetilde{\mathrm{Cont}_0}(S^3, \xi)$ is a uniformly perfect group (see [FPR18, Corollary 3.6 and Remark 3.7]), we have $Q(\widetilde{H}) = 0$. By Corollary 6.6 (2), the class $B\iota^*(\mathfrak{o}_H)_{\mathbb{R}}$ is unbounded. □

In Section 7, we provide another proof of Corollary 2.1 (2) by using a Milnor-Wood type inequality (Theorem 7.1) instead of Corollary 6.6 (2) (see Remark 7.4).

We end this section with the proof of Corollary 2.8. For this purpose, we need the following lemma.

Lemma 6.7. *For every topological group G whose universal covering group $\tilde{G}$ satisfies $H^1_{\mathrm{grp}}(\tilde{G}; \mathbb{R}) = 0$, the map*

$$\mathfrak{d}_b \colon Q(\tilde{G}) \to H^2_b(G; \mathbb{R})$$

is injective.

Proof. Using exact sequence (3.2) and the assumption $H^1_{\mathrm{grp}}(\tilde{G}; \mathbb{R}) = 0$, we obtain that the map $Q(\tilde{G}) \to H^2_b(\tilde{G}; \mathbb{R})$ is injective. Hence, for every homogeneous quasi-morphism $\mu \in Q(\tilde{G})$, the bounded cohomology class $[\delta\mu] \in H^2_b(\tilde{G}; \mathbb{R})$ is non-zero. Since

$$\pi^*(\mathfrak{d}_b(\mu)) = [\delta\mu] \in H^2_b(\tilde{G}; \mathbb{R}),$$

where $\pi^* \colon H^2_b(G; \mathbb{R}) \to H^2_b(\tilde{G}; \mathbb{R})$ is the homomorphism induced by the universal covering $\pi \colon \tilde{G} \to G$, the class $\mathfrak{d}_b(\mu)$ is also non-zero. □

Proof of Corollary 2.8. Because $\widetilde{\mathrm{Ham}}(M, \omega)$ is a perfect group ([Ban78]), Lemma 6.7 implies Corollary 2.8. □

7. Milnor-Wood type inequality and bundles with no flat structures

In this section, we show the existence of bundles over a surface which do not admit foliated (flat) structures. To this end, first we introduce a Milnor-Wood type inequality.

Let c be a universal characteristic class of foliated principal G-bundles. Then the characteristic class c is given as an element in $H^2(BG^\delta; \mathbb{R})$. For a foliated principal G-bundle $G \to E \to B$, the characteristic class $c(E)$ of E associated with c is defined by

$$c(E) = f^*c \in H^2(B; \mathbb{R}),$$

where $f \colon B \to BG^\delta$ is the classifying map of E.

Let Σ_h denote a closed oriented surface of genus $h \geq 1$ and $G \to E \to \Sigma_h$ be a foliated G-bundle. Further, let $\rho \colon \pi_1(\Sigma_h) \to G$ be the holonomy homomorphism of the bundle E. Then the classifying map of the bundle E is given by

$$B\rho \colon \Sigma_h \simeq B\pi_1(\Sigma_h) \to BG^\delta.$$

Theorem 7.1. *Let c be an element of $\mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*)$ and $[\Sigma_h] \in H_2(\Sigma_h;\mathbb{Z})$ be the fundamental class of Σ_h. Further, let $G \to E \to \Sigma_h$ be a foliated principal G-bundle. Then*

$$|\langle c(E), [\Sigma_h]\rangle| \leq D(\mu)(4h-4),$$

where $\mu \in Q(\tilde{G})$ is a homogeneous quasi-morphism satisfying $\mathfrak{d}(\mu) = c$.

Proof. By Theorem 6.4, there exists a homogeneous quasi-morphism $\mu \in Q(\tilde{G})$ satisfying $\mathfrak{d}(\mu) = [\mathfrak{D}(\mu)] = c$. Since $\pi^*\mathfrak{D}(\mu) = \delta\mu$, we have

$$\|\mathfrak{D}(\mu)\|_\infty = \|\delta\mu\|_\infty = D(\mu).$$

In particular, we have $\|\mathfrak{d}(\mu)\|_\infty \leq D(\mu)$. Let $\rho\colon \pi_1(\Sigma_h) \to G$ be a the holonomy homomorphism associated with the foliated bundle $G \to E \to \Sigma_h$. Since the operator norm of $\rho^*\colon H_b^2(G;\mathbb{R}) \to H_b^2(\pi_1(\Sigma_h);\mathbb{R})$ is less than or equal to 1, we have

$$\|\rho^*(\mathfrak{d}(\mu))\|_\infty \leq \|\mathfrak{d}(\mu)\|_\infty \leq D(\mu).$$

Note that the bounded cohomology of a topological space X is isometrically isomorphic to the bounded cohomology of the fundamental group $\pi_1(X)$ [Gro82]. Hence, we have

$$\|c(E)\|_\infty = \|\rho^*(\mathfrak{d}(\mu))\|_\infty \leq D(\mu).$$

Let $\|\Sigma_h\|$ denote the simplicial volume of Σ_h. Then we have $|\langle c, [\Sigma_h]\rangle| \leq \|c\|_\infty \|\Sigma_h\|$ ([Fri17, Proposition 7.10]). Finally, we obtain the inequality

$$|\langle c(E), [\Sigma_h]\rangle| \leq \|c\|_\infty \|\Sigma_h\| \leq D(\mu)(4h-4).$$

□

Theorem 7.2. *Let G be a topological group and Σ_h be a closed surface of genus $h \geq 1$. Assume that there exists a homogeneous quasi-morphism $\mu \in Q(\tilde{G})$ and $\gamma \in \pi_1(G)$ satisfying $\mu(\gamma) \neq 0$. Then there exist infinitely many isomorphism classes of principal G-bundles over Σ_h which do not admit foliated G-bundle structures.*

Proof. We normalize the homogeneous quasi-morphism μ as $\mu(\gamma) = 1$ by a non-zero constant multiple. If we set $c = \mathfrak{d}(\mu) = B\iota^*((\mu|_{\pi_1(G)})_*\mathfrak{o}) \in H^2(BG^\delta;\mathbb{R})$, then c belongs to $\mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*)$. Assume that a principal G-bundle $E \to \Sigma_h$ admits a foliated structure. Then there exists a continuous map $f_\delta\colon \Sigma_h \to BG^\delta$ such that $f = B\iota \circ f_\delta$, where $f\colon \Sigma_h \to BG$ is the classifying map of E. Let E_δ be the foliated G-bundle on Σ_h induced by f^δ. Then

$$c(E_\delta) = f_\delta^* c = f_\delta^*(B\iota^*\mu_*\mathfrak{o}) = \mu_*(f_\delta^* B\iota^*\mathfrak{o}) = \mu_*(f^*\mathfrak{o}) = \mu_*\mathfrak{o}(E).$$

Hence, Theorem 7.1 gives that

$$\langle (\mu|_{\pi_1(G)})_*\mathfrak{o}(E), [\Sigma_h]\rangle = \langle c(E_\delta), [\Sigma_h]\rangle \leq D(\mu)(4h-4). \tag{7.1}$$

For each $n \in \mathbb{Z}$, we now construct a principal G-bundle E_n over Σ_h whose characteristic number $\langle (\mu|_{\pi_1(G)})_*\mathfrak{o}(E_n), [\Sigma_h]\rangle = n$. Let us fix a triangulation $\mathcal{T}$ of Σ_h and take a triangle $\Delta \in \mathcal{T}$. For $n \in \mathbb{Z}$, we also take a loop $\{g_t\}_{0 \leq t \leq 1}$ in G which represents $\gamma^n \in \pi_1(G)$. Further, let $E \to \Sigma_h \setminus \mathrm{Int}(\Delta)$ and $E' \to \Delta$ be trivial G-bundles, where $\mathrm{Int}(\Delta)$ denotes the interior of Δ. Then we obtain a bundle E_n by gluing the bundles E and E' along $\partial\Delta \approx S^1$ with the transition function $S^1 \to G; t \to g_t$. Since the class $\mathfrak{o}(E_n)$ is the primary obstruction to the cross-sections (see Remark 3.4), we have $\langle \mathfrak{o}(E_n), [\Sigma_h]\rangle = \gamma^n$ and thereby obtain

$$\langle (\mu|_{\pi_1(G)})_*\mathfrak{o}(E_n), [\Sigma_h]\rangle = \mu(\gamma^n) = n.$$

Hence, from equation (7.1), for a sufficiently large n, E_n do not admit foliated G-bundle structures, which completes the proof. □

Remark 7.3. Any non-trivial principal G-bundle over the 2-sphere Σ_0 does not admit foliated structures, since the fundamental group of Σ_0 is trivial. Hence, if the order of $\pi_1(G)$ is infinite, there exist infinitely many isomorphism classes of principal G-bundles over Σ_0 which do not admit foliated structures.

Proof of Corollary 2.2. Ostrover's Calabi quasi-morphism satisfies the assumption in Theorem 7.2, so Theorem 7.2 implies the corollary. □

Remark 7.4. As an application of Theorem 7.1, one can prove Corollary 2.1 (2) by explicitly constructing a foliated H-bundle with an arbitrarily large characteristic number. Indeed, for every $N \in \mathbb{Z} = \pi_1(H)$, there exist $2k$ elements $\tilde{g}_1, \ldots, \tilde{g}_{2k}$ of $\widetilde{H}$ such that the equality $N = [\tilde{g}_1, \tilde{g}_2] \ldots [\tilde{g}_{2k-1}, \tilde{g}_{2k}]$ holds, since $\widetilde{H}$ is a uniformly perfect group. Note that the number k does not depend on N. We set $g_j = p(\tilde{g}_j) \in H$ for every j, where $p \colon \widetilde{H} \to H$ is the universal covering. Let Σ_k be a closed surface of genus k and $a_j \in \pi_1(\Sigma_k)$ be the canonical generators, with the relation

$$[a_1, a_2] \ldots [a_{2k-1}, a_{2k}] = 1.$$

Let $\varphi \colon \pi_1(\Sigma_k) \to \mathrm{Cont}_0(S^3, \xi)$ be the homomorphism defined by $\varphi(a_j) = g_j$ for every j. Then the characteristic number of the foliated H-bundle with the holonomy homomorphism φ is equal to N (this computation of the characteristic number is known as Milnor's algorithm [Mil58]).

8. Non-extendability of homomorphisms on $\pi_1(G)$ to homogeneous quasi-morphisms on $\tilde{G}$

In Section 2.1, we use the homogeneous quasi-morphisms on the universal covering $\tilde{G}$ to show the (un)boundedness of characteristic classes. In this section, in contrast, we use the (un)boundedness of characteristic classes to study the extension problem of homomorphisms on $\pi_1(G)$ to $\tilde{G}$. There have been previous studies on the extension problems of homomorphisms and homogeneous quasi-morphisms (see, for example, [Ish14], [Sht16], [KK19], [KKMM20], [KKMM21] and [Mar22]).

Let $T = S^1 \times S^1$ be the two-dimensional torus and $\mathrm{Homeo}_0(T)$ be the identity component of the homeomorphism group of T with respect to the compact-open topology. In [Ham65], it was shown that the fundamental group $\pi_1(\mathrm{Homeo}_0(T))$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Corollary 8.1. *Any non-trivial homomorphism in* $\mathrm{Hom}(\pi_1(\mathrm{Homeo}_0(T)), \mathbb{R})$ *cannot be extended to* $\widetilde{\mathrm{Homeo}}_0(T)$ *as a homogeneous quasi-morphism.*

Proof. It is enough to show that the equality

$$Q(\widetilde{\mathrm{Homeo}}_0(T)) = \pi^* Q(\mathrm{Homeo}_0(T))$$

holds, where $\pi \colon \widetilde{\mathrm{Homeo}}_0(T) \to \mathrm{Homeo}_0(T)$ is the universal covering. Because $\widetilde{\mathrm{Homeo}}_0(T)$ is a perfect group [KR11], Corollary 1.2 gives that

$$Q(\widetilde{\mathrm{Homeo}}_0(T))/\pi^* Q(\mathrm{Homeo}_0(T)) = \mathrm{Im}(c_G) \cap \mathrm{Im}(B\iota^*).$$

Because all non-zero classes in $\mathrm{Im}(B\iota^*)$ are unbounded [MR18], we have $Q(\widetilde{\mathrm{Homeo}}_0(T))/\pi^* Q(\mathrm{Homeo}_0(T)) = 0$, and the corollary holds. □

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Appendix A. Examples of (contact) Hamiltonian fibrations

Recall that $G_\lambda = \mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$, $H = \mathrm{Cont}_0(S^3, \xi)$, and the cohomology classes

$$\mathfrak{o}_{G_\lambda} \in H^2(BG_\lambda; \mathbb{Z}) \quad \text{and} \quad \mathfrak{o}_H \in H^2(BH; \mathbb{Z})$$

are the primary obstruction classes. Our main concern in this paper (e.g., Corollary 2.1) was the classes $B\iota^*(\mathfrak{o}_{G_\lambda})_\mathbb{R} \in H^2(BG_\lambda^\delta; \mathbb{R})$ and $B\iota^*(\mathfrak{o}_H)_\mathbb{R} \in H^2(BH^\delta; \mathbb{R})$. In this appendix, we instead use the classes $\mathfrak{o}_{G_\lambda}$ and $\mathfrak{o}_H$ to study (not necessarily foliated) Hamiltonian fibrations and contact Hamiltonian fibrations.

Let G and K be topological groups and $i\colon G \to K$ be a continuous homomorphism. Then the map i induces the continuous map $Bi\colon BG \to BK$ between the classifying spaces BG and BK. Let $G \to E \to B$ be a principal G-bundle. Then the map $i\colon G \to K$ induces a principal K-bundle $K \to E_i \to B$. More precisely, the bundle E_i is defined as the principal K-bundle whose classifying map is $Bi \circ f\colon B \to BK$, where $f\colon B \to BG$ is the classifying map of E. We begin with the following general proposition.

Proposition A.1. *Let G and K be topological groups and $i\colon G \to K$ be a continuous homomorphism. Assume that the universal covering $\tilde{G}$ is a perfect group. If there exists a non-trivial element $\tilde{g}$ of $\pi_1(G)$, then there exists a non-trivial principal G-bundle E over Σ_h. If we further assume that $i_*(\tilde{g}) = 0 \in \pi_1(K)$, then there exists a non-trivial principal G-bundle E over Σ_h such that the principal K-bundle E_i induced by $i\colon G \to K$ is trivial.*

Proof. Since $\tilde{G}$ is a perfect group, we can take $\tilde{g}_j \in \tilde{G}$ $(j = 1, \ldots, 2h)$ such that $\tilde{g} = [\tilde{g}_1, \tilde{g}_2] \cdots [\tilde{g}_{2h-1}, \tilde{g}_{2h}]$. Let us define a homomorphism $\rho\colon \pi_1(\Sigma_h) \to G$ by setting

$$\rho(a_j) = \pi(\tilde{g}_j) \tag{A.1}$$

for every j, where $\pi\colon \tilde{G} \to G$ is the universal covering. Then the principal G-bundle $G \to E_\rho \to \Sigma_h$ associated with the holonomy homomorphism ρ is non-trivial (see [Mil58]).

Let $K \to E_{i\circ\rho} \to \Sigma_h$ be the principal K-bundle associated with the holonomy homomorphism $i \circ \rho\colon \pi_1(\Sigma_h) \to K$. Note that the maps $B\rho\colon \Sigma_h \simeq B\pi_1(\Sigma_h) \to BG$ and $Bi \circ B\rho = B(i \circ \rho)\colon \Sigma_h \simeq B\pi_1(\Sigma_h) \to BK$ are the classifying maps of the bundles E_ρ and $E_{i\circ\rho}$, respectively. Hence the bundle $E_{i\circ\rho}$ is the principal K-bundle induced by the map i. We show that the principal K-bundle $E_{i\circ\rho}$ is trivial. By the assumption on $\tilde{g}$, we have

$$0 = i_*(\tilde{g}) = [i_*(\tilde{g}_1), i_*(\tilde{g}_2)] \cdots [i_*(\tilde{g}_{2h-1}), i_*(\tilde{g}_{2h})].$$

If we define $\widetilde{i \circ \rho}\colon \pi_1(\Sigma_h) \to \widetilde{K}$ by

$$\widetilde{i \circ \rho}(a_j) = i_*(\tilde{g}_j),$$

then this map $\widetilde{i \circ \rho}$ is a homomorphism satisfying $\pi \circ (\widetilde{i \circ \rho}) = i \circ \rho$, where $\pi\colon \widetilde{K} \to K$ is the universal covering. Thus, the classifying map $B(i\circ\rho)\colon B\pi_1(\Sigma_h) \to BG \to BK$ factors as

$$B(i \circ \rho) = B\pi \circ B(\widetilde{i \circ \rho})\colon \Sigma_h \simeq B\pi_1(\Sigma_h) \to B\widetilde{K} \to BK.$$

Since the fundamental group and second homotopy group of the classifying space $B\widetilde{K}$ are trivial, the map

$$B(\widetilde{i \circ \rho})\colon \Sigma_h \to B\widetilde{K}$$

is null-homotopic and so is the classifying map $B(i\circ\rho)$ of the bundle $E_{i\circ\rho}$. Thus, the bundle $E_{i\circ\rho}$ is a trivial bundle. $\square$

A.1. **Contact Hamiltonian fibrations.** Let M be a manifold with a contact structure ξ and $\mathrm{Cont}_0(M,\xi)$ be the group of contact Hamiltonian diffeomorphisms, that is, the identity component of the group

$$\mathrm{Cont}(M,\xi) = \{g \in \mathrm{Diff}(M) \mid g^*\xi = \xi\}$$

with the C^∞-topology. A fiber bundle $M \to E \to B$ is called a *contact Hamiltonian fibration* if the structure group reduces to the contact Hamiltonian diffeomorphism group.

The orientation preserving diffeomorphism group $\mathrm{Diff}_+(S^3)$ of the 3-sphere is homotopy equivalent to $SO(4)$ ([Hat83]). Hence, the fundamental group $\pi_1(\mathrm{Diff}_+(S^3))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let ξ be the standard contact structure on the 3-sphere. Then the fundamental group of $\mathrm{Cont}_0(S^3,\xi)$ is isomorphic to $\mathbb{Z}$ ([Eli92], [CS16]). Further, let $i\colon \mathrm{Cont}_0(S^3,\xi) \hookrightarrow \mathrm{Diff}_+(S^3)$ be the inclusion. It follows that the induced map

$$i_*\colon \pi_1(\mathrm{Cont}_0(S^3,\xi)) \cong \mathbb{Z} \to \pi_1(\mathrm{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$$

is surjective ([CS16]). If we let $\tilde{g} \in \pi_1(\mathrm{Cont}_0(S^3,\xi))$ be a non-zero even number in $\mathbb{Z} \cong \pi_1(\mathrm{Cont}_0(S^3,\xi))$, then $i_*(\tilde{g}) = 0 \in \pi_1(\mathrm{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$. Since $\widetilde{\mathrm{Cont}}_0(S^3,\xi)$ is a perfect group ([Ryb10]), Proposition A.1 gives the existence of a non-trivial principal $\mathrm{Cont}_0(S^3,\xi)$-bundle over a closed surface that is trivial as a principal $\mathrm{Diff}_+(S^3)$-bundle. In other words, there exists a sphere bundle that is non-trivial as a contact Hamiltonian fibration but trivial as an oriented sphere bundle.

For a contact Hamiltonian fibration $S^3 \to E \to \Sigma_h$, let $\mathfrak{o}(E) \in H^2(\Sigma_h;\mathbb{Z})$ be the obstruction class and $\chi(E) \in \mathbb{Z}$ denote the characteristic number:

$$\chi(E) = \langle \mathfrak{o}(E), [\Sigma_h]\rangle.$$

Then the following proposition can be shown.

Proposition A.2. *Let $S^3 \to E \to \Sigma_h$ be a foliated contact Hamiltonian fibration. If the characteristic number $\chi(E) \in \mathbb{Z}$ is even, the bundle E is trivial as an oriented sphere bundle. If $\chi(E)$ is odd, the bundle E is non-trivial as an oriented sphere bundle.*

Proof. Let $\rho\colon \pi_1(\Sigma_h) \to \mathrm{Cont}_0(S^3,\xi)$ be the holonomy homomorphism of E. Set $g_j = \psi(a_j)$ and take lifts $\tilde{g}_j \in \widetilde{\mathrm{Cont}}_0(S^3,\xi)$ of the g_j's, where $a_j \in \pi_1(\Sigma_h)$ are the generators. Further, set $\tilde{g} = [\tilde{g}_1,\tilde{g}_2]\cdots[\tilde{g}_{2h-1},\tilde{g}_{2h}]$. Then, by using the algorithm ([Mil58]) to compute the characteristic number of foliated bundles, we have

$$\chi(E) = [\tilde{g}_1,\tilde{g}_2]\cdots[\tilde{g}_{2g-1},\tilde{g}_{2g}] = \tilde{g} \in \mathbb{Z} \cong \pi_1(\mathrm{Cont}_0(S^3,\xi)).$$

If $\chi(E)$ is even, we have $i_*(\tilde{g}) = 0 \in \pi_1(\mathrm{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$. Thus, the bundle E is trivial as an oriented sphere bundle by the same arguments as in Proposition A.1. If $\chi(E)$ is odd, we have $i_*(\tilde{g}) = 1 \in \pi_1(\mathrm{Diff}_+(S^3)) \cong \mathbb{Z}/2\mathbb{Z}$. Since the characteristic number of E is non-zero, the bundle E is non-trivial as an oriented sphere bundle. $\square$

A.2. **Hamiltonian fibrations.** Let M be a manifold with a symplectic form ω. A fiber bundle $M \to E \to B$ is called a *Hamiltonian fibration* if the structure group reduces to the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms.

Let us consider the symplectic manifold $(S^2\times S^2,\omega_\lambda)$ for $\lambda > 1$. We take a non-zero element $\tilde{g}$ of $\pi_1(\mathrm{Ham}(S^2\times S^2,\omega_\lambda))$ such that the value $\mu^\lambda(\tilde{g})$ is non-zero (by Proposition 3.2). Since $\widetilde{\mathrm{Ham}}(S^2\times S^2,\omega_\lambda))$ is a perfect group, there exist

$\tilde{g}_1, \cdots, \tilde{g}_{2h} \in \widetilde{\mathrm{Ham}}(S^2 \times S^2, \omega_\lambda))$ such that $\tilde{g} = [\tilde{g}_1, \tilde{g}_2] \cdots [\tilde{g}_{2h-1}, \tilde{g}_{2h}]$. We define a homomorphism $\rho\colon \pi_1(\Sigma_h) \to \mathrm{Ham}(S^2 \times S^2, \omega_\lambda))$ by

$$\rho(a_j) = \pi(\tilde{g}_j)$$

for every $j = 1, \cdots, 2h$, where $a_j \in \pi_1(\Sigma_h)$ are the generators and $\pi\colon \widetilde{\mathrm{Ham}}(S^2 \times S^2, \omega_\lambda)) \to \mathrm{Ham}(S^2 \times S^2, \omega_\lambda))$ is the covering map. Let $p\colon E_\rho \to \Sigma_h$ be the Hamiltonian fibration whose holonomy homomorphism is ρ. Note that the Hamiltonian fibration E_ρ is non-trivial since $\tilde{g}$ is a non-zero element.

We can prove that the Hamiltonian fibration $p\colon E_\rho \to \Sigma_h$ is stably non-trivial in the following sense.

Proposition A.3. *Let $p\colon E_\rho \to \Sigma_h$ be the Hamiltonian fibration defined as above. Further, let (N, ω_N) be a closed symplectic manifold and $q\colon \epsilon_N = \Sigma_h \times N \to \Sigma_h$ be the trivial N-bundle. Then the fiberwise product*

$$E_\rho \times \epsilon_N \to \Sigma_h$$

is non-trivial as a Hamiltonian fibration.

To prove Proposition A.3, we use the following theorem essentially proved by Entov and Polterovich.

Theorem A.4 (Theorem 5.1 of [EP09])**.** *Let (N, ω_N) be a closed symplectic manifold. For $\lambda \geq 1$, we let $\omega_{\lambda,N}$ denote the symplectic form $\mathrm{pr}_1^*\omega_\lambda + \mathrm{pr}_2^*\omega_N$ where $\mathrm{pr}_1\colon S^2 \times S^2 \times N \to S^2 \times S^2$ and $\mathrm{pr}_2\colon S^2 \times S^2 \times N \to N$ are the first and second projections, respectively. Then there exists a function $\mu^{\lambda,N}\colon \widetilde{\mathrm{Ham}}(S^2 \times S^2 \times N, \omega_{\lambda,N}) \to \mathbb{R}$ such that*

$$\mu^{\lambda,N}(\tilde{\phi}_N) = \mu^\lambda(\tilde{\phi})$$

for every $\tilde{\phi} \in \widetilde{\mathrm{Ham}}(S^2 \times S^2, \omega_{\lambda,N})$.

Here, $\tilde{\phi}_N$ is the element of $\widetilde{\mathrm{Ham}}(S^2 \times S^2 \times N, \omega_{\lambda,N})$ represented by the path $\{\phi_N^t\}_{t\in[0,1]}$ defined by $\phi_N^t(x, y) = (\phi^t(x), y)$, where $\{\phi^t\}_{t\in[0,1]}$ is a path in $\mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$ representing $\tilde{\phi}$.

Remark A.5. The function $\mu^{\lambda,N}$ satisfies the conditions of a "partial Calabi quasi-morphism" ([Ent14, Theorem 3.2]). In particular, $\mu^{\lambda,N}$ maps the unit element to 0. However, the authors do not know whether the restriction of $\mu^{\lambda,N}$ to the fundamental group is a homomorphism.

Proof of Proposition A.3. Let $\{\tilde{g}^t\}_{t\in[0,1]}$ be a path in $\mathrm{Ham}(S^2 \times S^2, \omega_\lambda)$ which represents $\tilde{g} \in \pi_1(\mathrm{Ham}(S^2 \times S^2, \omega_\lambda))$, and define a loop $\tilde{g}_N = \{\tilde{g}_N^t\}_{t\in[0,1]}$ in $\mathrm{Ham}(S^2 \times S^2 \times N, \omega_{\lambda,N})$ by $\tilde{g}_N^t(x, y) = (g^t(x), y)$. Then, by Theorem A.4, we have that $\mu^{\lambda,N}(\tilde{g}_N) = \mu^\lambda(\tilde{g}) \neq 0$. Together with Remark A.5, we have that $\tilde{g}_N$ is a non-trivial element of $\pi_1\left(\mathrm{Ham}(S^2 \times S^2 \times N, \omega_{\lambda,N})\right)$. Then Proposition A.1 proves the proposition. □

References

[Anj02] Sílvia Anjos, *Homotopy type of symplectomorphism groups of $S^2 \times S^2$*, Geom. Topol. **6** (2002), 195–218.

[Ban78] Augustin Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. **53** (1978), no. 2, 174–227.

[Ban97] ———, *The structure of classical diffeomorphism groups*, Mathematics and its Applications, vol. 400, Kluwer Academic Publishers Group, Dordrecht, 1997.

[Ben91] D. J. Benson, *Representations and cohomology*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press, 1991.

[BG92] J. Barge and É. Ghys, *Cocycles d'Euler et de Maslov*, Math. Ann. **294** (1992), no. 2, 235–265.

[Bor13] Matthew Strom Borman, *Quasi-states, quasi-morphisms, and the moment map*, Int. Math. Res. Not. IMRN (2013), no. 11, 2497–2533.

[Bou95] Abdessalam Bouarich, *Suites exactes en cohomologie bornée réelle des groupes discrets*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 11, 1355–1359.

[Bra15] Michael Brandenbursky, *Bi-invariant metrics and quasi-morphisms on groups of Hamiltonian diffeomorphisms of surfaces*, Internat. J. Math. **26** (2015), no. 9, 1550066, 29.

[BZ15] Matthew Strom Borman and Frol Zapolsky, *Quasimorphisms on contactomorphism groups and contact rigidity*, Geom. Topol. **19** (2015), no. 1, 365–411.

[Cal04] Danny Calegari, *Circular groups, planar groups, and the Euler class*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 431–491.

[Cal09] ———, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009.

[Cas17] Alexander Caviedes Castro, *Calabi quasimorphisms for monotone coadjoint orbits*, J. Topol. Anal. **9** (2017), no. 4, 689–706.

[CMPSC11] Indira Chatterji, Guido Mislin, Christophe Pittet, and Laurent Saloff-Coste, *A geometric criterion for the boundedness of characteristic classes*, Math. Ann. **351** (2011), no. 3, 541–569.

[CS16] Roger Casals and Oldřich Spáčil, *Chern-Weil theory and the group of strict contactomorphisms*, J. Topol. Anal. **8** (2016), no. 1, 59–87.

[Dup78] Johan L. Dupont, *Curvature and characteristic classes*, Lecture Notes in Mathematics, Vol. 640, Springer-Verlag, Berlin-New York, 1978.

[Eli92] Yakov Eliashberg, *Contact* 3*-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 1-2, 165–192.

[Ent14] Michael Entov, *Quasi-morphisms and quasi-states in symplectic topology*, Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 1147–1171.

[EP03] Michael Entov and Leonid Polterovich, *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. (2003), no. 30, 1635–1676.

[EP09] ———, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), no. 3, 773–826.

[FOOO19] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono, *Spectral invariants with bulk, quasi-morphisms and Lagrangian Floer theory*, Mem. Amer. Math. Soc. **260** (2019), no. 1254, x+266.

[FPR18] Maia Fraser, Leonid Polterovich, and Daniel Rosen, *On Sandon-type metrics for contactomorphism groups*, Ann. Math. Qué. **42** (2018), no. 2, 191–214.

[Fri17] Roberto Frigerio, *Bounded cohomology of discrete groups*, Mathematical Surveys and Monographs, vol. 227, American Mathematical Society, Providence, RI, 2017.

[Gei08] Hansjörg Geiges, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics, vol. 109, Cambridge University Press, Cambridge, 2008.

[GG04] Jean-Marc Gambaudo and Étienne Ghys, *Commutators and diffeomorphisms of surfaces*, Ergodic Theory Dynam. Systems **24** (2004), no. 5, 1591–1617.

[Ghy01] Étienne Ghys, *Groups acting on the circle*, Enseign. Math. (2) **47** (2001), no. 3-4, 329–407.

[Giv90] A. B. Givental, *Nonlinear generalization of the Maslov index*, Theory of singularities and its applications, Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 71–103.

[GKT11] Światosł aw Gal, Jarek Kędra, and Aleksy Tralle, *On the algebraic independence of Hamiltonian characteristic classes*, J. Symplectic Geom. **9** (2011), no. 1, 1–9.

[Gro82] Michael Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. (1982), no. 56, 5–99 (1983).

[Ham65] Mary-Elizabeth Hamstrom, *The space of homeomorphisms on a torus*, Illinois J. Math. **9** (1965), 59–65.

[Hat83] Allen E. Hatcher, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. (2) **117** (1983), no. 3, 553–607.

[Ish14] Tomohiko Ishida, *Quasi-morphisms on the group of area-preserving diffeomorphisms of the 2-disk via braid groups*, Proc. Amer. Math. Soc. Ser. B **1** (2014), 43–51.

[JK02] Tadeusz Januszkiewicz and Jarek Kędra, *Characteristic classes of smooth fibrations*, 2002.

[KK19] Morimichi Kawasaki and Mitsuaki Kimura, *$\hat{G}$-invariant quasimorphisms and symplectic geometry of surfaces*, to appear in *Israel J. Math*, arXiv:1911.10855v2 (2019).

[KKMM20] Morimichi Kawasaki, Mitsuaki Kimura, Takahiro Matsushita, and Masato Mimura, *Bavard's duality theorem for mixed commutator length*, arXiv:2007.02257v3, to appear in Enseign. Math. (2020).

[KKMM21] Morimichi Kawasaki, Mitsuaki Kimura, Takahiro Matsushita, and Masato Mimura, *Commuting symplectomorphisms on a surface and the flux homomorphism*, preprint, arXiv:2102.12161v1 (2021).

[KR11] Agnieszka Kowalik and Tomasz Rybicki, *On the homeomorphism groups of manifolds and their universal coverings*, Cent. Eur. J. Math. **9** (2011), no. 6, 1217–1231.

[Man20] Kathryn Mann, *Unboundedness of some higher Euler classes*, Algebr. Geom. Topol. **20** (2020), no. 3, 1221–1234.

[Mar20] Shuhei Maruyama, *The dixmier-douady class, the action homomorphism, and group cocycles on the symplectomorphism group*, 2020.

[Mar22] ———, *Extensions of quasi-morphisms to the symplectomorphism group of the disk*, Topology Appl. **305** (2022), Paper No. 107880.

[McD04] Dusa McDuff, *Lectures on groups of symplectomorphisms*, Rend. Circ. Mat. Palermo (2) Suppl. (2004), no. 72, 43–78.

[McD10] ———, *Monodromy in Hamiltonian Floer theory*, Comment. Math. Helv. **85** (2010), no. 1, 95–133.

[Mil58] John Milnor, *On the existence of a connection with curvature zero*, Comment. Math. Helv. **32** (1958), 215–223.

[MN21] Nicolas Monod and Sam Nariman, *On the bounded cohomology of certain homeomorphism groups*, preprint, arXiv:2111.04365 (2021).

[MR18] Kathryn Mann and Christian Rosendal, *Large-scale geometry of homeomorphism groups*, Ergodic Theory Dynam. Systems **38** (2018), no. 7, 2748–2779.

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008.

[Oh05] Yong-Geun Oh, *Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds*, The breadth of symplectic and Poisson geometry, Progr. Math., vol. 232, Birkhäuser Boston, Boston, MA, 2005, pp. 525–570.

[Ost06] Yaron Ostrover, *Calabi quasi-morphisms for some non-monotone symplectic manifolds*, Algebr. Geom. Topol. **6** (2006), 405–434.

[OT09] Yaron Ostrover and Ilya Tyomkin, *On the quantum homology algebra of toric Fano manifolds*, Selecta Math. (N.S.) **15** (2009), no. 1, 121–149.

[Pon86] L. S. Pontryagin, *Selected works vol. 2, topological groups*, Translated from the Russian and with a preface by A. Brown. With additional material translated by P. S. V. Naidu., Gordon and Breach Science Publishers, Inc., New York-London-Paris, 1986.

[PR14] Leonid Polterovich and Daniel Rosen, *Function theory on symplectic manifolds*, CRM Monograph Series, vol. 34, American Mathematical Society, Providence, RI, 2014.

[Py06a] Pierre Py, *Quasi-morphismes de Calabi et graphe de Reeb sur le tore*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 5, 323–328.

[Py06b] ———, *Quasi-morphismes et invariant de Calabi*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 1, 177–195.

[Rez97] Alexander G. Reznikov, *Characteristic classes in symplectic topology*, Selecta Math. (N.S.) **3** (1997), no. 4, 601–642, Appendix D by Ludmil Katzarkov.

[Ryb10] Tomasz Rybicki, *Commutators of contactomorphisms*, Adv. Math. **225** (2010), no. 6, 3291–3326.

[Sch00] Matthias Schwarz, *On the action spectrum for closed symplectically aspherical manifolds*, Pacific J. Math. **193** (2000), no. 2, 419–461.

[She14] Egor Shelukhin, *The action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures*, Comment. Math. Helv. **89** (2014), no. 1, 69–123.

[Sht16] A. I. Shtern, *Extension of pseudocharacters from normal subgroups, III*, Proc. Jangjeon Math. Soc. **19** (2016), no. 4, 609–614.

[Sim07] Gabi Ben Simon, *The nonlinear Maslov index and the Calabi homomorphism*, Commun. Contemp. Math. **9** (2007), no. 6, 769–780. MR 2372458

[SS20] Yasha Savelyev and Egor Shelukhin, *K-theoretic invariants of Hamiltonian fibrations*, J. Symplectic Geom. **18** (2020), no. 1, 251–289.

[Thu72] William Thurston, *Noncobordant foliations of S^3*, Bull. Amer. Math. Soc. **78** (1972), 511–514.

[Thu74] ———, *Foliations and groups of diffeomorphisms*, Bull. Amer. Math. Soc. **80** (1974), 304–307.

[Ush11] Michael Usher, *Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasimorphisms*, Geom. Topol. **15** (2011), no. 3, 1313–1417.

[Via18] Renato Vianna, *Continuum families of non-displaceable Lagrangian tori in* $(\mathbb{C}P^1)^{2m}$, J. Symplectic Geom. **16** (2018), no. 3, 857–883.

[Whi78] George W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York-Berlin, 1978.

[Woo71] John W. Wood, *Bundles with totally disconnected structure group*, Comment. Math. Helv. **46** (1971), 257–273.

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