PSEUDO-SYMMETRIC ALMOST KENMOTSU 3-MANIFOLDS

JUN-ICHI INOGUCHI AND JI-EUN LEE

Abstract. We study semi-symmetry and pseudo-symmetry of almost Kenmotsu 3-manifolds. We prove that non-locally symmetric pseudo-symmetric H-almost Kenmotsu 3-manifolds are certain generalized almost Kenmotsu (κ, μ, ν)-spaces.

1. Introduction

The class of Riemannian symmetric spaces constitues an extremely magnificent class of Riemannian manifolds. Riemannian symmetric spaces have parallel Riemannian curvature. Namely, the Riemannian curvature is invariant under parallel translations. A Riemannian manifold $M = (M, g)$ is said to be *locally symmetric* if its Riemannian curvature R is parallel.

A Riemannian manifold M is of constant curvature c if and only if its Riemannian curvature R is of the form $R = cR_1$, where R_1 is a curvature-like tensor field (see Definition 2.1) defined by

$$R_1(X, Y)Z = (X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y. \tag{1}$$

The curvature formula $R = cR_1$ implies that a Riemannian manifold of constant curvature c is locally symmetric.

More generally, a Riemannian manifold M is said to be *semi-symmetric* if its Riemannian curvature R satisfies $R \cdot R = 0$. Here $R \cdot R$ is the derivation of R by itself (see Section 2.1). Obviously, the semi-symmetry is a generalization of local symmetry.

The semi-symmetry condition $R \cdot R = 0$ for Riemannian manifolds was recognized by Cartan [3, p. 265]. Note that every Riemannian 2-manifold is semi-symmetric. At the time of Cartan, the only known examples of semi-symmetric spaces are locally symmetric spaces and Riemannian 2-manifolds. The name "semi-symmetric space" was introduced by Sinjukov [26]. Szabó classified the local structure of semi-symmetric spaces [27]. Kowalski made a systematic study of foliated semi-symmetric 3-manifolds [16]. For more information on semi-symmetric spaces, we refer to [2].

The notion of pseudo-symmetry was introduced by Deszcz [5] as follows (see Definition 2.3): A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if its Riemannian curvature R satisfies $R \cdot R = L\, R_1 \cdot R$ for some smooth function L. Since then pseudo-symmetric Riemannian manifolds have been studied extensively. For the extrinsic analogs of pseudo-symmetry, we refer to [8].

Let us turn our attention to almost contact metric geometry. Contact metric manifolds are modeled on the odd-dimensional unit sphere $\mathbb{S}^{2n+1}(1)$ (Sasakian manifolds) and unit tangent sphere bundles. As a counterpart of the class of contact metric manifolds, the notion of almost

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Kenmotsu manifold was introduced by Janssens and Vanhecke [14]. An almost Kenmotsu manifold is said to be *Kenmotsu* if it is normal. In the classification of almost contact metric manifolds with automorphism group of maximal dimension due to Tanno [28], the warped product model of the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ equipped with a compatible homogeneous almost contact structure was discovered. Kenmotsu [15] clarified the almost contact structure of $\mathbb{H}^{2n+1}$ and introduced a class of almost contact metric manifold, nowadays called the class of *Kenmotsu manifold*.

In [7], Dileo and Pastore proposed the following question:

> *Is a locally symmetric almost Kenmotsu manifold either Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$?*

As a partial affirmative answer, they proved that locally symmetric almost Kenmotsu manifolds of dimension greater than 3 satisfying $R(X, Y)\xi = 0$ for all vector fields X and Y orthogonal to ξ are Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Here ξ is the characteristic vector field (see Section 3.1).

Next, Wang and Liu [30] proved that locally symmetric CR-integrable almost Kenmotsu manifolds of dimension greater than 3 are Kenmotsu of constant curvature -1 or locally isometric to the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Concerning on the question posed by Dileo and Pastore, the classification of locally symmetric almost Kenmotsu 3-manifold contains peculiar examples. Indeed, the classification result is stated as follows (see [11]):

Simply connected and complete locally symmetric almost Kenmotsu 3-manifolds are

- The hyperbolic 3-space $\mathbb{H}^3(-1)$ equipped with a homogeneous Kenmotsu structure.
- The Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$ equipped with a standard homogeneous Kenmotsu structure.
- The Riemannian product $\mathbb{H}^2(-4 - \gamma^2) \times \mathbb{R}$ equipped with a standard homogeneous Kenmotsu structure, where $\gamma \neq 0$.

This classification gives a negative answer to the question posed by Dileo and Pastore in dimension 3. It should be remarked that all of these spaces are realized as 3-dimensional non-unimodular Lie groups equipped with left invariant almost Kenmotsu structure.

These results motivate us to study pseudo-symmetry of almost Kenmotsu 3-manifolds.

In this paper, we study the semi-symmetry and pseudo-symmetry of almost Kenmotsu 3-manifolds. This paper is organized as follows. After recalling fundamental theory of pseudo-symmetry and almost Kenmotsu geometry in Sections 2, 3 and 4, we start our investigation on almost Kenmotsu 3-manifolds from Section 5. Perrone proved that every homogeneous almost Kenmotsu 3-manifold is a Lie group equipped with a left invariant almost Kenmotsu structure. According to the classification of homogeneous almost Kenmotsu 3-manifolds due to Perrone [21], there are two classes of homogeneous almost Kenmotsu Lie groups named as Type II Lie groups and Type IV Lie groups (see Example 5.2 and Example 5.6). First we prove that the semi-symmetry of a homogeneous almost Kenmotsu 3-manifold implies the local symmetry (Theorem 5.8). We give a complete classification of pseudo-symmetric homogeneous almost Kenmotsu 3-manifolds (Theorem 5.10).

In Section 5.4 we give some explicit examples of non-homogeneous pseudo-symmetric almost Kenmotsu 3-manifolds.

In Section 6, we study semi-symmetry and pseudo-symmetry of almost Kenmotsu 3-manifolds. However the full classification of those manifolds are far from the complete one. We focus on the property that the characteristic vector field of a Kenmotsu manifold is an eigenvector field of the Ricci operator. An almost Kenmotsu manifold is said to be *H-almost Kenmotsu* if its characteristic vector field of a Kenmotsu manifold is an eigenvector field of the Ricci operator (Definition 3.5 and Proposition 3.6). In subsection 6.4, we prove that proper semi-symmetric H-almost Kenmotsu 3-manifolds are certain generalized almost Kenmotsu (κ, μ, ν)-spaces (Theorem 6.6).

In subsection 6.5, we prove our main theorem (Theorem 6.9) which gives a partial classification of proper pseudo-symmetric almost Kenmotsu 3-manifolds. More precisely, we prove that proper pseudo-symmetric H-almost Kenmotsu 3-manifolds are certain generalized almost Kenmotsu (κ, μ, ν)-spaces.

2. Pseudo-symmetric spaces

2.1. **Semi-symmetric spaces.** Let $M = (M, g)$ be a Riemannian manifold with Levi-Civita connection ∇. We use the sign convention

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for the Riemannian curvature. We denote by ρ, S and r the Ricci tensor field, Ricci operator and the scalar curvature of M, respectively. Here we recall the notion of curvature-like tensor field (see *e.g.*, [17]).

Definition 2.1. A tensor field F of type $(1, 3)$ on a Riemannian manifold (M, g) is said to be *curvature-like* if it has all the properties of the Riemannian curvature R, that is, it satisfies:

(1) $F(X, Y)Z = -F(Y, X)Z$,
(2) $g(F(X, Y)Z, W) = -g(Z, F(X, Y)W)$,
(3) $g(F(X, Y)Z, W) = g(F(Z, W)X, Y)$,
(4) $F(X, Y)Z + F(Y, Z)X + F(Z, X)Y = 0$,
(5) $(\nabla_X F)(Y, Z)W + (\nabla_Y F)(Z, X)W + (\nabla_Z F)(X, Y)W = 0$

for all vector fields X, Y, Z, W on M.

Every curvature-like tensor field F acts on tensor fields as a derivation. For instance, the derivative $F \cdot R$ of the Riemannian curvature R by F is given by

$$\begin{aligned}(F \cdot R)(U, V, W; Y, X) = & \ (F(X, Y)R)(U, V, W) \\ = & \ F(X, Y)R(U, V)W - R(F(X, Y)U, V)W \\ & - R(U, F(X, Y)V)W - R(U, V)F(X, Y)W.\end{aligned}$$

A Riemannian manifold (M, g) is said to be *semi-symmetric* if $R \cdot R = 0$.

Definition 2.2. Let (M, g) be a Riemannian manifold. The *nullity vector space* of the Riemannian curvature R at a point $p \in M$ is a linear subspace

$$T_p^0 M = \{X \in T_p M \mid R(X, Y)Z = 0, \ \forall Y, Z \in T_p M \}$$

of $T_p M$. The *index of nullity at p* is $n_M(p) := \dim T_p^0 M$. The *index of conullity at p* is $u_M(p) := \dim M - n_M(p)$.

Now let (M^3, g) be a Riemannian 3-manifold with principal Ricci curvatures ρ_1, ρ_2 and ρ_3. Then the nullity vector space of R is rewritten as

$$T_p^0 M = \{X \in T_p M \mid S_p X = 0\}.$$

A Riemannian 3-manifold M is is semi-symmetric if and only if [25]:

$$(\rho_i - \rho_j)\{2(\rho_i + \rho_j) - r\} = 0, \quad i, j = 1, 2, 3,\ i \neq j.$$

Thus at each point $p \in M$, there are three possibilities

(1) $\rho_1 = \rho_2 = \rho_3 \neq 0$.
(2) $\rho_1 = \rho_2 \neq 0$ and $\rho_3 = 0$ up to numeration.
(3) $\rho_1 = \rho_2 = \rho_3 = 0$.

If $\rho_1 = \rho_2 = \rho_3 \neq 0$ at p, then this relation holds on a neighborhood U of p. On U, all of the principal Ricci curvatures are non-zero constants, hence U is of constant nonzero curvature. In particular, U is locally symmetric.

2.2. **Pseudo-symmetric spaces.** As a generalization of semi-symmetry, the notion of pseudo-symmetry was introduced by Deszcz.

Definition 2.3. A Riemannian manifold (M, g) is said to be *pseudo-symmetric* if there exists a function L such that $R \cdot R = LR_1 \cdot R$, where R_1 is the curvature-like tensor field defined by (1). The function L is called the *scale factor* of M.

Obviously, semi-symmetric spaces are pseudo-symmetric (with scale factor 0). When L is constant, then M is said to be of *constant type*. A pseudo-symmetric space is said to be *proper* if it is not semi-symmetric.

In 3-dimension, the following fundamental fact is well-known (see *e.g.*, [6]).

Proposition 2.4. *For a Riemannian* 3*-manifold M, the following properties are mutually equivalent*:

(1) *M is pseudo-symmetric with scale factor L. that is, there exists a function L such that $R \cdot R = L\, R_1 \cdot R$.*
(2) *there exists a function L such that $R \cdot S = LR_1 \cdot S$.*
(3) *the principal Ricci curvatures satisfy $\rho_1 = \rho_2$, and $\rho_3 = 2L$ up to numeration.*

3. Almost contact metric manifolds

In this section, we recall fundamental ingredients of almost contact metric geometry. For general information on almost contact metric geometry, we refer to Blair's monograph [1].

3.1. **Almost contact structures.** An *almost contact metric structure* of a $(2n+1)$-manifold M is a quadruple (φ, ξ, η, g) of structure tensor fields that satisfy:

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0,$$

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A $(2n+1)$-manifold $M = (M, \varphi, \xi, \eta, g)$ equipped with an almost contact metric structure is called an *almost contact metric manifold*. The vector field ξ is called the *characteristic vector field* of M. The 2-form

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the *fundamental* 2*-form* of M.

Here we recall auxiliary tensor fields h and h' which are very useful for the study of almost contact metric manifolds. The endomorphism fields h and h' are defined by $h = \pounds_\xi \varphi/2$ and $h' = h\varphi$, respectively. Here $\pounds_\xi$ denotes the Lie differentiation by ξ.

Definition 3.1. Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. A tangent plane Π_p at $p \in M$ is said to be *holomorphic* if it is invariant under φ_p.

It is easy to see that a tangent plane Π_p is holomorphic if and only if ξ_p is orthogonal to Π_p. The sectional curvature $K(\Pi_p)$ of a holomorphic plane Π_p is called the *holomorphic sectional curvature* (also called φ-sectional curvature) of Π_p. In case $\dim M = 3$, the holomorphic sectional curvature $K(\Pi_p)$ is denoted by H_p and called the *holomorphic sectional curvature* at p.

On an almost contact metric manifold M, we define a torsion tensor field N by

$$N(X, Y) := [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi, \quad X, Y \in \mathfrak{X}(M).$$

Here $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. An almost contact metric manifold M is said to be *normal* if $N(X, Y) = 0$ for all X, $Y \in \varGamma(TM)$.

3.2. **Almost Kenmotsu structure.**

Definition 3.2 ([14])**.** An almost contact metric manifold M is said to be *almost Kenmotsu* if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. An almost Kenmotsu manifold is said to be *Kenmotsu* if it is normal.

It should be remarked that every almost Kenmotsu $(2n+1)$-manifold satisfies $\operatorname{div} \xi = 2n$. Hence almost Kenmotsu manifolds can not be compact.

Proposition 3.3 ([7])**.** *If an almost Kenmotsu manifold is of constant curvature, then it is a Kenmotsu manifold of constant curvature* -1.

In the class of Kenmotsu manifolds, constancy of holomorphic sectional curvature is a too strong restriction. In fact, Kenmotsu [15] showed

Proposition 3.4. *Let M be a Kenmotsu manifold of dimension greater than* 3*. Then M is of constant holomorphic sectional curvature if and only if it is of constant curvature* -1.

Kenmotsu 3-manifolds of constant holomorphic sectional curvature are of constant sectional curvature -1 (see Proposition 4.2 below).

To close this subsection we introduce the following notion [12, 13]:

Definition 3.5. An almost Kenmotsu manifold whose characteristic vector field ξ is a harmonic unit vector field is called an *H-almost Kenmotsu manifold*.

Perrone showed the following fundamental fact ([22, Theorem 4.1], [21, Proposition 7]).

Proposition 3.6. *An almost Kenmotsu manifold M is H-almost Kenmotsu if and only if ξ is an eigenvector field of S.*

4. Kenmotsu 3-manifolds

Here we recall curvature properties of *Kenmotsu* 3*-manifolds*.

Proposition 4.1 ([9, 18])**.** *The Riemannian curvature R of a Kenmotsu 3-manifold M has the form*

$$R(X,Y)Z = \frac{r+4}{2}(X\wedge Y)Z + \frac{r+6}{2}[\xi \wedge \{(X\wedge Y)\xi\}]Z. \tag{2}$$

Here r is the scalar curvature. The Ricci operator S has the form

$$S = \frac{r+2}{2}\,\mathrm{I} - \frac{r+6}{2}\,\eta\otimes\xi.$$

The principal Ricci curvatures are $(r+2)/2$, $(r+2)/2$ and -2. The Ricci operator S commutes with φ. For a unit vector $X \in TM$ orthogonal to ξ, the sectional curvatures of planes $X\wedge\varphi X$ and $X\wedge\xi$ are given by

$$H = K(X\wedge\varphi X) = \frac{r}{2}+2, \quad K(X\wedge\xi) = -1.$$

Note that every Kenmotsu 3-manifold is H-almost Kenmotsu. Thus the notion of H-almost Kenmotsu is intermediate notion between Kenmotsu and almost Kenmotsu.

The local symmetry and Einstein conditions for Kenmotsu 3-manifolds are described as follows:

Proposition 4.2 ([9, 10])**.** (1) *Every Kenmotsu 3-manifold is a pseudo-symmetric space of constant type.*

(2) *The following properties of a Kenmotsu 3-manifold M are mutually equivalent.*
- *M is semi-symmetric.*
- *M is locally symmetric.*
- *M is η-Einstein, that is, $S = A\,\mathrm{I} + B\eta\otimes\xi$ for some constants A and B.*
- *the scalar curvature r is constant.*
- *the holomorphic sectional curvature H is constant.*
- *M is locally isomorphic to the hyperbolic 3-space $\mathbb{H}^3(-1)$ of curvature -1.*

5. Almost Kenmotsu 3-manifolds

5.1. **Fundamental formulas.** Let M be an almost Kenmotsu 3-manifold. Denote by $\mathcal{U}_1$ the open subset of M consisting of points p such that $h \neq 0$ around p. Next let $\mathcal{U}_0$ the open subset of M consisting of points $p \in M$ such that $h = 0$ around p. Since h is smooth, $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_0$ is an open dense subset of M. So any property satisfied in $\mathcal{U}$ is also satisfied in whole M. For any point $x \in \mathcal{U}$, there exits a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ around p, where e_1 is an eigenvector field of h.

Lemma 5.1 (*cf.* [4])**.** *Let M be an almost Kenmotsu 3-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ on $\mathcal{U}$ such that*

$$he_1 = \lambda e_1, \quad e_2 = \varphi e_1, \quad e_3 = \xi$$

for some locally defined smooth function λ. The Levi-Civita connection ∇ is described as

$$\begin{gathered}\nabla_{e_1}e_1 = -be_2 - \xi, \quad \nabla_{e_1}e_2 = be_1 + \lambda\xi, \quad \nabla_{e_1}e_3 = e_1 - \lambda e_2,\\ \nabla_{e_2}e_1 = ce_2 + \lambda\xi, \quad \nabla_{e_2}e_2 = -ce_1 - \xi, \quad \nabla_{e_2}e_3 = -\lambda e_1 + e_2,\\ \nabla_{e_3}e_1 = \alpha e_2, \quad \nabla_{e_3}e_2 = -\alpha e_1, \quad \nabla_{e_3}e_3 = 0,\end{gathered} \tag{3}$$

where

$$b = -\frac{1}{2\lambda}(e_2(\lambda)+\sigma(e_1)), \quad c = -\frac{1}{2\lambda}(e_1(\lambda)+\sigma(e_2)),$$

and σ is the 1-form metrically equivalent to $S\xi$, that is,

$$\sigma = g(S\xi, \cdot) = S(\xi, \cdot).$$

The covariant derivative $\nabla_\xi h$ of h by ξ is given by

$$\nabla_\xi h = -2\alpha h\varphi + \frac{\xi(\lambda)}{\lambda} h,$$

for $h \neq 0$ on the open subset $\mathcal{U}$.

The commutation relations are

$$[e_1, e_2] = be_1 - ce_2, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2.$$

The Jacobi identity is described as

$$e_1(\alpha - \lambda) + \xi(b) + c(\alpha - \lambda) + b = 0, \quad e_2(\alpha + \lambda) - \xi(c) + b(\alpha + \lambda) - c = 0.$$

Remark 1. On an almost Kenmotsu 3-manifold M with $h \neq 0$, the function $\alpha = g(\nabla_\xi W, \varphi W)$ is independent of the choice of unit eigenvector W of h.

The Riemannian curvature R is computed by the table of Levi-Civita connection in Lemma 5.1:

$$\begin{aligned}
&R(e_1, e_2)e_1 = -He_2 - \sigma(e_2)\xi, \quad R(e_1, e_2)e_2 = He_1 + \sigma(e_1)\xi,\\
&R(e_1, e_2)e_3 = \sigma(e_2)e_1 - \sigma(e_1)e_2, \qquad R(e_2, e_3)e_1 = \sigma(e_1)e_2 - \{\xi(\lambda) + 2\lambda\}\,\xi,\\
&R(e_2, e_3)e_2 = -\sigma(e_1)e_1 - K_{23}\xi, \qquad R(e_2, e_3)e_3 = \{\xi(\lambda) + 2\lambda\}\,e_1 + K_{23}e_2,\\
&R(e_3, e_1)e_1 = \sigma(e_2)e_2 + K_{13}\xi, \qquad R(e_3, e_1)e_2 = -\sigma(e_2)e_1 + \{\xi(\lambda) + 2\lambda\}\,\xi\\
&R(e_3, e_1)e_3 = -K_{13}e_1 - \{\xi(\lambda) + 2\lambda\}\,e_2.
\end{aligned}$$

Here the sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ are given by

$$H = K_{12} = \frac{r}{2} + 2(\lambda^2 + 1), \quad K_{13} = -(\lambda^2 + 2\alpha\lambda + 1), \quad K_{23} = -(\lambda^2 - 2\alpha\lambda + 1).$$

Next, the Ricci operator S of an almost Kenmotsu 3-manifold M is described as :

$$\begin{aligned}
Se_1 &= \left(\frac{r}{2} + \lambda^2 - 2\alpha\lambda + 1\right) e_1 + \{\xi(\lambda) + 2\lambda\}e_2 + \sigma(e_1)\xi,\\
Se_2 &= \{\xi(\lambda) + 2\lambda\}e_1 + \left(\frac{r}{2} + \lambda^2 + 2\alpha\lambda + 1\right) e_2 + \sigma(e_2)\xi,\\
Se_3 &= \sigma(e_1)e_1 + \sigma(e_2)e_2 - 2(\lambda^2 + 1)\xi.
\end{aligned} \tag{4}$$

Note that the scalar curvature r is computed as

$$r = -2\{e_1(c) + e_2(b) + b^2 + c^2 + \lambda^2 + 3\} \tag{5}$$

and

$$\sigma(e_1) = -e_2(\lambda) - 2\lambda b, \quad \sigma(e_2) = -e_1(\lambda) - 2\lambda c.$$

Remark 2. Since M is 3-dimensional, we have the relations

$$\rho_{11} = H + K_{13}, \quad \rho_{22} = H + K_{23}, \quad \rho_{33} = K_{13} + K_{23}.$$

5.2. **Two classes of homogeneous almost Kenmotsu 3-manifolds.** Perrone proved that every simply connected homogeneous almost Kenmotsu 3-manifold is a 3-dimensional non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. There are two classes of 3-dimensional almost Kenmotsu Lie groups.

- The characteristic vector field ξ is orthogonal to the unimodular kernel (Type II Lie groups).
- The characteristic vector field ξ is transversal to the unimodular kernel but not orthogonal (Type IV Lie groups).

After performing normalization procedure, those Lie group has the Lie algebra determined by the following commutation relations:

- The type II Lie algebra $\mathfrak{g} = \mathfrak{g}(\lambda, \alpha)$ is generated by
$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2,$$
where λ, $\alpha \in \mathbb{R}$.
- The type IV Lie algebra $\mathfrak{g} = \mathfrak{g}[\alpha, \gamma]$ is generated by
$$[e_1, e_2] = \gamma e_1, \quad [e_2, e_3] = 2\alpha e_1, \quad [e_3, e_1] = -2e_1,$$
where α, $\gamma \in \mathbb{R}$ and $\gamma \neq 0$.

In this subsection we exhibit these Lie groups in detail. For more information on these examples, we refer to [11].

5.2.1. *Type II Lie groups.*

Example 5.2 (Type II Lie groups)**.** Let $G(\lambda, \alpha)$ be a 3-dimensional non-unimodular Lie group with Lie algebra $\mathfrak{g}(\lambda, \alpha)$ generated by the orthonormal basis $\{e_1, e_2, e_3\}$ with commutation relations

$$[e_1, e_2] = 0, \quad [e_2, e_3] = (\alpha - \lambda)e_1 + e_2, \quad [e_3, e_1] = -e_1 + (\alpha + \lambda)e_2. \tag{6}$$

Then a left invariant almost contact structure (φ, ξ, η) compatible to the left invariant metric g is defined by

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad \xi = e_3, \quad \eta = g(e_3, \cdot). \tag{7}$$

Then (φ, ξ, η, g) is a left invariant almost Kenmotsu structure. Note that $\{e_1, e_2, e_3\}$ is regarded as a global orthonormal frame field, as in Lemma 5.1 under the choice $b = c = 0$. Hence the Ricci operator S of $G(\lambda, \alpha)$ is given by

$$Se_1 = -2(1+\lambda\alpha)e_1+2\lambda e_2, \quad Se_2 = 2\lambda e_1-2(1-\lambda\alpha)e_2, \quad Se_3 = -2(1+\lambda^2)\xi, \quad r = -2(3+\lambda^2).$$

The principal Ricci curvatures are

$$\rho_1 = -2 + 2\lambda\sqrt{1+\alpha^2}, \quad \rho_2 = -2 - 2\lambda\sqrt{1+\alpha^2}, \quad \rho_3 = -2(1+\lambda^2).$$

From these one can check that $G(\lambda, \alpha)$ is locally symmetric if and only if $\lambda = 0$ or $\alpha = \lambda^2 - 1 = 0$. In the former case $G(0, \alpha)$ is Kenmotsu and isometric to $\mathbb{H}^3(-1)$. In the latter case $G(\pm 1, 0)$ is isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$. Moreover the semi-symmetry is equivalent to the local symmetry. These Lie groups $G(0, \alpha)$ will be discussed in the next example.

Example 5.3 (Kenmotsu Lie groups)**.** The universal covering of the non-unimodular Lie group $G(0,\alpha)$ is realized as the following linear Lie group

$$\mathcal{S}(\alpha)=\left\{\left.\begin{pmatrix} e^{-z}\cos(\alpha z) & -e^{-z}\sin(\alpha z) & x\\ e^{-z}\sin(\alpha z) & e^{-z}\cos(\alpha z) & y\\ 0 & 0 & 1\end{pmatrix}\ \right|\ x,y,z\in\mathbb{R}\right\},\quad \alpha\in\mathbb{R}$$

equipped with a left invariant metric $e^{2z}(dx^2+dy^2)+dz^2$. Then $\mathcal{S}(\alpha)$ is isometric to the hyperbolic 3-space $\mathbb{H}^3(-1)$ of constant curvature -1 (see the preceding example). Take a left invariant orthonormal frame field

$$e_1=e^{-z}\cos(\alpha z)\frac{\partial}{\partial x}+e^{-z}\sin(\alpha z)\frac{\partial}{\partial y},\quad e_2=-e^{-z}\sin(\alpha z)\frac{\partial}{\partial x}+e^{-z}\cos(\alpha z)\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}=:\xi.$$

Then $\{e_1,e_2,e_3\}$ satisfies the commutation relations (6) with $\lambda=0$. Define an almost Kenmotsu structure by (7), then the structure is Kenmotsu and the Ricci operator S is given by $S=-2\mathrm{I}$ with scalar curvature $r=-6$. Note that at every point (x,y,z), the nullity space $T^0_{(x,y,z)}\mathcal{S}(\alpha)$ of R is $\{0\}$. Thus index of nullity is 0. In particular, the Lie group

$$\mathcal{S}(0)=\left\{\left.\begin{pmatrix} e^{-z} & 0 & x\\ 0 & e^{-z} & y\\ 0 & 0 & 1\end{pmatrix}\ \right|\ x,y,z\in\mathbb{R}\right\}$$

is isomorphic to the solvable part $\mathcal{S}$ of $\mathrm{SL}_2\mathbb{C}$ according to the *Iwasawa decomposition* $\mathrm{SL}_2\mathbb{C}=\mathrm{SU}_2\cdot\mathcal{S}$. The homogeneous Kenmotsu 3-manifold $\mathbb{H}^3(-1)$ is represented as $\mathbb{H}^3(-1)=\mathcal{S}(\alpha)/\{I\}$.

Example 5.4 (Strictly almost Kenmotsu Lie group)**.** The simply connected Lie group $G(\lambda,0)$ with Lie algebra $\mathfrak{g}(\lambda,0)$ given in Example 5.2 is realized as the linear Lie group

$$G(\lambda,0)=\left\{\left.\begin{pmatrix} e^{-z}\cosh(\lambda z) & e^{-z}\sinh(\lambda z) & x\\ e^{-z}\sinh(\lambda z) & e^{-z}\cosh(\lambda z) & y\\ 0 & 0 & 1\end{pmatrix}\ \right|\ x,y,z\in\mathbb{R}\right\},\quad \lambda\in\mathbb{R}.$$

We equip a left invariant metric

$$e^{2z}\cosh(2\lambda z)(dx^2+dy^2)-2e^{2z}\sinh(2\lambda z)dx\,dy+dz^2.$$

Note that $G(0,0)=\mathcal{S}(0)$. Take a left invariant orthonormal frame field

$$e_1=e^{-z}\left(\cosh(\lambda z)\frac{\partial}{\partial x}+\sinh(\lambda z)\frac{\partial}{\partial y}\right),\quad e_2=e^{-z}\left(\sinh(\lambda z)\frac{\partial}{\partial x}+\cosh(\lambda z)\frac{\partial}{\partial y}\right),\quad e_3=\frac{\partial}{\partial z}=:\xi.$$

Then $\{e_1,e_2,e_3\}$ satisfies the commutation relations (6) with $\alpha=0$. Define an almost Kenmotsu structure by (7), then the Ricci operator S is given by

$$Se_1=-2e_1+2\lambda e_2,\quad Se_2=2\lambda e_1-2e_2,\quad Se_3=-2(1+\lambda^2)e_3,\quad r=-2(3+\lambda^2).$$

Hence $G(\lambda,0)$ is H-almost Kenmotsu. The principal Ricci curvatures are

$$-2+2\lambda,\quad -2-2\lambda,\quad -2-2\lambda^2.$$

Thus $G(\lambda,0)$ is locally symmetric if and only if $\lambda=0$ or $\lambda^2=1$. In the former case, $G(\lambda,0)$ is of constant curvature -1. In the latter case, the principal Ricci curvatures are $\{-4,-4,0\}$. Moreover $G(\pm1,0)$ is isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$. The homogeneous almost Kenmotsu space representation of $\mathbb{H}^2(-4)\times\mathbb{R}$ is $\mathbb{H}^2(-4)\times\mathbb{R}=G(\pm1,0)/\{I\}$. This homogeneous Riemannian space is locally symmetric, but not a Riemannian symmetric space. Note that $(G(\lambda,0),\varphi,\xi,\eta,g)$ is strictly almost Kenmotsu for $\lambda\neq0$. The locally symmetric space $\mathbb{H}^2(-4)\times\mathbb{R}=G(\pm1,0)$ is a strictly almost Kenmotsu $(-2,0,2)$-space.

5.2.2. *Type IV Lie groups.*

Example 5.5 (Type IV Lie groups)**.** Let us consider a 3-dimensional non-unimodular Lie group $G = G[\alpha, \gamma]$ of type IV equipped with a left invariant almost Kenmotsu structure. The Lie algebra $\mathfrak{g} = \mathfrak{g}[\beta, \gamma]$ is determined by the commutation relations:

$$[e_1, e_2] = \gamma e_1, \quad [e_2, e_3] = 2\alpha e_1, \quad [e_3, e_1] = -2e_1, \quad \alpha \in \mathbb{R}, \quad \gamma \neq 0. \tag{8}$$

Then the Levi-Civita connection is described as

$$\begin{gathered} \nabla_{e_1} e_1 = -\gamma e_2 - 2e_3, \quad \nabla_{e_1} e_2 = \gamma e_1 - \alpha e_3, \quad \nabla_{e_1} e_3 = 2e_1 + \alpha e_2, \\ \nabla_{e_2} e_1 = -\alpha e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = \alpha e_1, \\ \nabla_{e_3} e_1 = \alpha e_2, \quad \nabla_{e_3} e_2 = -\alpha e_1, \quad \nabla_{e_3} e_3 = 0. \end{gathered} \tag{9}$$

Note that $\{e_1, e_2, e_3\}$ does not satisfies the assumption of Lemma 5.1. Introduce a left invariant almost contact metric structure by (7), then the Lie group is strictly almost Kenmotsu. The unimodular kernel is spanned by e_1 and $\xi + \frac{2}{\gamma}(e_1 - e_2)$. The operators h and h' are given by

$$he_1 = -\alpha e_1 + e_2, \quad he_2 = e_1 + \alpha e_2, \quad h'e_1 = e_1 + \alpha e_2, \quad h'e_2 = \alpha e_1 - e_2.$$

The eigenvalues of h are 0, $\lambda = \sqrt{\alpha^2 + 1}$ and $-\lambda = -\sqrt{\alpha^2 + 1}$. The covariant derivative $\nabla_\xi h$ is computed as

$$\nabla_\xi h = -2\alpha h \varphi.$$

Hence $\nabla_\xi h = 0$ holds when and only when $\alpha = 0$.

The Ricci operator S is described as (see [11, § 9.6]):

$$\begin{gathered} \rho_{11} = 2\alpha^2 - \gamma^2 - 4, \quad \rho_{12} = -4\alpha, \quad \rho_{13} = 2\alpha\gamma, \\ \rho_{22} = -2\alpha^2 - \gamma^2, \quad \rho_{23} = -2\gamma, \quad \rho_{33} = -2\alpha^2 - 4. \end{gathered}$$

Since $\gamma \neq 0$, $G[\alpha, \gamma]$ is *not* H-almost Kenmotsu. The principal Ricci curvatures are

$$-4 - \alpha^2 - \gamma^2, \quad -4 - \alpha^2 - \gamma^2, \quad \alpha^2.$$

Hence $G[\alpha, \gamma]$ is pseudo-symmetric. In particular, $G[0, \gamma]$ is locally symmetric.

Here we give an explicit model for semi-symmetric Lie groups $G[0, \gamma]$.

Example 5.6 (Strictly almost Kenmotsu Lie groups)**.** Let us consider the Lie group

$$G[0, \gamma] = \left\{ \left(\begin{array}{ccc} \exp(-\sqrt{\gamma^2 + 4}\, z) & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x, y, z \in \mathbb{R} \right\}, \quad \gamma \in \mathbb{R}$$

equipped with a left invariant Riemannian metric

$$g = \exp(2\sqrt{\gamma^2 + 4}\, z) dx^2 + dy^2 + dz^2.$$

One can check that $(G[0, \gamma], g)$ is isometric to $\mathbb{H}^2(-4 - \gamma^2) \times \mathbb{R}$. Hence $G[0, \gamma]$ is locally symmetric for any γ. Note that as a Riemannian manifold, $G[0, \gamma]$ is a warped product

$$\mathbb{R}^2 \times_f \mathbb{R} = (\mathbb{R}^2(y, z) \times \mathbb{R}(x), dy^2 + dz^2 + f(z)^2\, dx^2)$$

with warping function $f(z) = \exp(\sqrt{\gamma^2 + 4} z)$.

Take a global orthonormal frame field:

$$e_1 = \exp(-\sqrt{\gamma^2 + 4} z) \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{\sqrt{\gamma^2 + 4}} \left(-2\frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z} \right), \quad e_3 = \frac{1}{\sqrt{\gamma^2 + 4}} \left(\gamma \frac{\partial}{\partial y} + 2\frac{\partial}{\partial z} \right) := \xi.$$

Then $\{e_1, e_2, e_3\}$ satisfies the commutation relations (8) with $\alpha = 0$. Define an almost Kenmotsu structure by (7), then the operators h and h' are given by

$$he_1 = e_2, \quad he_2 = e_1, \quad h'e_1 = e_1, \quad h'e_2 = -e_2.$$

Thus h and h' have eigenvalues ± 1 and 0. The covariant derivative $\nabla_\xi h$ is computed as $\nabla_\xi h = 0$. The Ricci operator S is described as (*cf.* [11, Example 9.8]):

$$\rho_{11} = -\gamma^2 - 4, \quad \rho_{12} = 0, \quad \rho_{13} = -0,$$
$$\rho_{22} = -\gamma^2, \quad \rho_{23} = -2\gamma, \quad \rho_{33} = -4.$$

It should be remarked that $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ (with $\gamma \neq 0$) is *not* almost H-Kenmotsu, since $\rho_{23} \neq 0$.

5.3. **The classification of pseudo-symmetric homogeneous almost Kenmotsu 3-manifolds.** Perrone [21] classified homogeneous almost Kenmotsu 3-manifolds.

Theorem 5.7. *A simply connected homogeneous almost Kenmotsu* 3*-manifold is a Lie group equipped with a left invariant almost Kenmotsu structure. It is isomorphic to one of the following Lie groups.*

(1) *The Kenmotsu Lie group $\mathcal{S}(\alpha)$.*
(2) *Strictly almost Kenmotsu Lie group $G(\lambda, \alpha)$.*
(3) *Strictly almost Kenmotsu Lie group $G[\alpha, \gamma]$.*

From this classification we obtain the following classification of semi-symmetric homogeneous almost Kenmotsu 3-manifolds.

Theorem 5.8. *A homogeneous almost Kenmotsu* 3*-manifold is semi-symmetric if and only if it is locally symmetric and locally isomorphic to the following homogeneous almost Kenmotsu Lie groups*:

(1) *The Kenmotsu Lie group $\mathcal{S}(\alpha)$.*
(2) *The strictly almost Kenmotsu Lie group $G(\pm 1, 0)$ which is isometric to $\mathbb{H}^2(-4) \times \mathbb{R}$.*
(3) *The strictly almost Kenmotsu Lie group $G[0, \gamma]$ which is isometric to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$.*

The semi-symmetry condition can be weakened for almost Kenmotsu Lie groups of type IV as follows:

Proposition 5.9. *Let $G[\alpha, \gamma]$ be a strictly almost Kenmotsu Lie group of type IV. Then the following properties for $G[\alpha, \gamma]$ are mutually equivalent*:

- *$G[\alpha, \gamma]$ is semi-symmetric.*
- *$G[\alpha, \gamma]$ is locally symmetric.*
- *$G[\alpha, \gamma]$ satisfies $\nabla_\xi S = 0$.*
- *$G[\alpha, \gamma]$ satisfies $(\nabla_\xi S)\xi = 0$.*

Proof. Direct computation show that

$$(\nabla_{e_1} S)e_1 = -4\alpha^2 \left\{ \frac{1}{2}\gamma e_2 + e_3 \right\}, \quad (\nabla_{e_1} S)e_2 = 2\alpha \left\{ -\alpha\gamma e_1 + 2\gamma e_2 - (\frac{1}{2}\gamma^2 - 2)e_3 \right\},$$

$$(\nabla_{e_1} S)e_3 = 2\alpha \left\{ (-2\alpha)e_1 + (2 - \frac{1}{2}\gamma^2)e_2 - 2\gamma e_3 \right\},$$

$$(\nabla_{e_2} S)e_1 = -2\alpha \left\{ (-2\alpha)\gamma e_1 + \gamma e_2 + \frac{1}{2}(4\alpha^2 - \gamma^2)e_3 \right\}, \quad (\nabla_{e_2} S)e_2 = -2\alpha(\gamma e_2 - 2\alpha e_3),$$

$$(\nabla_{e_2}S)e_3 = -2\alpha\left\{\frac{1}{2}(4\alpha^2-\gamma^2)e_1 - 2\alpha e_2 + 2\alpha\gamma e_3)\right\},$$

$$(\nabla_{e_3}S)e_1 = -2\alpha\left\{-4\alpha e_1 - 2(\alpha^2-1)e_2 - \gamma e_3\right\},\ (\nabla_{e_3}S)e_2 = 2\alpha\left\{2(\alpha^2-1)e_1 - 4\alpha e_2 + \alpha\gamma e_3\right\},$$

$$(\nabla_{e_3}S)e_3 = 2\alpha\gamma(e_1+\alpha e_2).$$

From these we obtain

$$\nabla S = 0 \Longleftrightarrow (\nabla_\xi S)\xi = 0 \Longleftrightarrow \alpha = 0.$$

Hence $G[\alpha,\gamma]$ is locally symmetric if and only if $\alpha = 0$. □

Now we give a classification of pseudo-symmetric homogeneous almost Kenmotsu 3-manifolds.

Theorem 5.10. *A homogeneous almost Kenmotsu 3-manifold is pseudo-symmetric if and only it it is locally isomorphic to the following homogeneous almost Kenmotsu Lie groups*:

1. *The Kenmotsu Lie group $\mathcal{S}(\alpha)$ (locally symmetric).*
2. *The strictly almost Kenmotsu Lie group $G(\lambda,\alpha)$ satisfying $\lambda^2-\alpha^2=1$. The simply conncted one is realized as the linear Lie group*

$$\left\{\left(\begin{array}{ccc} e^{-z}\cosh z & (\lambda-\alpha)e^{-z}\sinh z & x \\ (\lambda+\alpha)e^{-z}\sinh z & e^{-z}\cosh z & y \\ 0 & 0 & 1 \end{array}\right)\ \middle|\ x,y,z\in\mathbb{R}\right\}.$$

   The almost Kenmotsu Lie group $G(\lambda,\alpha)$ satisfying $\lambda^2-\alpha^2=1$ is a pseudo-symmetric space of constant type. It is proper if and only if $\alpha\neq 0$ and $\lambda^2\neq 1$. The almost Kenmotsu Lie groups $G(\pm 1,0)$ are locally symmetric and isometric to $\mathbb{H}^2(-4)\times\mathbb{R}$.
3. *The strictly almost Kenmotsu Lie groups $G[\alpha,\gamma]$ with $\alpha\neq 0$. These Lie groups are proper pseudo-symmetric spaces of constant type.*
4. *The strictly almost Kenmotsu Lie groups $G[0,\gamma]$. The almost Kenmotsu Lie group $G[0,\gamma]$ is isometric to $\mathbb{H}^2(-4-\gamma^2)\times\mathbb{R}$ and locally-symmetric.*

Proof. First, we consider the Lie group $G(\lambda,\alpha)$ of type II. The principal Ricci curvatures are

$$\rho_1 = -2+2\lambda\sqrt{1+\alpha^2},\quad \rho_2 = -2-2\lambda\sqrt{1+\alpha^2},\quad \rho_3 = -2(1+\lambda^2).$$

Thus we get

- $\rho_1=\rho_2 \Longleftrightarrow \lambda=0$,
- $\rho_1=\rho_3$ or $\rho_2=\rho_3 \Longleftrightarrow \lambda=0$ or $\lambda^2-\alpha^2=1$.

Thus $G(\lambda,\alpha)$ is pseudo-symmetric if and only if $\lambda=0$ (locally symmetric) or $\lambda^2-\alpha^2=1$. In the latter case, one can confirm that $G(\lambda,\alpha)$ with $\lambda^2-\alpha^2=1$ satisfies $R\cdot S = H\,R_1\cdot S$. Thus the scale factor L is given by $L=H=-(1-\lambda^2)$.

Next, we consider the Lie group $G[\alpha,\gamma]$ of type IV. As we saw before $G[\alpha,\gamma]$ is pseudo-symmetric. One can check that $G[\alpha,\gamma]$ satisfies $R\cdot S=\alpha^2\,R_1\cdot S$. In particular $G[\alpha,\gamma]$ is semi-symmetric if and only if it is locally symmetric. □

5.4. **Generalized almost Kenmotsu (κ,μ,ν)-spaces.** Next we give some examples of *non-homogeneous* pseudo-symmetric almost Kenmotsu 3-manifolds.

Definition 5.11. An almost Kenmotsu 3-manifold M is said to be a *generalized almost Kenmotsu (κ,μ,ν)-space* if there exist three functions κ, μ and ν such that

$$R(X,Y)\xi = \kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\}+\nu\{\eta(Y)\varphi hX-\eta(X)\varphi hY\}$$

for all vector fields X and Y on M. In particular, M is said to be an *almost Kenmotsu* (κ,μ,ν)*-space* if it is a generalized almost Kenmotsu (κ,μ,ν)-space all of κ, μ and ν are constant.

Definition 5.12. Let M be a generalized almost Kenmotsu (κ,μ,ν)-space. If all the functions κ, μ and ν are constants, then M is called an *almost Kenmotsu* (κ,μ,ν)*-space*. A generalized almost Kenmotsu (κ,μ,ν)-space is said to be *proper* if $|d\kappa|^2+|d\mu|^2+|d\nu|^2\neq 0$.

Theorem 5.13 ([12, 19])**.** *Let M be an almost Kenmotsu 3-manifold. If M is a generalized almost Kenmotsu (κ,μ,ν)-space, then M is an H-almost Kenmotsu manifold. Conversely if M is an H-almost Kenmotsu manifold, then M satisfies the generalized (κ,μ,ν)-condition on an open dense subset. In such a case we have*

$$\kappa=-(\lambda^2+1),\quad \mu=-2\alpha,\quad \lambda\nu=2\lambda+\xi(\lambda).$$

The Ricci operator has the form

$$S=\left(\frac{r}{2}-\kappa\right)\mathrm{I}-\left(\frac{r}{2}-3\kappa\right)\eta\otimes\xi+\mu h+\nu\varphi h.$$

Moreover, we have

$$S\xi=2\kappa\xi,\quad \mathrm{tr}\,(h^2)=-2(\kappa+1).$$

Corollary 5.14 ([12])**.** *For an almost Kenmotsu 3-manifold M, the following three properties are mutually equivalent*:

- *M satisfies $S\varphi=\varphi S$.*
- *M is weakly η-Einstein, that is, $S=A\,\mathrm{I}+B\eta\otimes\xi$ for some functions A and B.*
- *M is a generalized almost Kenmotsu $(\kappa,0)$-space satisfying $d\kappa\wedge\eta=0$.*

Remark 3. In Proposition 4.2, we know that every Kenmotsu 3-manifold is H-almost Kenmotsu and pseudo-symmetric, especially η-Einstein. However the converse statement does not hold. Indeed, there exists a strictly H-almost Kenmotsu 3-manifolds which are weakly η-Einstein (and hence pseudo-symmetric) mentioned in Corollary 5.14. Such examples will be given in Example 5.17 below.

Example 5.15 (The strictly almost Kenmotsu Lie group $G(\lambda)$)**.** For any $\lambda\neq 0$, the strictly almost Kenmotsu Lie group $G(\lambda)$ is an almost Kenmotsu $(-1-\lambda^2,0,2)$-space. As we saw before, $G(\lambda)$ is pseudo-symmetric if and only if $\lambda^2=1$. Moreover, $G(\pm 1)=\mathbb{H}^2(-4)\times\mathbb{R}$ are locally symmetric.

Example 5.16 ([20])**.** Let $M=\{(x,y,z)\in\mathbb{R}^3\,|\,z>0\}$ be the upper half space. We introduce an almost contact Riemannian structure on M by

$$\xi=\frac{\partial}{\partial z},\quad \eta=dz,\quad g=ze^{2z}dx^2+\frac{e^{2z}}{z}dy^2+dz^2,$$

$$\varphi\frac{\partial}{\partial x}=z\frac{\partial}{\partial y},\quad \varphi\frac{\partial}{\partial y}=-\frac{1}{z}\frac{\partial}{\partial x},\quad \varphi\frac{\partial}{\partial z}=0.$$

Then $M=(M,\varphi,\xi,\eta,g)$ is a strictly almost Kenmotsu 3-manifold. The Ricci operator and the scalar curvature are computed as ([11, Example 7.1], [12, Example 2])

$$S=-2I-\frac{1}{2z^2}\eta\otimes\xi+(2-\frac{1}{z})\varphi h,\quad r=-\frac{12z^2+1}{2z^2}=-2(\lambda^2+3)=2(\kappa-2).$$

One can check that (M,φ,ξ,η,g) is a generalized almost Kenmotsu $(\kappa,0,\nu)$-space with

$$\kappa = -1 - \frac{1}{4z^2} < -1, \quad \nu = 2 - \frac{1}{z}.$$

In particular κ and ν satisfy $d\kappa\wedge\eta = d\nu\wedge\eta = 0$. The generalized almost Kenmotsu $(\kappa,0,\nu)$-space M has principal Ricci curvatures

$$\rho_1 = -2 + \frac{2z-1}{2z^2}, \quad \rho_2 = -2 - \frac{2z-1}{2z^2}, \quad \rho_3 = -2 - \frac{1}{2z^2}.$$

Hence M is *not* pseudo-symmetric.

Example 5.17 ([20])**.** On the Cartesian 3-space $\mathbb{R}^3(x,y,z)$, we define an almost contact metric structure (φ,ξ,η,g) by

$$\xi = \frac{\partial}{\partial z}, \quad \eta = dz, \quad g = e^{f(z)+2z}dx^2 + e^{2z-f(z)}dy^2 + dz^2,$$

$$\varphi\frac{\partial}{\partial x} = e^{f(z)}\frac{\partial}{\partial y}, \quad \varphi\frac{\partial}{\partial y} = -e^{-f(z)}\frac{\partial}{\partial x}, \quad \varphi\frac{\partial}{\partial z} = 0,$$

where $f(z) = -e^{-2z}$. The non-zero eigenvalues of h are $\lambda = e^{-2z}$ and $-\lambda$. Then resulting almost contact Riemannian 3-manifold $M = (\mathbb{R}^3,\varphi,\xi,\eta,g)$ is a generalized almost Kenmotsu $(\kappa,0)$-space with $\kappa = -1 - e^{-4z}$. In particular M satisfies $d\kappa\wedge\eta = 0$.

The Ricci operator S and the scalar curvature r are computed as ([11, Example 7.2], [12, Example 3]):

$$S = -2\,\mathrm{I} - 2\lambda^2\eta\otimes\xi, \quad r = -2e^{-4z} - 6 = -2(\lambda^2+3) = 2(\kappa-2).$$

Thus M is really weakly η-Einstein, and hence it is proper pseudo-symmetric.

Example 5.18 ([24])**.** We construct a generalized almost Kenmotsu (κ,μ)-space with prescribed function μ. Let M be an open submanifold of $\mathbb{R}^3(x,y,z)$ defined by $z < -1$. Take arbitrary smooth functions $\mu(z)$, $q_1(z)$ and $q_2(z)$ of z. We put

$$f_1(x,y,z) = x - \frac{y(\mu(z) + 2\sqrt{-1-z}\,)}{2} + q_1(z), \quad f_2(x,y,z) = y + \frac{x(\mu(z) - 2\sqrt{-1-z})}{2} + q_2(z),$$
$$f_3(z) = -4(1+z).$$

We introduce an almost contact Riemannian structure (φ,ξ,η,g) by

$$\xi = f_1(x,y,z)\frac{\partial}{\partial x} + f_2(x,y,z)\frac{\partial}{\partial y} + f_3(z)\frac{\partial}{\partial z}, \quad \eta = \frac{dz}{f_3(z)},$$

$$g = dx^2 + dy^2 + (1 + f_1^2 + f_2^2)\eta\otimes\eta - f_1(dx\otimes\eta + \eta\otimes dx) - f_2(dy\otimes\eta + \eta\otimes dy),$$

$$\varphi\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad \varphi\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}, \quad \varphi\frac{\partial}{\partial z} = \frac{f_2}{f_3}\frac{\partial}{\partial x} - \frac{f_1}{f_3}\frac{\partial}{\partial y}.$$

Then $M(\mu,q_1,q_2) = (\mathbb{R}^3,\varphi,\xi,\eta,g)$ is a strictly almost Kenmotsu 3-manifold. The orthonormal frame field

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \xi$$

is a global orthonormal frame field as in Lemma 5.1 satisfying $b = c = 0$, $\alpha = -\mu(z)/2$ and $\lambda = \sqrt{-1-z}$. One can see that

$$R(X,Y)\xi = z\{\eta(Y)X - \eta(X)Y\} + \mu(z)\{\eta(Y)hX - \eta(X)hY\}.$$

Thus $M(\mu, q_1, q_2)$ is a generalized almost Kenmotsu (κ, μ)-space. Note that $\kappa = z$ and $\mu = \mu(z)$ satisfy $d\kappa \wedge \eta = 0$ and $d\mu \wedge \eta = 0$. The Ricci operator S and the scalar curvature r are computed as ([11, Example 7.3], [12, Example 4]):

$$Se_1 = (-2+\lambda\mu)e_1, \quad Se_2 = (-2-\lambda\mu)e_2, \quad Se_3 = -2(1+\lambda^2)e_3, \quad r = -2(\lambda^2+3) = 2(\kappa-2),$$

Thus $M(\mu, q_1, q_2)$ is H-almost Kenmotsu. Hence $M(\mu, q_1, q_2)$ is pseudo-symmetric if and only if

$$\mu(z) = 0, \quad \text{or} \quad \mu(z) = \pm 2\lambda(z) = \pm 2\sqrt{-1-z}.$$

In particular, when $\mu(z) = 0$, then $M(0, q_1, q_2)$ is weakly η-Einstein with $S = -2\mathrm{I} - 2\lambda^2 \eta \otimes \xi$. In particular $M(0, q_1, q_2)$ is proper pseudo-symmetric. Moreover, $M(\mu, q_1, q_2)$ with $\mu(z) = \pm 2\sqrt{-1-z}$ is also proper pseudo-symmetric.

Saltarelli proved the following local classifications.

Theorem 5.19 ([24]). *Let M be a 3-dimensional generalised almost Kenmotsu (κ, μ)-space satisfying $d\kappa \wedge \eta = 0$ and $\kappa < -1$. Then M is locally isomorphic to the $M(\mu, q_1, q_2)$ in Example 5.18.*

Remark 4. Local coordinate changes between Example 5.17 and Example 5.18 with $\mu = 0$ can be seen in [24].

6. The pseudo symmetry of almost Kenmotsu 3-manifolds

6.1. **Classification: Locally symmetric spaces.** In our previous work [11] we obtained the following result.

Proposition 6.1 ([11]). *Locally symmetric almost Kenmotsu 3-manifolds satisfy $\nabla_\xi h = 0$.*

Next, locally symmetric almost Kenmotsu 3-manifolds are classified as follows ([11], see also [23]):

Theorem 6.2. *Let M be an almost Kenmotsu 3-manifold. Then M is locally symmetric if and only if M is one of the following spaces:*

(1) *If M is H-almost Kenmotsu, then M is a Kenmotsu manifold of constant curvature -1 or locally isomorphic to $\mathbb{H}^2(-4) \times \mathbb{R}$ or*
(2) *If M is non H-almost Kenmotsu, then M is locally locally isomorphic to $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ for some $\gamma \neq 0$.*

The example $\mathbb{H}^2(-4-\gamma^2) \times \mathbb{R}$ was discovered in [23].

Corollary 6.3. *Every complete locally symmetric almost Kenmotsu 3-manifold is realized as a Lie group equipped with a left invariant almost Kenmotsu structure.*

6.2. **The system of semi-symmetry.** The derivative $R(e_1, e_2) \cdot S$ is computed as

$$\begin{aligned}
(R(e_1,e_2)S)e_1 =\ & 2\{(\xi(\lambda)+2\lambda)H + \rho_{13}\rho_{23}\}e_1 - \{(\rho_{13})^2 - (\rho_{23})^2 - 4\alpha\lambda H\}e_2 \\
& + \{K_{23}\rho_{23} + (\xi(\lambda)+2\lambda)\rho_{13}\}e_3, \\
(R(e_1,e_2)S)e_2 =\ & -\{(\rho_{13})^2 - (\rho_{23})^2 - 4\alpha\lambda H\}e_1 - 2\{(\xi(\lambda)+2\lambda)H + \rho_{13}\rho_{23}\}e_2 \\
& - \{K_{13}\rho_{13} + (\xi(\lambda)+2\lambda)\rho_{23}\}e_3 \\
(R(e_1,e_2)S)e_3 =\ & \{K_{23}\rho_{23} + (\xi(\lambda)+2\lambda)\rho_{13}\}e_1 - \{K_{13}\rho_{13} + (\xi(\lambda)+2\lambda)\rho_{23}\}e_2.
\end{aligned}$$

Hence $R(e_1, e_2) \cdot S = 0$ if and only if

$$(\xi(\lambda) + 2\lambda)H + \rho_{13}\rho_{23} = 0, \tag{10}$$

$$(\rho_{13})^2 - (\rho_{23})^2 - 4\alpha\lambda H = 0, \tag{11}$$

$$K_{23}\rho_{23} + (\xi(\lambda) + 2\lambda)\rho_{13} = 0, \tag{12}$$

$$K_{13}\rho_{13} + (\xi(\lambda) + 2\lambda)\rho_{23} = 0. \tag{13}$$

Next, the derivative $R(e_2, e_3) \cdot S$ is computed as

$$\begin{aligned}
(R(e_2,e_3)S)e_1 = & \ \{K_{13}\rho_{13} + (\xi(\lambda)+2\lambda)\rho_{23}\}e_2 - \{(\xi(\lambda)+2\lambda)H + \rho_{13}\rho_{23}\}e_3, \\
(R(e_2,e_3)S)e_2 = & \ \{K_{13}\rho_{13} + (\xi(\lambda)+2\lambda)\rho_{23}\}e_1 + 2\{(\xi(\lambda)+2\lambda)\rho_{13} + K_{23}\rho_{23}\}e_2 \\
& + \{(\rho_{13})^2 - (\xi(\lambda)+2\lambda)^2 - K_{23}(H - K_{13})\}e_3, \\
(R(e_2,e_3)S)e_3 = & - \{(\xi(\lambda)+2\lambda)H + \rho_{13}\rho_{23}\}e_1 + \{(\rho_{13})^2 - (\xi(\lambda)+2\lambda)^2 - K_{23}(H - K_{13})\}e_2 \\
& - 2\{(\xi(\lambda)+2\lambda)\rho_{13} + K_{23}\rho_{23}\}e_3.
\end{aligned}$$

Thus $R(e_2, e_3) \cdot S = 0$ holds if and only if (10), (12), (13) and

$$(\rho_{13})^2 - (\xi(\lambda) + 2\lambda)^2 - K_{23}(H - K_{13}) = 0 \tag{14}$$

hold.

Finally, the derivative $R(e_3, e_1) \cdot S$ is computed as

$$\begin{aligned}
(R(e_3,e_1)S)e_1 = & - 2\{K_{13}\rho_{13} + (\xi(\lambda)+2\lambda)\rho_{23}\}e_1 - \{(K_{23}\rho_{23} + (\xi(\lambda)+2\lambda)\rho_{13}\}e_2 \\
& + \{(\xi(\lambda)+2\lambda)^2 - (\rho_{23})^2 + K_{13}(H - K_{23})\}e_3 \\
(R(e_3,e_1)S)e_2 = & - \{(K_{23}\rho_{23} + (\xi(\lambda)+2\lambda)\rho_{13}\}e_1 + \{(\xi(\lambda)+2\lambda)H + \rho_{13}\rho_{23}\}e_3 \\
(R(e_3,e_1)S)e_3 = & \ \{(\xi(\lambda)+2\lambda)^2 - (\rho_{23})^2 + K_{13}(H - K_{23})\}e_1 + \{(\xi(\lambda)+2\lambda)H + \rho_{13}\rho_{23}\}e_2 \\
& + 2\{K_{13}\rho_{13} + (\xi(\lambda)+2\lambda)\rho_{23}\}e_3.
\end{aligned}$$

Thus $R(e_3, e_1) \cdot S = 0$ if and only if (10), (12), (13) and

$$(\rho_{23})^2 - (\xi(\lambda) + 2\lambda)^2 - K_{13}(H - K_{23}) = 0 \tag{15}$$

hold. The equation (11) is a consequence of (14) and (15). Hence we obtain the the following criterion.

Proposition 6.4 ([11])**.** *A strictly almost Kenmotsu* 3*-manifold M is semi-symmetric if and only if M satisfies the following system* (10), (12), (13), (14) *and* (15)*.*

Multiplying ρ_{13} to (12) and combining (10) and (14), we get

$$(\xi(\lambda) + 2\lambda)\{(\xi(\lambda) + 2\lambda)^2 - K_{13}K_{23}\} = 0.$$

This equation is rewritten as

$$\rho_{12}\{(\rho_{12})^2 - K_{13}K_{23}\} = 0. \tag{16}$$

6.3. **The case $\nabla_\xi h = 0$.** We know that the local symmetry of an almost Kenmotsu 3-manifold implies the invariance $\nabla_\xi h = 0$. The converse statement does not hold. Indeed the non-unimodular Lie group $G(\lambda)$ with $\lambda \neq 0, \pm 1$ exhibited in Example 5.4 satisfies $\nabla_\xi h = 0$ but not locally symmetric.

Here we study semi-symmetry of almost Kenmotsu 3-manifolds under the invariance condition $\nabla_\xi h = 0$.

Theorem 6.5. *Let M be an almost Kenmotsu 3-manifold satisfying $\nabla_\xi h = -2\alpha h\varphi$ for some constant α. Then M is semi-symmetric if and only if it is locally symmetric.*

Proof. Let M be an almost Kenmotsu 3-manifold. In case M is Kenmotsu, semi-symmetry implies the local symmetry, we assume that $\mathcal{U}_1$ is non-empty and take a local orthonormal frame field $\{e_1, e_2, e_3\}$ as in Lemma 5.1 on $\mathcal{U}$. Let us assume that $\nabla_\xi h = -2\alpha h\varphi$ and $d\alpha = 0$, then the system of semi-symmetry reduces to the following system:

$$2\lambda H + \rho_{13}\rho_{23} = 0, \tag{17}$$

$$K_{23}\rho_{23} + 2\lambda\rho_{13} = 0, \tag{18}$$

$$K_{13}\rho_{13} + 2\lambda\rho_{23} = 0, \tag{19}$$

$$(\rho_{13})^2 - 4\lambda^2 - K_{23}(H - K_{13}) = 0, \tag{20}$$

$$(\rho_{23})^2 - 4\lambda^2 - K_{13}(H - K_{23}) = 0. \tag{21}$$

The equation (16) reduces to

$$\lambda(4\lambda^2 - K_{13}K_{23}) = 0.$$

Thus on $\mathcal{U}_1$, we have

$$K_{13}K_{23} = 4\lambda^2.$$

This equation is rewritten as

$$(\lambda^2 + 2\alpha\lambda - 1)(\lambda^2 - 2\alpha\lambda - 1) = 0.$$

Thus λ is a non-zero constant on $\mathcal{U}_1$ and hence K_{13} and K_{23} are also constant. Moreover we have

$$\rho_{13} = \sigma(e_1) = -2\lambda b, \quad \rho_{23} = \sigma(e_2) = -2\lambda c.$$

Now we consider the case $\lambda^2 - 2\alpha\lambda - 1 = 0$. The case $\lambda^2 + 2\alpha\lambda - 1 = 0$ is treated in a similar manner. In this case $\lambda = \alpha \pm \sqrt{\alpha^2 + 1}$ and

$$K_{13} = -2(2\alpha\lambda + 1), \quad K_{23} = -2.$$

Inserting these into (18), we get $c = \lambda b$.

From the Jacobi identity, we have

$$\xi(b) + b\lambda(\alpha - \lambda) + b = 0, \quad \lambda\xi(b) = b\alpha.$$

Multiplying 2λ to the second equation and combining the first equation as well as $2\lambda\alpha = \lambda^2 - 1$, we get $b(\lambda^2 - 1) = 0$. Thus we deduce that $b = 0$ or $\lambda^2 = 1$.

(1) In case $b = 0$ on $\mathcal{U}_1$, then $c = 0$. Then M satisfies H-almost Kenmotsu condition on $\mathcal{U}_1$. By (17), we have $H = 0$. This implies that $r = -4(1 + \lambda^2)$. On the other hand, scalar curvature r is computed as $r = -2(\lambda^2 + 3)$. Thus we have $\lambda^2 = 1$ and hence $r = -8$. This together with the assumption $\lambda^2 - 2\alpha\lambda - 1 = 0$, we obtain $\alpha = 0$. The Ricci operator is expressed as

$$S = \begin{pmatrix} -2 & 2\lambda & 0 \\ 2\lambda & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

relative to $\{e_1, e_2, e_3\}$. The principal Ricci curvatures are $\{-4, -4, 0\}$. One can check that $\mathcal{U}_1$ is locally symmetric and locally isomorphic to $G(\lambda) = \mathbb{H}^2(-4) \times \mathbb{R}$ as an almost Kenmotsu manifold.

(2) Next we consider the case $\lambda^2 = 1$. Since we have already investigated the case $b = 0$ and $\lambda^2 = 1$. It suffices to to study the case $\lambda^2 = 1$ and $b \neq 0$.

Since we assumed $\lambda^2 - 2\alpha\lambda - 1 = 0$, we have $\alpha = 0$. Hence

$$K_{13} = K_{23} = -2, \quad H = \frac{r}{2} + 4, \quad \rho_{13} = -2\lambda b, \quad \rho_{23} = -2\lambda c.$$

Then from (18), we get $c = \lambda b$. Next, from (17), we have $H = -2b^2$. Hence the scalar curvature is $-4b^2 - 8$. The Ricci operator has the componets

$$S = \begin{pmatrix} -2-2b^2 & 2\lambda & -2\lambda b \\ 2\lambda & -2-2b^2 & -2b \\ -2\lambda b & -2b & -4 \end{pmatrix}, \quad \lambda = \pm 1.$$

It should be remarked that the local vector field $V_0 = (e_1 + \lambda e_2) - b\lambda\xi$ satisfies $SV_0 = 0$. Thus S has 0 as an eigenvalue. Moreover one can see that the principal Ricci curvatures are

$$-2(b^2 + 2), \quad -2(b^2 + 2), \quad 0.$$

Hence the nullity vector space of R is given by

$$T_p^0 M = \{X \in T_p M \mid S_p X = 0\} = \mathbb{R}V_0.$$

Thus on $\mathcal{U}_1$, the index of nullity is constant 1 and the index of conullity is constant 2. One can check that the vector field X_0 satisfies $\nabla_{V_0} V_0 = 0$, namely the integral curves of V_0 are geodesics. Hence V_0 has constant length. The square norm of V_0 is $b^2 + 2$. Thus we proved that b is constant.

Here we introduce another local orthonormal frame field $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ by

$$\bar{e}_1 = \frac{1}{\sqrt{2}}(e_1 - \lambda e_2), \quad \bar{e}_2 = \frac{1}{\sqrt{2}}(e_1 + \lambda e_2), \quad \bar{e}_3 = e_3 = \xi.$$

Then we have

$$[\bar{e}_1, \bar{e}_2] = \sqrt{2} b\lambda \bar{e}_1, \quad [\bar{e}_2, \bar{e}_3] = 0, \quad [\bar{e}_3, \bar{e}_1] = -2\bar{e}_1.$$

We notice that the distribution spanned by $\bar{e}_2$ and $\bar{e}_3$ is integrable. Moreover the integral surfaces of this distribution is flat and totally geodesic. Thus $\mathcal{U}_1$ is locally foliated by Euclidean planes.

From the commutation relations of $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$, we notice that $\mathcal{U}_1$ is locally isomorphic to a non-unimodular Lie group whose Lie algebra is of type IV. Since $\mathcal{U}_1$ is semi-symmetric, the non-unimodular Lie group is locally isomorphic to the warped product model

$$\mathbb{H}^2(-4 - 2b^2) \times \mathbb{R} = \mathbb{R}^2 \times_f \mathbb{R}$$

of the Riemannian symmetric space $\mathbb{H}^2(-4 - 2b^2) \times \mathbb{R}$ as described in Example 5.6. Thus $\mathcal{U}_1$ is locally symmetric. Note that the distribution spanned by $\bar{e}_2$ and $\bar{e}_3$ is the left invariant distribution generated by the unimodular kernel. The integral surfaces are obtained by left translations from the normal subgroup corresponding to the unimodular kernel (which is intrinsically flat).

□

Wang [29, Theorem 5.1] claimed that there exist proper semi-symmetric almost Kenmostu 3-manifolds satisfying $\nabla_\xi h = -2\alpha h\varphi$ and $d\alpha = 0$. The above result corrects [29, Theorem 5.1].

6.4. **Semi-symmetric H-almost Kenmotsu 3-manifolds.** Next we study semi-symmetric H-almost Kenmotsu 3-manifolds. First of all we recall that semi-symmetric Kenmotsu 3-manifolds are of constant curvature -1.

Assume that M is H-almost Kenmotsu, then the system of semi-symmetry reduces to

$$(\xi(\lambda)+2\lambda)H=0, \tag{22}$$

$$(\xi(\lambda)+2\lambda)^2+K_{23}(H-K_{13})=0, \tag{23}$$

$$(\xi(\lambda)+2\lambda)^2+K_{13}(H-K_{23})=0 \tag{24}$$

over $\mathcal{U}=\mathcal{U}_0\cup\mathcal{U}_1$. Note that on $\mathcal{U}_0$, we have $h=0$ and hence the above system implies $r=-6$. Thus $\mathcal{U}_0$ is of constant curvature -1, especially it is locally symmetric. This conclusion is consistent with Proposition 4.2.

Since we assumed that M is strictly H-almost Kenmotsu, $\mathcal{U}_1$ is non-empty. Let us work on $\mathcal{U}_1$. From (23) and (24) we have

$$H(K_{13}-K_{23})=0$$

which is equivalent to $H\alpha=0$ (over $\mathcal{U}_1$). Since M is H-almost Kenmotsu, it is locally a generalized almost Kenmotsu (κ,μ,ν)-space with

$$\kappa=-1-\lambda^2<-1, \quad \mu=-2\alpha, \quad \lambda\nu=\xi(\lambda)+2\lambda.$$

Let us consider open sets

$$\begin{aligned}\mathcal{U}_2 =&\{p\in\mathcal{U}_1 \mid H=0 \text{ in a neighborhood of } p\},\\ \mathcal{U}_3 =&\{p\in\mathcal{U}_1 \mid H\neq 0 \text{ in a neighborhood of } p\}.\end{aligned}$$

Then $\mathcal{U}_2\cup\mathcal{U}_3$ is open and dense in $\mathcal{U}_1$.

(1) On $\mathcal{U}_2$ we have $r=-4(1+\lambda^2)=4\kappa$ and

$$S=\begin{pmatrix}\kappa+\lambda\mu & \lambda\nu & 0\\ \lambda\nu & \kappa-\lambda\mu & 0\\ 0 & 0 & 2\kappa\end{pmatrix}.$$

Thus principal Ricci curvatures are

$$\rho_1=\kappa+\lambda\sqrt{\mu^2+\nu^2}, \quad \rho_2=\kappa-\lambda\sqrt{\mu^2+\nu^2}, \quad \rho_3=2\kappa.$$

The possibilities for $\mathcal{U}_2$ to be semi-symmetric are (i) $\rho_1=\rho_3$ and $\rho_2=0$ or (ii) $\rho_2=\rho_3$ and $\rho_1=0$. For the former case we have $\rho_1=\rho_3 \Longleftrightarrow \rho_2=0 \Longleftrightarrow \kappa=\lambda\sqrt{\mu^2+\nu^2}$. For the latter case, we have $\rho_2=\rho_3 \Longleftrightarrow \rho_1=0 \Longleftrightarrow \kappa=-\lambda\sqrt{\mu^2+\nu^2}$. In case $\mathcal{U}_2$ is semi-symmetric, then the equation $\kappa=\pm\lambda\sqrt{\mu^2+\nu^2}$ implies the equation

$$\lambda^4+(2-\mu^2-\nu^2)\lambda^2+1=0.$$

Since λ^2 is a positive real valued function, μ and ν should satisfy $\mu^2+\nu^2\geq 4$ and we obtain

$$\lambda^2=\frac{\mu^2+\nu^2-2\pm\sqrt{\mu^2+\nu^2}\,\sqrt{\mu^2+\nu^2-4}}{2}.$$

One can see that

$$1\leq\frac{\mu^2+\nu^2-2+\sqrt{\mu^2+\nu^2}\,\sqrt{\mu^2+\nu^2-4}}{2}=\lambda^2,$$

and

$$0<\lambda^2=\frac{\mu^2+\nu^2-2-\sqrt{\mu^2+\nu^2}\,\sqrt{\mu^2+\nu^2-4}}{2}\leq 1.$$

(2) On $\mathcal{U}_3$, we have $\alpha=0$, equivalently $\mu=0$. From (22), $\lambda\nu=\xi(\lambda)+2\lambda=0$. Thus $\mathcal{V}_3$ is a generalized $(\kappa,0)$-space. Hence the Ricci operator has the form

$$S=\begin{pmatrix}\frac{r}{2}-\kappa & 0 & 0\\ 0 & \frac{r}{2}-\kappa & 0\\ 0 & 0 & 2\kappa\end{pmatrix},$$

and $\mathcal{U}_3$ is not semi-symmetric.

Henceforth we proved the following theorem.

Theorem 6.6. *Let M be an H-almost Kenmotsu 3-manifold. If M is proper semi-symmetric, then it is a generalized almost Kenmotsu (κ,μ,ν)-space satisfying $\kappa^2=\lambda^2(\mu^2+\nu^2)$, $\mu^2+\nu^2\geq 4$ and $r=4\kappa$, where $\kappa<-1$ and $\mu\neq 0$.*

Assume that $d(\mathrm{tr}(h^2))\wedge\eta=0$, then we have $e_1(\lambda)=e_2(\lambda)=0$ over $\mathcal{U}_1$. Since M is a H-almost Kenmotsu manifold, $b=c=0$ and $r=-2(\lambda^2+3)$. From the proper semi-symmetric condition, we have $r=4\kappa=-4(\lambda^2+1)$ and hence we have $\lambda^2=1$. Using $\kappa^2=\lambda^2(\mu^2+\nu^2)$, we get $\lambda^2+\alpha^2=1$. So, we have $\alpha=0$ (*i.e.*, $\mu=0$) and it is a contradiction. Hence we have the following Corollary.

Corollary 6.7. *There does not exist a proper semi-symmetric H-almost Kenmotsu 3-manifold M with $d(\mathrm{tr}(h^2))\wedge\eta=0$.*

Note that $\mathbb{H}^2(-4)\times\mathbb{R}$ is locally symmetric and it is an almost Kenmotsu $(-2,0,2)$-space. This example satisfies $\kappa^2=\lambda^2(\mu^2+\nu^2)$, $\mu^2+\nu^2=4$ and $r=4\kappa$.

6.5. **The system of pseudo-symmetry.** In this section we study pseudo-symmetry of almost Kenmotsu 3-manifolds.

First, the derivative $(e_1\wedge e_2)\cdot S$ is computed as

$$\begin{aligned}((e_1\wedge e_2)S)e_1 &= 2(\xi(\lambda)+2\lambda)e_1+4\alpha\lambda e_2+\rho_{23}e_3,\\ ((e_1\wedge e_2)S)e_2 &= 4\alpha\lambda e_1-2(\xi(\lambda)+2\lambda)e_2-\rho_{13}e_3,\\ ((e_1\wedge e_2)S)e_3 &= \rho_{23}e_1-\rho_{13}e_2.\end{aligned}$$

Hence $R(e_1,e_2)\cdot S=LR_1(e_1,e_2)\cdot S$ holds if and only if

$$(\xi(\lambda)+2\lambda)H+\rho_{13}\rho_{23}=L(\xi(\lambda)+2\lambda), \tag{25}$$

$$(\rho_{13})^2-(\rho_{23})^2-4\alpha\lambda H=-4\alpha\lambda L, \tag{26}$$

$$K_{23}\rho_{23}+(\xi(\lambda)+2\lambda)\rho_{13}=L\rho_{23}, \tag{27}$$

$$K_{13}\rho_{13}+(\xi(\lambda)+2\lambda)\rho_{23}=L\rho_{13}. \tag{28}$$

Next, the derivative $(e_2\wedge e_3)\cdot S$ is computed as

$$\begin{aligned}((e_2\wedge e_3)S)e_1 &= \rho_{13}e_2-(\xi(\lambda)+2\lambda)e_3,\\ ((e_2\wedge e_3)S)e_2 &= \rho_{13}e_1+2\rho_{23}e_2-(H-K_{13})\,e_3,\\ ((e_2\wedge e_3)S)e_3 &= -(\xi(\lambda)+2\lambda)e_1-(H-K_{13})\,e_2-2\rho_{23}e_3.\end{aligned}$$

Thus $R(e_2,e_3)\cdot S=0$ holds if and only if (25), (27), (28) and

$$(\rho_{13})^2-(\xi(\lambda)+2\lambda)^2-K_{23}(H-K_{13})=-L\,(H-K_{13}) \tag{29}$$

hold.

Finally, the derivative $(e_3 \wedge e_1) \cdot S$ is computed as

$$\begin{aligned}((e_3 \wedge e_1)S)e_1 =& -2\rho_{13}e_1 - \rho_{23}e_2 + (H - K_{23})\, e_3,\\ ((e_3 \wedge e_1)S)e_2 =& -\rho_{23}e_1 + (\xi(\lambda) + 2\lambda)e_3,\\ ((e_3 \wedge e_1)S)e_3 =& (H - K_{23})\, e_1 + (\xi(\lambda) + 2\lambda)e_2 + 2\rho_{13}e_3.\end{aligned}$$

Thus $R(e_3, e_1) \cdot S = L(e_3 \wedge e_1) \cdot S$ if and only if (25), (27), (28) and

$$(\rho_{23})^2 - (\xi(\lambda) + 2\lambda)^2 - K_{13}(H - K_{23}) = -L\,(H - K_{23})\,. \tag{30}$$

hold. The equation (26) is a consequence of (29) and (30). Hence we obtain the the following criterion.

Proposition 6.8. *An almost Kenmotsu* 3*-manifold* M *is pseudo-symmetric if and only if it satisfies the system* (25)*,* (27)*,* (28)*,* (29) *and* (30)*.*

6.6. **Pseudo-symmetric H-almost Kenmotsu 3-manifolds.** Let us consider an H-almost Kenmotsu 3-manifold M. As we mentioned before, every Kenmotsu 3-manifold is H-almost Kenmotsu and pseudo-symmetric. Let us assume that M is H-almost Kenmotsu. Then the pseudo-symmetry of M is the system:

$$(\xi(\lambda) + 2\lambda)(L - H) = 0, \tag{31}$$

$$(\xi(\lambda) + 2\lambda)^2 + K_{23}(H - K_{13}) = L\,(H - K_{13})\,, \tag{32}$$

$$(\xi(\lambda) + 2\lambda)^2 + K_{13}(H - K_{23}) = L\,(H - K_{23}) \tag{33}$$

over $\mathcal{U}$.

First, on $\mathcal{U}_0$, $h = 0$ and hence $\mathcal{U}_0$ is Kenmotsu and hence it is pseudo-symmetric with $L = -1$. Let us investigate the system of pseudo-symmetry over $\mathcal{U}_0$. The above system reduces to $L = -1$ or $r = -6$. The first case is just a consequence of $h = 0$. In the latter case, $\mathcal{U}_0$ is of constant curvature -1, especially locally symmetric. Thus we observe that the system of pseudo-symmetry is valid on $\mathcal{U}_0$.

Next, we assume that $\mathcal{U}_1$ is non-empty. From the last two equations, we deduce that $\alpha(L - H) = 0$ on $\mathcal{U}_1$. Since we assumed that M is H-almost Kenmotsu, $\mathcal{U}_1$ satisfies the generalized almost Kenmotsu (κ, μ, ν)-condition with

$$\kappa = -1 - \lambda^2, \quad \mu = -2\alpha, \quad \lambda\nu = \xi(\lambda) + 2\lambda.$$

Let us consider open sets

$$\begin{aligned}\mathcal{U}_4 =& \{p \in \mathcal{U}_1 \mid H = L \text{ in a neighborhood of } p\,\},\\ \mathcal{U}_5 =& \{p \in \mathcal{U}_1 \mid H \neq L \text{ in a neighborhood of } p\,\}.\end{aligned}$$

Then $\mathcal{U}_4 \cup \mathcal{U}_5$ is open and dense in $\mathcal{U}_1$.

(1) On $\mathcal{U}_4$ we have $L = H = \frac{r}{2} + 2(\lambda^2 + 1)$ and

$$S = \begin{pmatrix} \rho_{11} & \lambda\nu & 0 \\ \lambda\nu & \rho_{22} & 0 \\ 0 & 0 & 2\kappa \end{pmatrix},$$

where

$$\rho_{11} = \frac{r}{2} - \kappa + \lambda\mu, \quad \rho_{22} = \frac{r}{2} - \kappa - \lambda\mu.$$

Then the principal Ricci curvatures are computed as

$$\rho_1 = \frac{r}{2} - \kappa + \lambda\sqrt{\mu^2+\nu^2}, \quad \rho_2 = \frac{r}{2} - \kappa - \lambda\sqrt{\mu^2+\nu^2}, \quad \rho_3 = 2\kappa.$$

The possibilities for $\mathcal{U}_4$ to be pseudo-symmetric are (i) $\rho_1 = \rho_3 \neq \rho_2$ or (ii) $\rho_2 = \rho_3 \neq \rho_1$. For the first case we have $L = H = \kappa - \lambda\sqrt{\mu^2+\nu^2}$. For the second case, we have $L = H = \kappa + \lambda\sqrt{\mu^2+\nu^2}$.

(2) On $\mathcal{U}_5$, we have $\alpha = 0$, equivalently $\mu = 0$. From (22), $\lambda\nu = \xi(\lambda) + 2\lambda = 0$. Thus $\mathcal{U}_5$ is a generalized $(\kappa, 0)$-space. Hence the Ricci operator has the form

$$S = \begin{pmatrix} \frac{r}{2} - \kappa & 0 & 0 \\ 0 & \frac{r}{2} - \kappa & 0 \\ 0 & 0 & 2\kappa \end{pmatrix},$$

and $\mathcal{U}_5$ is pseudo-symmetric and $L = \kappa$. If $r \neq 6\kappa$ then it is proper pseudo-symmetric.

Hence we have

Theorem 6.9. *Let M be an H-almost Kenmotsu 3-manifold. If M is proper pseudo-symmetric, then M is locally isomorphic to one of the following spaces*:

(1) *a Kenmotsu 3-manifold of non-constant curvature. In this case $L = -1$.*
(2) *a generalized almost Kenmotsu $(\kappa, 0, 0)$-space with $\kappa < -1$, $r \neq 6\kappa$ and $L = \kappa$.*
(3) *a generalized almost Kenmotsu (κ, μ, ν)-space with $\kappa < -1$ and $L = H = \kappa \pm \lambda\sqrt{\mu^2+\nu^2}$, where the holomorphic sectional curvature $H = \frac{r}{2} - 2\kappa \neq 0$.*

It should be remarked that there are no examples of proper semi-symmetric almost Kenmotsu 3-manifolds in this list because of Theorem 6.6. Next, there exits a proper pseudo-symmetric almost Kenmotsu 3-manifold belonging to the second class. Indeed, the generalized almost Kenmotsu $(\kappa, 0)$-space exhibited in Example 5.17 is proper pseudo-symmetric, weakly η-Einstein and $r = 2(\kappa - 2)$. Moreover Example 5.17 satisfies $d\kappa \wedge \eta = 0$. This facts motives us to require an additional condition $d\kappa \wedge \eta = 0$ for proper pseudo-symmetric almost Kenmotsu 3-manifolds. Under the additional conditions $d\kappa \wedge \eta = 0$ and $\kappa < -1$, proper pseudo-symmetric almost Kenmotsu 3-manifolds in the class (2) and (3) satisfy $a = b = 0$ and $r = 2\kappa - 4$. Note that every Kenmotsu 3-manifold satisfies $R(X,Y)\xi = (-1)\{\eta(Y)X - \eta(X)Y\}$. Thus it is regarded as a (κ, μ, ν)-space with $\kappa = -1$ and $h = 0$. Hence $d\kappa \wedge \eta = 0$ holds.

Corollary 6.10. *Let M be an H-almost Kenmotsu 3-manifold. If M is proper pseudo-symmetric and satisfies $d(\mathrm{tr}(h^2)) \wedge \eta = 0$, then M is locally isomorphic to one of the following spaces*:

(1) *a Kenmotsu 3-manifold of non-constant curvature. In this case $L = -1$.*
(2) *a generalized almost Kenmotsu $(\kappa, 0, 0)$-space with $\kappa < -1$ and $L = \kappa$.*
(3) *a generalized almost Kenmotsu (κ, μ, ν)-space with $\kappa < -1$ and $L = H = \kappa \pm \lambda\sqrt{\mu^2+\nu^2}$, where the holomorphic sectional curvature $H = -\kappa - 2 \neq 0$.*

In particular, if M is an H-almost Kenmotsu 3-manifold with constant $\mathrm{tr}\, h^2$, then M locally satisfies the generalized almost Kenmotsu (κ, μ, ν)-condition with

$$\kappa = -1 - \lambda^2, \quad \mu = -2\alpha, \quad \nu = 2.$$

First, there does not exist the first and second case of Corollary 6.10 in H-almost Kenmotsu 3-manifold with constant $\mathrm{tr}\, h^2$. Next, for the third case of Corollary 6.10, since $\lambda \neq 0$, from $\kappa \pm \lambda\sqrt{\mu^2+\nu^2} = -\kappa - 2$, we have $\lambda^2 - \alpha^2 = 1$ and $\alpha \neq 0$. This corresponds to the second case of Theorem 5.10. Hence we have

Corollary 6.11. *Let M be an H-almost Kenmotsu 3-manifold. If M is proper pseudo-symmetric and $\lambda^2 := \operatorname{tr} h^2/2$ is a constant on M, then M is locally isomorphic to the strictly almost Kenmotsu Lie group $G(\lambda,\alpha)$ satisfying $\lambda^2 - \alpha^2 = 1$ and $\alpha \neq 0$. It is a pseudo-symmetric space of constant type. The simply connected one is realized as the linear Lie group*

$$\left\{ \left(\begin{array}{ccc} e^{-z}\cosh z & (\lambda-\alpha)e^{-z}\sinh z & x \\ (\lambda+\alpha)e^{-z}\sinh z & e^{-z}\cosh z & y \\ 0 & 0 & 1 \end{array} \right) \middle| \; x,y,z \in \mathbb{R} \right\}.$$

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(J. I.) Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan
Email address: `inoguchi@math.sci.hokudai.ac.jp`

(J.-E. L.) Department of Mathematics, Chonnam National University, Gwangju 61186, Korea
Email address: `jieunlee12@jnu.ac.kr`