MINIMAL SUBMANIFOLDS IN $\mathbb{H}^3 \times \mathbb{E}^1$

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Abstract. In this paper minimal invariant, minimal totally real and minimal CR-submanifolds in the 4-dimensional model space $\mathbb{H}^3 \times \mathbb{E}^1$ equipped with standard globally conformal Kähler structures are studied.

Introduction

The space $\mathbb{H}^3 \times \mathbb{E}^1$ is one of the 19 model spaces in four dimensional geometries (see [12]). Among these, the following 14 spaces admit complex structure compatible to the geometric structure [23, Theorem 2.1].

Complex space form	Hermitian symmetric	Kähler	GCK Vaisman	GCK non-Vaisman
$\mathbb{E}^4$, $\mathbb{C}P^2$, $\mathbb{C}H^2$	$\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{E}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$	F^4	$\mathbb{S}^3 \times \mathbb{E}^1$, $\mathrm{Nil}_3 \times \mathbb{E}^1$ $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$	Sol_0^4,Sol_1^4

There are five locally conformal Käher model spaces (LCK model spaces, in short), $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathrm{Nil}_3 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$, Sol_0^4 and Sol_1^4. Since the model spaces are simply connected, all of these are globally conformal Kähler surfaces (GCK surfaces, in short). On every LCK surface, there exist two peculiar vector field; Lee field and anti-Lee field. LCK surfaces with a parallel Lee field are called *Vaisman surfaces*. Among the above GCK model spaces, $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathrm{Nil}_3 \times \mathbb{E}^1$ and $\widetilde{\mathrm{SL}}_2\mathbb{R} \times \mathbb{E}^1$ are Vaisman. On the other hand both Lee field and anti-Lee field of Sol_0^4 and Sol_1^4 are *non-parallel*. Thus there are no (non-Kähler) GCK model spaces with a parallel anti-Lee field.

The model space $\mathbb{H}^3 \times \mathbb{E}^1$ admits orthogonal complex structures but these are *not* compatible to the geometric structure. Thus the space $\mathbb{H}^3 \times \mathbb{E}^1$ does not belong to the list of Wall consisting of 14 spaces.

It should be remarked that the hyperbolic 3-space $\mathbb{H}^3$ is identified with the solvable part $\mathrm{B}_2^+\mathbb{C}$ of $\mathrm{SL}_2\mathbb{C}$. As a result $\mathbb{H}^3 \times \mathbb{E}^1$ is realized as a solvable Lie group $\mathrm{B}_2^+\mathbb{C} \times \mathbb{R}$ equipped with a left invariant locally symmetric Riemannian metric. As a Riemannian symmetric space, $\mathbb{H}^3 \times \mathbb{E}^1$ is represented as a coset manifold $(\mathrm{SL}_2\mathbb{C} \times \mathbb{R})/\mathrm{SU}_2$.

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As we mentioned, our ambiant model space does not admit complex structures compatible to the geometric structure, *i.e.,* orthogonal complex structures invariant under $\mathrm{SL}_2\mathbb{C}\times\mathbb{R}$-action [24, Theorem 1.1]. However there exist some left invariant orthogonal complex structures.

In locally conformal Kähler geometry (LCK geometry, in short), Kashiwada [15] showed that the Riemannian product $\overline{N}\times\mathbb{E}^1$ of a β-Kenmotsu manifold $\overline{N}$ and the Euclidean line admits a locally conformal Kähler structure with a *parallel anti-Lee field*. Since the hyperbolic 3-space $\mathbb{H}^3$ admits left invariant (±1)-Kenmotsu structures, $\mathbb{H}^3\times\mathbb{E}^1$ has GCK structures induced from the (±1)-Kenmotsu structures of $\mathbb{H}^3$. Matsushita [18] obtained a pair (J,J') of orthogonal complex structures on $\mathbb{H}^3\times\mathbb{E}^1$. Both complex structures are left invariant and have opposite orientation to each other. One can see that Matsushita's complex structures coincide with the complex structures of GCK structures of $\mathbb{H}^3\times\mathbb{E}^1$. From an LCK geometric viewpoint, the GCK surface $\mathbb{H}^3\times\mathbb{E}^1$ is understood as a standard example of a GCK-surface with a parallel anti-Lee field.

Wall's list motivates us to develop submanifold geometry in GCK-model spaces with respect to the complex structure. The study of submanifolds in GCK model spaces in terms of complex structure began with the study of J-trajectories. By a J-trajectory, we mean a curve γ in an almost Hermitian manifold (M,J,g) satisfying the *J-trajectory equation*:

$$\nabla_{\dot\gamma}\dot\gamma = qJ\dot\gamma,$$

where ∇ is the Levi-Civita connection and q a constant (called the *charge* or *strength*). Obviously, J-trajectories with $q=0$ are geodesics. In particular, when the ambient space is an almost Kähler manifold, then J-trajectories are nothing but Kähler magnetic curves.

Ateş, Munteanu and Nistor [1] studied J-trajectories in $\mathbb{S}^3\times\mathbb{E}^1$. The model space $\mathbb{S}^3\times\mathbb{E}^1$ is a Vaisman surface. Inspired by the article [1], Lee and the second named author of the present paper developed a general theory of J-trajectories in LCK-surfaces [14]. The present authors investigated J-trajectories in GCK model spaces Sol^4_0 [8] and Sol^4_1 [9], respectively. In consecutive works [10, 11], we studied holomorphic curves, totally real surfaces and real hypersurfaces in Sol^4_0 and Sol^4_1 that are minimal. We found motivation for these works in Munteanu's article [19] where the author considered minimal submanifolds in $\mathbb{R}^4$ equipped with a certain non-Vaisman GCK structure.

Our primary interest is to open up submanifold geometry of $\mathbb{H}^3\times\mathbb{E}^1$ in term of GCK structure. As a first step, J-trajectories in $\mathbb{H}^3\times\mathbb{E}^1$ are investigated in our previous work [13]. The purpose of the present paper is to study minimal surfaces and minimal hypersurfaces in $\mathbb{H}^3\times\mathbb{E}^1$. More precisely we study minimal invariant surfaces, minimal totally real surfaces and minimal real hypersurfaces in $\mathbb{H}^3\times\mathbb{E}^1$. For submanifold geometry in $\mathbb{H}^3\times\mathbb{E}^1$ from a purely Riemannian geometric viewpoint, we refer to [3, 4, 21].

Notation: Throughout this article, we use the following notational convention.

Let M be a submanifold in $\mathbb{H}^3\times\mathbb{E}^1$. For any vector field V on $\mathbb{H}^3\times\mathbb{E}^1$, we denote the components of V tangent to M and normal to M by $\tan(V)$ and $\mathrm{nor}(V)$, respectively.

1. Preliminaries

1.1. **LCK surfaces.** Let $N = (N, J, g)$ be a Hermitian surface with fundamental 2-form Ω:

$$\Omega(X, Y) = g(X, JY).$$

Then N is said to be a *locally conformal Kähler surface* (LCK surface, in short) if there exits an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of N and a family of smooth functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that

$$d(e^{-\sigma_\alpha}\Omega) = 0 \;\text{ on }\; U_\alpha$$

for all α. Namely the local conformal change $(U_\alpha, J|_{U_\alpha}, e^{-\sigma_\alpha} g)$ is Kähler.

In case $U_\alpha = N$, then N is said to be a *globally conformal Kähler surface* (GCK, in short). Notice that any simply connected LCK manifold is GCK.

On an LCK manifold $\omega = d\sigma_\alpha$ is globally defined and satisfies

$$d\Omega = \omega \wedge \Omega. \tag{1.1}$$

The closed 1-form ω is called the *Lee form*.

Let us denote by B the vector field metrically dual to ω and call it the *Lee field* or *Lee vector field*. The vector field $A = JB$ is called the *anti-Lee field* (or *anti-Lee vector field*). An LCK surface is called a *Vaisman surface* if its Lee field is parallel.

Even if the conformal change $e^{-\sigma_\alpha} g$ of g is locally defined, the Levi-Civita connection D of $e^{-\sigma_\alpha} g$ is globally defined on N. The connection D is called the *Weyl connection* of (N, J, g) and it is given by

$$D_X Y = \nabla_X Y - \frac{1}{2}\Big(\omega(X)Y + \omega(Y)X - g(X, Y)B\Big). \tag{1.2}$$

Since $DJ = 0$, we get

$$(\nabla_X J)Y = \frac{1}{2}\Big(\omega(JY)X - \omega(Y)JX + g(X, Y)A - \Omega(X, Y)B\Big). \tag{1.3}$$

1.2. **LCK manifolds with parallel anti-Lee field.** Let $\overline{N}$ be a 3-manifold. A quartet $(\varphi, \xi, \eta, \bar{g})$ consisting of an endomorphism field φ, a vector field ξ, a 1-form η and a Riemannian metric $\bar{g}$ is said to be an *almost contact Riemannian structure* if it satisfies

$$\varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \varphi = 0$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A 3-manifold $(\bar{N}, \varphi, \xi, \eta, \bar{g})$ together with an almost contact Riemannian structure is called an *almost contact Riemannian 3-manifold*. The 2-form Φ defined by

$$\Phi(X, Y) = \bar{g}(X, \varphi Y)$$

is called the *fundamental 2-form* of $\overline{N}$.

We extend φ to the Riemannian product $N = (\overline{N} \times \mathbb{E}^1, \bar{g} + dw^2)$, by the rule

$$\begin{cases} JX = \varphi X, \quad X \in \mathfrak{X}(\overline{N}),\ X \perp \xi, \\ J\xi = \frac{\partial}{\partial w}, \\ J\frac{\partial}{\partial w} = -\xi. \end{cases} \tag{1.4}$$

An almost contact Riemannian 3-manifold $(\overline{N}, \varphi, \xi, \eta, \bar{g})$ is said to be *normal* if J is integrable.

Definition 1.1. An almost contact Riemannian 3-manifold $(\overline{N}, \varphi, \xi, \eta, \bar{g})$ is said to be an *almost β-Kenmotsu 3-manifold* if

$$d\eta = 0, \quad d\Phi = 2\beta\eta \wedge \Phi.$$

Here β is a non-zero constant. An almost β-Kenmotsu 3-manifold is called a *β-Kenmotsu* 3-manifold if it is normal.

1-Kenmotsu 3-manifolds are simply called Kenmotsu 3-manifolds. The hyperbolic 3-space $\mathbb{H}^3$ is a standard example of (± 1)-Kenmotsu manifold (see the next section).

Kashiwada [15] proved the following fact.

Theorem 1.1. *Let N be an LCK surface with parallel anti-Lee field A. Then the following integrability result holds.*

- *The distribution $\mathcal{D}_1$ defined by the Pfaff equation $\omega \circ J = 0$ is integrable. The integral hypersurface are non-compact totally geodesic real hypersurfaces with an induced β-Kenmotsu structure, where $|B| = 2|\beta|$.*
- *The distribution $\mathcal{D}_2$ defined by the Pfaff equation $\omega = 0$ and $\omega \circ J = 0$ is integrable. The integral surfaces are totally umbilical and J-invariant surfaces.*

Such LCK structures naturally appear on the product of β-Kenmotsu 3-manifolds and the Euclidean line.

Let $\overline{N}$ be a β-Kenmotsu 3-manifold, then the Riemannian product $N = (\overline{N} \times \mathbb{E}^1, \bar{g} + dw^2)$ equipped with the complex structure J defined by (1.4) is an LCK surface. The exterior derivative $d\Omega$ of the fundamental 2-form satisfies $d\Omega = 2(\beta\eta) \wedge \Omega$. Hence the Lee form is $\omega = 2\beta\eta$. The Lee field and anti-Lee field are

$$B = 2\beta\xi, \quad A = 2\beta\frac{\partial}{\partial w}.$$

The covariant derivative ∇J is is given by [13, Proposition 2.3]:

$$(\nabla_X J)Y = -\beta(\eta(Y)\varphi X + \Phi(X,Y)\xi - g(\varphi X, \varphi Y)\partial_w).$$

1.3. Curve theory.

Definition 1.2. If γ is a curve in a Riemannian n-manifold (N, g), parametrized by arc length t, we say that γ is a *Frenet curve of osculating order* $r(\leq n)$ when there exist orthonormal vector fields $E_1, E_2, \dots, E_r$ along γ such that

$$\begin{aligned}&\dot{\gamma} = E_1, \nabla_{\dot{\gamma}}E_1 = \kappa_1 E_2, \nabla_{\dot{\gamma}}E_2 = -\kappa_1 E_1 + \kappa_2 E_3, \dots, \\ &\nabla_{\dot{\gamma}}E_{r-1} = -\kappa_{r-2}E_{r-2} + \kappa_{r-1}E_r, \nabla_{\dot{\gamma}}E_r = -\kappa_{r-1}E_{r-1},\end{aligned} \tag{1.5}$$

where $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ are positive C^∞ functions of t. The function κ_j is called the j-th *curvature* of γ.

A *geodesic* is regarded as a Frenet curve of osculating order 1. A *Riemannian circle* is defined as a Frenet curve of osculating order 2 with *constant* κ_1. A *helix* of order r is a Frenet curve of osculating order r, such that all the curvatures $\kappa_1, \kappa_2, \dots, \kappa_{r-1}$ are constant.

For Frenet curves in Hermitian surfaces, we recall the following notion:

Definition 1.3. Let $\gamma(t)$ be a Frenet curve of osculating order $r > 0$ in a Hermitian surface (N, J, g). The *complex torsions* τ_{ij} $(1 \leq i < j \leq r \leq 4)$ are smooth functions along γ defined by $\tau_{ij} = g(E_i, JE_j)$ A helix of order r in (N, J, g) is said to be a *holomorphic helix* of order r if all complex torsions are constant. In particular holomorphic helices of order 2 are called *holomorphic circles.*

A smooth curve $\gamma(u)$ in an almost Hermitian 4-manifold $N = (N, J, g)$ is said to be a *J-trajectory* with strength q if it satisfies

$$\nabla_{\dot\gamma}\dot\gamma = qJ\dot\gamma \tag{1.6}$$

for some constant q. One can see that every J-trajectory has constant speed. Thus hereafter we parametrize J-trajectories by arc length parameter t.

Note that when N is an almost Kähler 4-manifold, then its fundamental 2-form Ω is regarded as a static magnetic field on N. J-trajectories are called *Kähler magnetic trajectories* with respect to the magnetic field $-\Omega$ (called the Kähler magnetic field).

Proposition 1.1 ([13])**.** *Let γ be a J-trajectory with strength $q \neq 0$ parameterized by arc length in an LCK surface N with parallel anti-Lee field. Then the first and second curvatures of γ are given by*

$$\kappa_1 = |q|, \quad \kappa_2 = \frac{1}{2}\sqrt{|A|^2 - \omega(\dot\gamma)^2 - \omega(J\dot\gamma)^2}.$$

- *The osculating order is at most* 3.
- *A J-trajectory is a Frenet curve of osculating order* 2 *if and only if* $(\nabla_{\dot\gamma}\omega)\dot\gamma = 0$.
- *Assume that γ is a J-trajectory of osculating order* 3. *If* $\omega(\dot\gamma) = 0$, *then the second curvature κ_2 is constant.*
- *All the complex torsions of a non-geodesic J-trajectory are constant.*

2. The model space $\mathbb{H}^3 \times \mathbb{E}^1$

2.1. **Hyperbolic 3-space.** Let us realize the hyperbolic 3-space $\mathbb{H}^3$ as the half space

$$\mathbb{R}^3_+ = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid x^1 > 0\}$$

equipped with the Poincaré metric

$$\bar g = \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(x^1)^2}.$$

The hyperbolic 3-space $\mathbb{H}^3$ is represented by $\mathrm{SL}_2\mathbb{C}/\mathrm{SU}_2$ as a Riemannian symmetric space. Let us recall the polar decomposition $\mathrm{SL}_2\mathbb{C} = \mathrm{B}_2^+\mathbb{C} \cdot \mathrm{SU}_2$ of $\mathrm{SL}_2\mathbb{C}$. Here $\mathrm{B}_2^+\mathbb{C}$ is a solvable Lie group given by

$$\mathrm{B}_2^+\mathbb{C} = \left\{ \begin{pmatrix} \sqrt{x^1} & (x^2 + ix^3)/\sqrt{x^1} \\ 0 & 1/\sqrt{x^1} \end{pmatrix} \mid x^1, x^2, x^3 \in \mathbb{R}, x^1 > 0 \right\}, \quad i = \sqrt{-1}.$$

The solvable Lie group $\mathrm{B}_2^+\mathbb{C}$ is identified with $\mathbb{H}^3$ via the imbedding

$$(x^1, x^2, x^3) \longmapsto \begin{pmatrix} \sqrt{x^1} & (x^2 + ix^3)/\sqrt{x^1} \\ 0 & 1/\sqrt{x^1} \end{pmatrix}.$$

The Lie algebra $\mathfrak{b}_2^+\mathbb{C}$ of $\mathrm{B}_2^+\mathbb{C}$ is spanned by the basis

$$\bar{e}_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{e}_3 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

The left invariant vector fields derived from $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$ are given by

$$\bar{e}_1 = x^1\frac{\partial}{\partial x^1}, \quad \bar{e}_2 = x^1\frac{\partial}{\partial x^2}, \quad \bar{e}_3 = x^1\frac{\partial}{\partial x^3},$$

respectively. The left invariant Maurer-Cartan form is computed as

$$\begin{pmatrix} \sqrt{x^1} & (x^2+ix^3)/\sqrt{x^1} \\ 0 & 1/\sqrt{x^1} \end{pmatrix}^{-1} d\begin{pmatrix} \sqrt{x^1} & (x^2+ix^3)/\sqrt{x^1} \\ 0 & 1/\sqrt{x^1} \end{pmatrix} = \bar{\vartheta}^1\bar{e}_1 + \bar{\vartheta}^2\bar{e}_2 + \bar{\vartheta}^3\bar{e}_3,$$

where

$$\bar{\vartheta}^1 = \frac{dx^1}{x^1}, \quad \bar{\vartheta}^2 = \frac{dx^2}{x^1}, \quad \bar{\vartheta}^3 = \frac{dx^3}{x^1}.$$

The Poincaré metric of $\mathbb{H}^3$ is characterized as the left invariant metric determined by the condition $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is orthonormal. Hereafter we orient $\mathbb{H}^3$ by the volume element

$$dv_{\bar{g}} = \frac{1}{(x^1)^3}dx^1 \wedge dx^2 \wedge dx^3.$$

Let us parametarize the Lie algebra $\mathfrak{b}_2^+\mathbb{C}$ as

$$\mathfrak{b}_2^+\mathbb{C} = \left\{ \begin{pmatrix} u^1/2 & u^2+iu^3 \\ 0 & -u^1/2 \end{pmatrix} \mid u^1, u^2, u^3 \in \mathbb{R} \right\}.$$

Then the exponential map is surjective and given explicitly by

$$\exp\begin{pmatrix} u^1/2 & u^2+iu^3 \\ 0 & -u^1/2 \end{pmatrix} = \begin{pmatrix} e^{u^1/2} & (u^2+iu^3)(e^{u^1}-1)/u^1 \\ 0 & e^{-u^1/2} \end{pmatrix}.$$

Note that $\lim_{u^1\to 0}\{(e^{u^1}-1)/u^1\} = 1$.

The polar decomposition of $\mathrm{SL}_2\mathbb{C}$ induces a *dressing action* on $\mathbb{H}^3$. One can verify the following lemma.

Lemma 2.1. *Let G be a Lie group which admits a Lie group decomposition $G = G_1 \cdot G_2$. Then G acts on the Lie subgroup G_1 by the dressing action*

$$g\sharp h = (gh)_1, \quad g \in G, \quad h \in G_1.$$

Here $(gh)_1$ denotes the G_1-component of gh.

Now we apply this lemma to the hyperbolic 3-space. Take a matrix $A \in \mathrm{SL}_2\mathbb{C}$ and $X \in \mathrm{B}_2^+\mathbb{C}$. Decompose AX as $AX = (AX)_B(AX)_U$ according to the polar decomposition, then the dressing action of $\mathrm{SL}_2\mathbb{C}$ is defined by $(A, X) \longmapsto A\sharp X := (AX)_B$. The dressing action is transitive. For any $X \in \mathrm{B}_2^+\mathbb{R}$, there exists a matrix $A \in \mathrm{SL}_2\mathbb{C}$. In fact, take any $U \in \mathrm{SU}_2$ and set $A = XU$, then $A\sharp I = X$. The stabilizer at the origin I is the special unitary group SU_2. Thus we obtain a homogeneous space representation $\mathbb{H}^3 = \mathrm{SL}_2\mathbb{C}/\mathrm{SU}_2$, again.

2.2. **The (-1)-Kenmotsu structure.** We define left invariant endomorphism fields $\varphi^{\mathsf{H}}_{\pm}$ on $\mathbb{H}^3$ by

$$\varphi^{\mathsf{H}}_{+}\bar{e}_1 = 0, \quad \varphi^{\mathsf{H}}_{+}\bar{e}_2 = e_3, \quad \varphi^{\mathsf{H}}_{+}\bar{e}_3 = -e_2,$$
$$\varphi^{\mathsf{H}}_{-}\bar{e}_1 = 0, \quad \varphi^{\mathsf{H}}_{-}\bar{e}_2 = -e_3, \quad \varphi^{\mathsf{H}}_{-}\bar{e}_3 = e_2.$$

Next we set $\eta_{\mathsf{H}} = \bar{\vartheta}^1$ and $\xi_{\mathsf{H}} = \bar{e}_1$. Then we obtain left invariant almost contact structures $(\varphi^{\mathsf{H}}_{+}, \xi_{\mathsf{H}}, \eta_{\mathsf{H}}, \bar{g})$ and $(\varphi^{\mathsf{H}}_{-}, \xi_{\mathsf{H}}, \eta_{\mathsf{H}}, \bar{g})$. Let us denote by $\Phi^{\mathsf{H}}_{\pm}$ the fundamental 2-form of $(\varphi^{\mathsf{H}}_{\pm}, \xi_{\mathsf{H}}, \eta_{\mathsf{H}}, \bar{g})$;

$$\Phi^{\mathsf{H}}_{\pm}(X, Y) = \bar{g}(X, \varphi^{\mathsf{H}}_{\pm}Y), \quad X, Y \in \mathfrak{X}(\mathbb{H}^3).$$

Then we have

$$d\Phi^{\mathsf{H}}_{\pm} = (-2)\eta_{\mathsf{H}} \wedge \Phi^{\mathsf{H}}_{\pm}$$

and hence both the structures are (-1)-Kenmotsu.

2.3. **The product space.** Now let us consider the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. This Riemannian product is expressed as a half space

$$\mathbb{R}^4_+ = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 \mid x^1 > 0\}$$

equipped with a homogenous Riemannian metric

$$g = \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(x^1)^2} + (dx^4)^2.$$

Take a global orthonormal frame field

$$e_1 = x^1 \frac{\partial}{\partial x^1}, \ e_2 = x^1 \frac{\partial}{\partial x^2}, \ e_3 = x^1 \frac{\partial}{\partial x^3}, \ e_4 = \frac{\partial}{\partial x^4}, \tag{2.1}$$

Then the Levi-Civita connection ∇ is described as [20, (2.2)]:

$$\nabla_{e_2} e_2 = \nabla_{e_3} e_3 = e_1, \quad \nabla_{e_2} e_1 = -e_2, \quad \nabla_{e_3} e_1 = -e_3.$$

Other components are all zero.

Note that $\mathbb{H}^3 \times \mathbb{E}^1$ is represented by $(\mathrm{SL}_2\mathbb{C} \times \mathbb{R})/\mathrm{SU}_2$ as a Riemannian symmetric space. An almost complex structure J on $\mathbb{H}^3 \times \mathbb{E}^1 = (\mathrm{SL}_2\mathbb{C} \times \mathbb{R})/\mathrm{SU}_2$ is said to be *compatible to the geometric structure of* $\mathbb{H}^3 \times \mathbb{E}^1$ if it is invariant under the $\mathrm{SL}_2\mathbb{C} \times \mathbb{R}$-action.

2.4. **The solvable Lie group model.** The product manifold $\mathbb{H}^3 \times \mathbb{R}$ is identified with the linear Lie group

$$\left\{ \begin{pmatrix} \sqrt{x^1} & (x^2 + ix^3)/\sqrt{x^1} & 0 \\ 0 & 1/\sqrt{x^1} & 0 \\ 0 & 0 & x^4 \end{pmatrix} \mid x^1, x^2, x^3, x^4 \in \mathbb{R}, x^1 > 0 \right\} = \mathrm{B}^+_2\mathbb{C} \times \mathbb{R}.$$

The global orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ is left invariant and span the Lie algebra $\mathfrak{b}^+_2\mathbb{C} \oplus \mathbb{R}$. At the origin, $\{e_1, e_2, e_3, e_4\}$ has the values:

$$e_1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The solvable Lie group $\mathrm{B}^+_2\mathbb{C} \times \mathbb{R}$ is realized as the half space $\mathbb{R}^4_+$ with multiplication

$$(x^1, x^2, x^3, x^4)(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \tilde{x}^4) = \left(x^1\tilde{x}^1, x^2 + x^1\tilde{x}^2, x^3 + x^1\tilde{x}^3, x^4 + \tilde{x}^4 \right).$$

2.5. **Almost complex structure.** We extend the endomorphism fields $\varphi_\pm$ to almost complex structures $J_\pm$ on the product manifold $\mathbb{H}^3\times\mathbb{R}$ in the following manner (see (1.4)):

$$J_\pm\bar{X}=\varphi_\pm^{\mathsf{H}}\bar{X},\quad \bar{X}\in\mathfrak{X}(\mathbb{H}^3),\ X\perp\xi,\quad J_\pm\xi_{\mathsf{H}}=\frac{\partial}{\partial x^4},\quad J_\pm\frac{\partial}{\partial x^4}=-\xi_{\mathsf{H}}.$$

Explicitly, we have

$$J_\pm e_1=e_4,\quad J_\pm e_2=\pm e_3,\quad J_\pm e_3=\mp e_2,\quad J_\pm e_4=-e_1. \tag{2.2}$$

One can see that J_+ is the complex structure J and J_- is the opposite one J' of Matsushita [18], respectively.

Denote by $\Omega_\pm$ the fundamental 2-form of $(\mathbb{H}^3\times\mathbb{R},J_\pm,g)$, then we have

$$d\Omega_\pm=\frac{-2dx^1}{x^1}\wedge\Omega_\pm.$$

This formula shows that $(\mathbb{H}^3\times\mathbb{R},J_\pm,g)$ is a globally conformal Kähler surface with Lee form $\omega=-2\frac{dx^1}{x^1}$. The corresponding Lee field $B_\pm$ and anti-Lee field $A_\pm$ are

$$A_\pm=-2\frac{\partial}{\partial x^4}=:A,\quad B_\pm=-2\xi_{\mathsf{H}}=-2x^1\frac{\partial}{\partial x^1}=:B.$$

One can see that A is parallel.

The identity component of the automorphism group of the GCK structure $(J_\pm,g)$ is $\mathbb{H}^3\times\mathbb{E}^1=\mathrm{B}_2^+\mathbb{C}\times\mathbb{R}$.

Thus $J_\pm$ are *not* compatible to the geometric structure of $\mathbb{H}^3\times\mathbb{E}^1$.

Remark 1. The notion of 4-dimensional geometry is introduced as follows [12, 23, 24]:

A pair (X,G) is said to be a 4-*dimensional geometry* if it satisfies

(1) X is a complete and simply connected Riemannian 4-manifold.
(2) G is a group of isometries of X.
(3) G acts transitively on X and
(4) G contains a discrete subgroup Γ with $\Gamma\backslash X$ of finite volume.

The Riemannian product $\mathbb{H}^3\times\mathbb{E}^1$ is one of the model spaces of 4-dimensional geometries. It is worth pointing out that $\mathbb{H}^3\times\mathbb{E}^1$ does not admit complex structures compatible to the 4-dimensional geometric structure [24, p. 275]. More precisely, in [23] it is proved that if $\mathbb{H}^3\times\mathbb{E}^1$ admits an orthogonal complex structure J, then it is not invariant under the $\mathrm{SL}_2\mathbb{C}\times\mathbb{R}$-action [24, p. 275].

3. J-trajectories

We start our investigation on submanifold geometry in terms of $J_\pm$ with curve theory.

3.1. **J_+-trajectories.** Let us consider a J_+-trajectory

$$\gamma(s)=(x^1(t),x^2(t),x^3(t),x^4(t))$$

in $\mathbb{H}^3 \times \mathbb{E}^1$ parametrized by arc length. Then the unit tangent vector field is given by

$$\begin{aligned}\dot\gamma(t) =& \dot x^1(t)\frac{\partial}{\partial x^1} + \dot x^2(t)\frac{\partial}{\partial x^2} + \dot x^3(t)\frac{\partial}{\partial x^3} + \dot x^4(t)\frac{\partial}{\partial x^4}\\ =&\frac{\dot x^1(t)}{x^1(t)}e_1 + \frac{\dot x^2(s)}{x^1(t)}e_2 + \frac{\dot x^3(t)}{x^1(t)}e_3 + \dot x^4(t)e_4.\end{aligned}$$

Thus we obtain

$$J_+\dot\gamma = -\dot x^4(t)e_1 - \frac{\dot x^3(t)}{x^1(t)}e_2 + \frac{\dot x^2(s)}{x^1(t)}e_3 + \frac{\dot x^1(t)}{x^1(t)}e_4. \tag{3.1}$$

Next we compute the covariant derivative $\nabla_{\dot\gamma}\dot\gamma$. It follows (see [7, p. 1504])

$$\begin{aligned}\nabla_{\dot\gamma}\dot\gamma =& \left(\frac{\ddot x^1}{x^1} + \frac{-(\dot x^1)^2 + (\dot x^2)^2 + (\dot x^3)^2}{(x^1)^2}\right)\cdot e_1 + \left(\frac{\ddot x^2}{x^1} - \frac{2\dot x^1\dot x^2}{(x^1)^2}\right)\cdot e_2\\ &+ \left(\frac{\ddot x^3}{x^1} - \frac{2\dot x^1\dot x^3}{(x^1)^2}\right)\cdot e_3 + \ddot x^4\cdot e_4.\end{aligned} \tag{3.2}$$

Hence from the J_+-trajectory equation, using (3.1) and (3.2), we obtain the following system of differential equations

$$\begin{aligned}&\frac{\ddot x^1}{x^1} + \frac{-(\dot x^1)^2 + (\dot x^2)^2 + (\dot x^3)^2}{(x^1)^2} = -q\dot x^4,\\ &\frac{\ddot x^2}{x^1} - \frac{2\dot x^1\dot x^2}{(x^1)^2} = -\frac{q\dot x^3}{x^1},\\ &\frac{\ddot x^3}{x^1} - \frac{2\dot x^1\dot x^3}{(x^1)^2} = \frac{q\dot x^2}{x^1},\\ &\ddot x^4 = \frac{q\dot x^1}{x^1}.\end{aligned} \tag{3.3}$$

Introducing substitutions $p = \dfrac{\dot x^1}{x^1},\ r = \dfrac{\dot x^2}{x^1},\ s = \dfrac{\dot x^3}{x^1},\ u = \dot x^4$ we obtain the following system of ODE's

$$\begin{aligned}&\dot p + r^2 + s^2 = -qu,\\ &\dot r = pr - qs,\\ &\dot s = qr + ps,\\ &\dot u = qp.\end{aligned} \tag{3.4}$$

The arc length condition is

$$p^2 + r^2 + s^2 + u^2 = 1. \tag{3.5}$$

From Proposition 1.1, the second curvature κ_2 is computed as

$$\kappa_2 = \sqrt{r^2 + s^2}.$$

Since $\mathbb{H}^3 \times \mathbb{E}^1$ is homogeneous, we may assume that $\gamma(t)$ starts at the origin. We set the initial condition

$$\gamma(0) = (1, 0, 0, 0), \quad \dot\gamma(0) = (p_0, r_0, s_0, u_0).$$

First we observe the case $\kappa_2 = 0$. In this case, γ lies in the totally geodesic Euclidean half plane (see Example 4.1)

$$M(1,4;0,0) = \left(\{(x^1,0,0,x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}, \frac{(dx^1)^2}{(x^1)^2} + (dx^4)^2 \right).$$

and the system is reduced to

$$\dot{p} = -qu, \quad \dot{u} = qp, \quad p^2 + u^2 = 1.$$

This system is solved as

$$p(t) = p_0 \cos(qt) - u_0 \sin(qt), \quad u(t) = p_0 \sin(qt) + u_0 \cos(qt).$$

Thus we get

$$\begin{aligned} x^1(t) &= \exp\left\{ \frac{1}{q}\left(p_0 \sin(qt) + u_0 \cos(qt)\right) - \frac{u_0}{q} \right\}, \\ x^4(t) &= \frac{1}{q}\left(-p_0 \cos(qt) + u_0 \sin(qt)\right) + \frac{p_0}{q}. \end{aligned}$$

Since $p_0^2 + u_0^2 = 1$, we set

$$p_0 = \cos\alpha, \quad u_0 = \sin\alpha,$$

then γ is rewritten as

$$x^1(t) = \exp\left\{ \frac{1}{q}\left(\sin(qt+\alpha) - \sin\alpha\right) \right\}, \quad x^4(t) = -\frac{1}{q}\left(\cos(qt+\alpha) - \cos\alpha\right).$$

Under the isometry

$$\mathsf{X} = x^4, \quad \mathsf{Y} = \log x^1,$$

The Euclidean half plane $M(1,4;0,0)$ is isometric to the Euclidean plane $\mathbb{E}^2(\mathsf{X},\mathsf{Y})$ with metric $d\mathsf{X}^2 + d\mathsf{Y}^2$. The J_+-trajectory is transformed as

$$\mathsf{X} = -\frac{1}{q}\left(\cos(qt+\alpha) - \cos\alpha\right), \quad \mathsf{Y} = \frac{1}{q}\left(\sin(qt+\alpha) - \sin\alpha\right). \tag{3.6}$$

This is a Euclidean circle of curvature $|q|$.

According to [13], the system of J_+-trajectory (3.4) can be solved. Since the case $r^2 + s^2 = 0$ was already treated, we may assume that $r^2 + s^2 \neq 0$. From the second and the third equation of (3.4) it follows,

$$2(r\dot{r} + s\dot{s}) = 2(pr^2 - qrs + qrs + ps^2) = 2p(r^2 + s^2).$$

Therefore we obtain

$$\frac{1}{r^2+s^2}\frac{d}{dt}(r^2+s^2) = 2p.$$

Integrating this equation we have

$$\log(r(t)^2 + s(t)^2) = 2\int p(t)dt = 2\int \frac{dx^1}{x^1} = 2\log x^1(t) + \text{constant}.$$

Thus,

$$r(t)^2 + s(t)^2 = \left(Rx^1(t)\right)^2, \quad R \in \mathbb{R}.$$

So we may put

$$r(t) = Rx^1(t)\cos\psi(t), \quad s(t) = Rx^1(t)\sin\psi(t)$$

for some function $\psi(t)$.

If $\cos\psi(t) = 0$, then $r(t) = 0$. From the second equation of (3.4), we get $q = 0$. It is a geodesic. Since we are interested in non-geodesic J_+-trajectories, we assume that $\cos\psi(t) \neq 0$. It follows

$$\begin{aligned}\frac{1}{\cos^2\psi}\frac{d\psi}{dt} =& \frac{d}{dt}\tan\psi = \frac{d}{dt}\frac{s}{r} = \frac{\dot{s}r - s\dot{r}}{r^2} = \frac{q(r^2+s^2)}{r^2} \\ =& q\left(1 + \frac{s^2}{r^2}\right) = q(1+\tan^2\psi) = \frac{q}{\cos^2\psi}.\end{aligned}$$

Therefore we get

$$\psi(t) = qt + \psi_0.$$

The constant ψ_0 is determined by the initial condition as

$$r_0 = R\cos\psi_0, \quad s_0 = R\sin\psi_0.$$

The x^2-coordinate and x^3-coordinate are computed as

$$\begin{aligned}x^2(t) &= \int \frac{dx^2}{dt}dt = R\int x^1(t)^2\cos(qt+\psi_0)dt, \\ x^3(t) &= \int \frac{dx^3}{dt}dt = R\int x^1(t)^2\sin(qt+\psi_0)dt.\end{aligned}$$

From the fourth equation of (3.4),

$$u(t) = \int qp(t)dt = q\log x^1(t) + u_0,$$

because we demand the initial condition $x^1(0) = 1$ and $x^4(0) = 0$. Hence

$$x^4(t) = q\int_0^t \log x^1(t)dt + u_0 t.$$

Theorem 3.1 ([13])**.** *Let $x^1(t)$ be the solution of*

$$\frac{d^2x^1}{dt^2} = \frac{1}{x^1}\left(\frac{dx^1}{dt}\right)^2 - x^1(q^2\log x^1 + qu^0 + (Rx^1)^2). \tag{3.7}$$

satisfying the initial condition $x^1(0) = 1$ and $\dot{x}^1(0) = p_0$. Then the curve $\gamma(t) = (x^1(t), x^2(t), x^3(t), x^4(t))$ defined by

$$\begin{aligned}x^2(t) =& R\int_0^t x^1(t)^2\cos(qt+\psi_0)dt, \\ x^3(t) =& R\int_0^t x^1(t)^2\sin(qt+\psi_0)dt, \\ x^4(t) =& q\int_0^t \log x^1(t)dt + u_0 t.\end{aligned}$$

is a J_+-trajectory in $\mathbb{H}^3 \times \mathbb{E}^1$.

Proof. From the first equation of (3.4),

$$\dot{p}(t) = -qu(t) - r(t)^2 - s(t)^2 = -q(q\log x^1(t) + u_0) - (Rx^1(t))^2.$$

This is rewritten as

$$\frac{d^2x^1}{dt^2} = \frac{1}{x^1}\left(\frac{dx^1}{dt}\right)^2 - x^1(q^2\log x^1 + qu^0 + (Rx^1(t))^2).$$

Thus $x^1(t)$ is a solution to this second order ODE satisfying the initial condition $x^1(0) = 1$ and $\dot{x}^1(0) = p_0$. □

3.2. **J_--trajectories.** Next we consider J_--trajectories. Since

$$J_-\dot{\gamma} = -\dot{x}^4(t)e_1 + \frac{\dot{x}^3(t)}{x^1(t)}e_2 - \frac{\dot{x}^2(s)}{x^1(t)}e_3 + \frac{\dot{x}^1(t)}{x^1(t)}e_4, \tag{3.8}$$

the J_--trajectory equation yields the system

$$\begin{aligned}
&\dot{p} + r^2 + s^2 = -qu,\\
&\dot{r} = pr + qs,\\
&\dot{s} = -qr + ps,\\
&\dot{u} = qp.
\end{aligned} \tag{3.9}$$

The arc length condition is (3.5) and the second curvature is $\kappa_2 = \sqrt{r^2+s^2}$. Note that the case $\kappa_2 = 0$ leads to the Euclidean circle of constant curvature given by (3.6).

The system (3.9) is analogous to the system (3.4). Hence, we apply approach similar to used in the previous subsection. We may assume that $r^2+s^2 \neq 0$. By analogous procedure applied for J_+-trajectories, from the second and the third equation of (3.9), we get

$$r(t) = Rx^1(t)\cos\psi(t), \quad s(t) = Rx^1(t)\sin\psi(t)$$

for some function $\psi(t)$. The main difference between procedures in J_+ and J_- case is that here instead of $\cos\psi(t)$ we consider $\sin\psi(t)$.

If $\sin\psi(t) = 0$, then $s(t) = 0$. From the third equation of (3.9), we get $q = 0$, i.e J_--trajectory is a geodesic.

Since we are interested in non-geodesic J_--trajectories, we assume that $\sin\psi(t) \neq 0$. It follows

$$\begin{aligned}
\frac{1}{\sin^2\psi}\frac{d\psi}{dt} =& \frac{d}{dt} - \cot\psi = -\frac{d}{dt}\frac{r}{s} = \frac{\dot{s}r - s\dot{r}}{s^2} = -\frac{q(r^2+s^2)}{s^2}\\
=& -q\left(1 + \frac{r^2}{s^2}\right) = -q(1+\cot^2\psi) = -\frac{q}{\sin^2\psi}.
\end{aligned}$$

Hence we get

$$\psi(t) = -qt + \psi_0.$$

The further consideration is quit analogous to the previous subsection. Thus, we generalize result given in Theorem 3.1.

Theorem 3.2. *Let $x^1(t)$ be the solution of*

$$\frac{d^2x^1}{dt^2} = \frac{1}{x^1}\left(\frac{dx^1}{dt}\right)^2 - x^1(q^2\log x^1 + qu^0 + (Rx^1)^2).$$

satisfying the initial condition $x^1(0) = 1$ *and* $\dot{x}^1(0) = p_0$. *Then the curve* $\gamma(t) = (x^1(t), x^2(t), x^3(t), x^4(t))$ *defined by*

$$x^2(t) = R \int_0^t x^1(t)^2 \cos(\pm qt + \psi_0) dt,$$
$$x^3(t) = R \int_0^t x^1(t)^2 \sin(\pm qt + \psi_0) dt,$$
$$x^4(t) = q \int_0^t \log x^1(t) dt + u_0 t.$$

is a $J_\pm$*-trajectory in* $\mathbb{H}^3 \times \mathbb{E}^1$.

4. Invariant surfaces in $\mathbb{H}^3 \times \mathbb{E}^1$

In this section we consider minimal invariant surfaces (or minimal holomorphic curves) in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$.

Definition 4.1. An immersed submanifold M of $\mathbb{H}^3 \times \mathbb{E}^1$ is said to be $J_\pm$-*invariant* (or holomorphic) if its tangent bundle is invariant under $J_\pm$, i.e.,

$$J_\pm X \in TM, \quad \forall X \in TM.$$

One can see that every invariant submanifold M inherits a complex structure from the ambient space. As a result, the inclusion map $\iota : M \to \mathbb{H}^3 \times \mathbb{E}^1$ is a holomorphic immersion. On this reason, invariant submanifolds are often called *complex submanifolds*. The real dimension of an invariant submanifold M is 2 or 4. Obviously in the latter case, M is an open submanifold of $\mathbb{H}^3 \times \mathbb{E}^1$. On the former case M is called a *holomorphic curve* in $\mathbb{H}^3 \times \mathbb{E}^1$.

4.1. Typical examples.

Example 4.1 (Invariant flat half plane)**.** For any $b, c \in \mathbb{R}$,

$$M(1,4;b,c) = \{(x^1, b, c, x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

is a totally geodesic and flat surface. Moreover $M(1,4;b,c)$ is $J_\pm$-invariant. The induced metric is

$$\frac{(dx^1)^2}{(x^1)^2} + (dx^4)^2.$$

In particular $M(1,4;0,0)$ is an abelian Lie subgroup of $\mathbb{H}^3 \times \mathbb{E}^1$. Under the isometry

$$\mathsf{X} = x^4, \quad \mathsf{Y} = \log x^1,$$

the flat half plane $M(1,4;b,c)$ is isometric to the Euclidean plane $\mathbb{E}^2(\mathsf{X},\mathsf{Y})$ with metric $d\mathsf{X}^2 + d\mathsf{Y}^2$.

Example 4.2 (Totally umbilical invariant surface)**.** For any $a > 0$, $d \in \mathbb{R}$,

$$M(2,3;a,d) = \{(a, x^2, x^3, d) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

is an $J_\pm$-invariant surface. Since the induced metric is a flat metric $\{(dx^2)^2 + (dx^3)^2\}/a^2$, $M(2,3;a,d)$ is isometric to Euclidean plane. The mean curvature vector field is e_1. The vector-valued second fundamental form is given by gH. Hence $M(2,3;a,d)$ is a totally umbilical and non-minimal. This fact is consistent with Theorem 1.1. Indeed, $M(2,3;a,d)$ is an integral surface of the distribution $\mathcal{D}_2$ defined by the Pfaff equations $\omega = 0$ and $\omega \circ J = 0$ discussed in Theorem 1.1.

It should be remarked that $M(2, 3; a, d)$ is a *horosphere* which lies in the leaf (hypersurface)

$$M(1, 2, 3; d) = \{(x^1, x^2, x^3, d) \in \mathbb{H}^3 \times \mathbb{E}^1\} = \mathbb{H}^3 \times \{d\}$$

of $\mathbb{H}^3 \times \mathbb{E}^1$. In particular $M(2, 3; 1, 0)$ is an abelian Lie subgroup of $\mathbb{H}^3 \times \mathbb{E}^1$. As is well known a horosphere $M(1, 2, 3; d)$ is isometric to a sphere missing the tangent point which tangents to the ideal boundary as drawn in Figure 1.

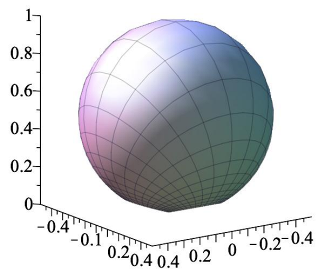

FIGURE 1. Horosphere: The vertical axis is the x^1-axis

4.2. **Minimal invariant surfaces.** We classify minimal proper invariant submanifolds of $\mathbb{H}^3 \times \mathbb{E}^1$.

Theorem 4.1. *The only minimal $J_\pm$-invariant surface in $\mathbb{H}^3 \times \mathbb{E}^1$ is an open portion of a flat totally geodesic half plane $M(1, 4; b, c)$ exhibited in Example 4.1 for some $b, c \in \mathbb{R}$.*

Proof. A $J_\pm$-invariant surface of $\mathbb{H}^3 \times \mathbb{E}^1$ is minimal if and only if its Lee field B is tangent to the surface [6, 17]. Based on this fact, let us consider a proper invariant submanifold M tangent to Lee field B and anti-Lee field A.

Assume that this M is given by the implicit equations $f_i(x^1, x^2, x^3, x^4) = 0$, $i = 1, 2$, where f_1, f_2 are smooth functions on $\mathbb{H}^3 \times \mathbb{E}^1$ verifying that the Jacobi matrix

$$\mathrm{Jac} := \frac{D(f_1, f_2)}{D(x^1, x^2, x^3, x^4)} = \begin{pmatrix} \partial_{x^1} f_1 & \partial_{x^2} f_1 & \partial_{x^3} f_1 & \partial_{x^4} f_1 \\ \partial_{x^1} f_2 & \partial_{x^2} f_2 & \partial_{x^3} f_2 & \partial_{x^4} f_2 \end{pmatrix}$$

has rank 2 at every point of M. Since $A = -2e_4$ and $B = -2e_1$ are tangent to M, they belong to $\mathrm{Ker}(df_i)$, $i =$ 1,2. Hence, we have

$$\frac{\partial f_i}{\partial x^4} = 0 \quad \text{and} \quad \frac{\partial f_i}{\partial x^1} = 0, \ i = 1, 2,$$

and Jacobi matrix has the form

$$\mathrm{Jac} = \begin{pmatrix} 0 & \partial_{x^2} f_1 & \partial_{x^3} f_1 & 0 \\ 0 & \partial_{x^2} f_2 & \partial_{x^3} f_2 & 0 \end{pmatrix}. \tag{4.1}$$

Let us pay our attention to the second column $(\partial_{x^2} f_1, \partial_{x2} f_2)$ of Jac. We investigate the rank of the column vector-valued function $(\partial_{x^2} f_1, \partial_{x^2} f_2)$. Take a point $p_0 \in M$, then on an open domain $\mathcal{U}$ around p_0, we have the following three possibilities:

Case 1: $\frac{\partial f_1}{\partial x^2} = \frac{\partial f_2}{\partial x^2} = 0$ on $\mathcal{U}$.

This implies a contradiction to the fact that Jacobi matrix has rank 2 on $\mathcal{U}$.

Case 2: $\frac{\partial f_1}{\partial x^2} \neq 0$ and $\frac{\partial f_2}{\partial x^2} \neq 0$ on $\mathcal{U}$.

In this case, by using the implicit function theorem we can assume $f_i = x^2 - F_i(x^3)$ for some locally defined smooth functions F_1 and F_2 defined on some open domain $\mathcal{V}$ contained in $\mathcal{U}$. Since $f_1 = x^2 - F_1(x^3) = 0$ and $f_2 = x^2 - F_2(x^3) = 0$, we obtain $F_1 = F_2$ which again implies contradiction regarding the fact that Jacobi matrix has rank 2 on $\mathcal{U}$.

Case 3: $\frac{\partial f_1}{\partial x^2} \neq 0$ and $\frac{\partial f_2}{\partial x^2} = 0$ on $\mathcal{U}$ (without loss of the generality).

In this case, the implicit function theorem implies $f_1 = x^2 - F_1(x^3)$. Having in mind $\partial_{x^2} f_2 = 0$ and the fact that Jacobi matrix has to have rank 2, by (4.1) directly follows $f_2 = x^3 - c = 0$. Hence, $x^3 = \text{const.}$ which implies $F_1 = \text{const.}$ Finally, $f_1 = x^2 - b = 0$ and $f_2 = x^3 - c = 0$, $b, c \in \mathbb{R}$.

Thus, the obtained invariant surface, tangent to A and B, is a flat totally geodesic half plane given in Example 4.1. □

5. Totally real surfaces

In this section we consider totally real submanifolds in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$.

Definition 5.1. A *totally real submanifold* M in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$ is a submanifold such that for all vector field X tangent to M, $J_\pm X$ is normal to M.

It is obvious that any curve is a totally real submanifold, so we are interested in 2-dimensional totally real submanifolds in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$.

5.1. **Typical examples.**

Example 5.1 (Totally real hyperbolic plane)**.** For any $c, d \in \mathbb{R}$,

$$M(1, 2; c, d) = \{(x^1, x^2, c, d) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

is totally real with respect to $J_\pm$ with induced metric

$$\frac{(dx^1)^2 + (dx^2)^2}{(x^1)^2}.$$

Obviously $M(1, 2; c, d)$ is isometric to $\mathbb{H}^2$. One can see that $M(1, 2; c, d)$ is a totally geodesic.

Example 5.2 (Totally real hyperbolic plane)**.** For any $b, d \in \mathbb{R}$,

$$M(1, 3; b, d) = \{(x^1, b, x^3, d) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

is totally real with respect to $J_\pm$ with induced metric

$$\frac{(dx^1)^2 + (dx^3)^2}{(x^1)^2}.$$

Obviously $M(1,3;b,d)$ is isometric to $\mathbb{H}^2$. One can see that $M(1,3;b,d)$ is a totally geodesic.

Example 5.3 (Totally real Euclidean plane)**.** For any $a>0$, $c\in\mathbb{R}$,

$$M(2,4;a,c)=\{(a,x^2,c,x^4)\in\mathbb{H}^3\times\mathbb{E}^1\}$$

is $J_\pm$-totally real and has the induced metric

$$\frac{(dx^2)^2}{a^2}+(dx^4)^2.$$

Thus $M(2,4;a,c)$ is isometric to $\mathbb{E}^2$. The mean curvature vector field is $H=e_1/2$.

Example 5.4 (Totally real Euclidean plane)**.**

$$M(3,4;a,b)=\{(a,b,x^3,x^4)\in\mathbb{H}^3\times\mathbb{E}^1\}$$

is $J_\pm$-totally real and has the induced metric

$$\frac{(dx^3)^2}{a^2}+(dx^4)^2.$$

Thus $M(3,4;a,b)$ is isometric to $\mathbb{E}^2$. The mean curvature vector field is $H=e_1/2$.

Example 5.5 (Totally real cylindrical surfaces in Euclidean 3-space)**.** Take a regular curve $\gamma(s)=(x(s),y(s))$ in the Euclidean plane $\mathbb{E}^2(x,y)$ and consider an immersion of a product manifold $\gamma\times\mathbb{R}(t)$ into $\mathbb{H}^3\times\mathbb{E}^1$ by

$$\iota_{E,a}:\gamma\times\mathbb{R}\to\mathbb{H}^3\times\mathbb{E}^1;\quad \iota_{E,a}(s,t)=(a,ax(s),ay(s),t).$$

Here a is a positive constant. We denote the resulting surface by $M_E(\gamma,a)$. Since the left translation by $(a,0,0,0)$ is an isometry of $\mathbb{H}^3\times\mathbb{E}^1$, $M_E(\gamma):=M_E(\gamma,1)$ is congruent to $M_E(\gamma,a)$. Thus without loss of generality, it suffices to discuss $M_E(\gamma)$ parametrized by

$$(1,x^2(s),x^3(s),x^4)$$

hereafter. Let us recall the totally umbilical invariant Euclidean plane $M(2,3;1,0)$ exhibited in Example 4.2. The induced metric of $M(2,3;1,0)$ is $(dx^2)^2+(dx^3)^3$. If we regard the image of the curve $\gamma(s)$ as a curve in $M(2,3;1,0)$, then we can reparametrize $(x^2(s),x^3(s))$ by the arc length parameter with respect to the Euclidean metric $(dx^2)^2+(dx^3)^2$.

Then the induced metric (the first fundamental form) of $M_E(\gamma)$ is $ds^2+(dx^4)^2$. Thus $M_E(\gamma)$ is intrinsically flat. The tangent bundle of $M_E(\gamma)$ is spanned by

$$E_1=(x^2)'(s)e_2+(x^3)'(s)e_3,\quad E_2=\frac{\partial}{\partial x^4}=e_4.$$

The normal bundle is spanned by

$$\nu_1=-(x^3)'(s)e_2+(x^2)'(s)e_3=J_+E_1=-J_-E_1,\quad \nu_2=e_1=-B/2=-J_\pm E_2.$$

Thus $M_E(\gamma)$ is totally real.

Notice that $A=-2e_4$ is tangent to $M_E(\gamma)$ and $B=-2e_1$ is normal to $M_E(\gamma)$. We have,

$$\begin{aligned}\nabla_{E_1}E_1=&e_1+x^1\left((x^2)''e_2+(x^3)''e_3\right)\\ =&x^1\left(((x^2)'(x^2)''+(x^3)'(x^3)'')E_1+((x^2)'(x^3)''-(x^2)''(x^3)')\nu_1+\nu_2\right)\\ =&a\kappa_{\mathsf{E}}(s)\nu_1+\nu_2,\\ \nabla_{E_2}E_2=&0.\end{aligned}$$

Here $\kappa_{\mathsf{E}}(s)$ is the Euclidean curvature of $\gamma(s)$. The mean curvature vector field

$$H = \frac{1}{2}\left(\mathrm{nor}(\nabla_{E_1}E_1) + \mathrm{nor}(\nabla_{E_2}E_2)\right).$$

is then given by $H = (a\kappa_{\mathsf{E}}(s)\nu_1 + \nu_2)/2$. Thus the totally real surface $M_E(\gamma)$ is non-minimal.

Example 5.6 (Totally real cylindrical surfaces in hyperbolic 3-space)**.** Take a regular curve $\gamma(s) = (x(s), y(s))$ in the Euclidean plane $\mathbb{E}^2(x, y)$ and consider an immersion of a product manifold $\gamma \times \mathbb{R}(t)$ into $\mathbb{H}^3 \times \mathbb{E}^1$ by

$$\iota_{H,d} : \gamma \times \mathbb{R} \to \mathbb{H}^3 \times \mathbb{E}^1; \quad \iota_{H,d}(s,t) = (t, x(s), y(s), d).$$

Here d is a constant. We denote the resulting surface by $M_H(\gamma, d)$. Since the left translation by $(0, 0, 0, d)$ is an isometry of $\mathbb{H}^3 \times \mathbb{E}^1$, $M_H(\gamma) := M_H(\gamma, 0)$ is congruent to $M_H(\gamma, d)$. Thus without loss of generality, it suffices to discuss $M_H(\gamma)$ parameterized as the form $(x^1, x^2(s), x^3(s), d)$, hereafter. As in the previous example, the induced metric of $M(2, 3; 1, 0)$ is $(dx^2)^2 + (dx^3)^3$ and if we regard the image of the curve $\gamma(s)$ as a curve in $M(2, 3; 1, 0)$, then we can reparametarize $(x^2(s), x^3(s))$ by the arc length parameter with respect to the Euclidean metric $(dx^2)^2 + (dx^3)^2$.

If $\gamma(s)$ is defined for all $s \in \mathbb{R}$, then $M_H(\gamma)$ is identified with

$$\left(\mathbb{R}^+(x^1) \times \mathbb{R}(s), \frac{(dx^1)^2 + ds^2}{(x^1)^2}\right).$$

This is nothing but the hyperbolic plane of curvature -1. The tangent bundle of $M_H(\gamma)$ is spanned by

$$E_1 = (x^2)'(s)e_2 + (x^3)'(s)e_3, \quad E_2 = x^1\frac{\partial}{\partial x^1} = e_1.$$

The normal bundle is spanned by

$$\nu_1 = -(x^3)'(s)e_2 + (x^2)'(s)e_3 = J_+E_1 = -J_-E_1, \quad \nu_2 = e_4 = -A/2 = J_\pm E_2.$$

Thus $M_H(\gamma)$ is totally real with respect to both J_+ and J_-. Notice that B is tangent to $M_H(\gamma)$ and A is normal to $M_H(\gamma)$.

Then we have

$$\begin{aligned}\nabla_{E_1}E_1 =& e_1 + x^1\left((x^2)''e_2 + (x^3)''e_3\right) = E_2 + x^1\kappa_{\mathsf{E}}(s)\nu_1,\\ \nabla_{E_2}E_2 =& 0.\end{aligned}$$

where $\kappa_{\mathsf{E}}(s)$ is the Euclidean curvature of $\gamma(s)$.

The mean curvature vector field is given by

$$H = \frac{1}{2}x^1\kappa_{\mathsf{E}}(s)\nu_1.$$

Thus the totally real surface $M_H(\gamma)$ is minimal if and only if $\gamma(s)$ is a line in $\mathbb{E}^2(x, y)$. In particular, $M(1, 2; c, d)$ in Example 5.1 and $M(1, 3; b, d)$ in Example 5.2 are totally real minimal surfaces.

5.2. **Totally real surfaces in terms of A and B.** Mentioned simple examples motivate us to study the following two cases:

(1) B is normal to M and thus A is tangent to M,
(2) B is tangent to M and thus A is normal to M.

5.2.1. *B **is normal and** A **is tangent to** M.* Let

$$E_1 = -\frac{1}{2}A = e_4$$

be the normalized vector field of the anti-Lee field A. We are looking for E_2 unitary, orthogonal to E_1 and tangent to M. From the relations $g(E_2, E_2) = 1$, $g(E_2, E_1) = 0$, $g(E_2, B) = 0$ we have

$$E_2 = \cos\psi e_2 + \sin\psi e_3$$

for some function ψ. The *integrability condition* for submanifold tangent to E_1 and E_2 is $[E_1, E_2] \in \operatorname{span}\{E_1, E_2\}$. Using Levi-Civita connection we obtain

$$[E_1, E_2] = \nabla_{E_1}E_2 - \nabla_{E_2}E_1 = \partial_{x^4}\psi\,(-\sin\psi e_2 + \cos\psi e_3)\,.$$

This formula implies that the only possibility of the integrability is $[E_1, E_2]\|E_2$. If we assume the condition $[E_1, E_2]\|E_2$, then we obtain $\partial_{x^4}\psi = 0$. Hence, we conclude $\psi = \psi(x^1, x^2, x^3)$ and $[E_1, E_2] = 0$.

Let us consider a totally real surface M given by $f_i(x^1, x^2, x^3, x^4) = 0$, $i = 1, 2$, where f_1, f_2 are smooth functions on $\mathbb{H}^3 \times \mathbb{E}^1$ verifying $\operatorname{rank}\big(D(f_1, f_2)/D(x^1, x^2, x^3, x^4)\big) = 2$ at every point of M.

Since E_1 and E_2 are tangent to M, they belong to $\operatorname{Ker}(df_i)$, $i = 1, 2$. Hence, we have

$$\frac{\partial f_i}{\partial x^4} = 0 \quad \text{and} \quad \cos\psi\frac{\partial f_i}{\partial x^2} + \sin\psi\frac{\partial f_i}{\partial x^3} = 0,\ i = 1, 2. \tag{5.1}$$

Next, without loss of generality, we consider three cases.

Case A1: $\frac{\partial f_1}{\partial x^1} = \frac{\partial f_2}{\partial x^1} = 0$.

Notice that Jacobi matrix in this case is the same as (4.1). Thus we can repeat consideration from the previous section. The obtained solution $x^2 = \text{const.}$ and $x^3 = \text{const.}$ does not define totally real surface because the second equation of (5.1) is not satisfied. However, as we proved, it defines an invariant surface in $\mathbb{H}^3 \times \mathbb{E}^1$.

Case A2: $\frac{\partial f_1}{\partial x^1} \neq 0$ and $\frac{\partial f_2}{\partial x^1} \neq 0$.

In this case, by using the implicit function theorem we can assume $f_i = x^1 - F_i(x^2, x^3) = 0$ which by analogues reasoning to the given in previous section (Case 2) leads to the contradiction.

Case A3: $\frac{\partial f_1}{\partial x^1} \neq 0$ and $\frac{\partial f_2}{\partial x^1} = 0$.

In this case the Jacobi matrix is given by

$$\mathrm{Jac} = \begin{pmatrix} \partial_{x^1}f_1 & \partial_{x^2}f_1 & \partial_{x^3}f_1 & 0 \\ 0 & \partial_{x^2}f_2 & \partial_{x^3}f_2 & 0 \end{pmatrix}. \tag{5.2}$$

Obviously, $f_2 = f_2(x^2, x^3)$ and by the implicit function theorem it follows $f_1 = x^1 - F_1(x^2, x^3)$.

Further, we have three subcases.

Subcase A3a: $\frac{\partial f_1}{\partial x^2} = \frac{\partial f_2}{\partial x^2} = 0$.

Since $\operatorname{rank}\mathrm{Jac} = 2$, it follows $\partial_{x^3}f_2 \neq 0$. This implies $f_2 = x^3 - c = 0$. Because $x^3 = \text{const.}$, we obtain $f_1 = x^1 - a = 0$. So, we have the first solution, a part of Euclidean plane in $\mathbb{H}^3 \times \mathbb{E}^1$

given by $x^1 = a$ and $x^3 = c$ (see Example 5.3). Notice that by the second equation of (5.1) this is solution only for $\sin\psi = 0$.

Subcase A3b: $\frac{\partial f_1}{\partial x^2} \neq 0, \quad \frac{\partial f_2}{\partial x^2} \neq 0.$

From $\frac{\partial f_2}{\partial x^2} \neq 0$, it follows $f_2 = x^2 - F_2(x^3) = 0$. This implies $f_1 = x^1 - F_1(x^3)$. Substituting f_1 in the second equation of (5.1), we obtain $\sin\psi = 0$ or $F_1 = \text{const}$. Substituting f_2 in the second equation of (5.1), we have $\cos\psi + \sin\psi(-\partial_{x^3}F_2) = 0$. Combining this relation with $\sin\psi = 0$ we obtain a contradiction. In the other case, when $F_1 = \text{const}$, we obtain the equations which determine the second totally real surface

$$M \ldots \begin{cases} f_1 = x^1 - a = 0, \\ f_2 = x^2 - \int \cot\psi(x^3)dx^3 = 0. \end{cases}$$

Note that this totally real surface is a cylindrical surface over a curve $f_2(x^2, x^3) = 0$ lying in the hypersurface isometric to the Euclidean 3-space $M(2, 3, 4; a)$ (see Example 6.4).

Remark 2. Notice that for $\cos\psi = 0$, we obtain the totally real Euclidean plane $M(3, 4; a, b)$. If $\psi(x^3) = x^3$, we obtain a cylindrical surface

$$M \ldots \begin{cases} x^1 = a \text{ (const.)}, \\ x^2 = \ln|\sin x^3| + \text{const.} \end{cases}$$

which lies in the Euclidean 3-space $M(2, 3, 4; a)$.

Subcase A3c: $\frac{\partial f_1}{\partial x^2} = 0, \quad \frac{\partial f_2}{\partial x^2} \neq 0.$

From the subcase conditions it follows $f_1 = x^1 - F_1(x^3)$ and $f_2 = x^2 - F_2(x^3)$. So, the study and results are the same as in the Subcase A3b.

Subcase A3d: $\frac{\partial f_1}{\partial x^2} \neq 0, \quad \frac{\partial f_2}{\partial x^2} = 0.$

Since $\frac{\partial f_2}{\partial x^2} = 0$, the study and results are the same as in the Subcase A3a.

Since the functions f_1 and f_2 are in the same position in (5.2) with respect to variables x^2 and x^3, we have to investigate four new subcases:

$$\frac{\partial f_1}{\partial x^3} = \frac{\partial f_2}{\partial x^3} = 0, \quad \frac{\partial f_1}{\partial x^3} \neq 0, \frac{\partial f_2}{\partial x^3} \neq 0, \quad \frac{\partial f_1}{\partial x^3} = 0, \frac{\partial f_2}{\partial x^3} \neq 0, \quad \frac{\partial f_1}{\partial x^3} \neq 0, \frac{\partial f_2}{\partial x^3} = 0.$$

In the first and the fourth subcase, by analogous procedure applied in the Subcase A3a, we have a totally real surface $M(3, 4; a, b)$ (see Example 5.4). Notice that this is a particular case of the solution obtained in the Subcase A3b.

In the second (and the third) subcase, from $\frac{\partial f_2}{\partial x^3} \neq 0$, it follows $f_2 = x^3 - F_2(x^2) = 0$. This implies $f_1 = x^1 - F_1(x^2)$. Substituting f_1 in the second equation of (5.1), we obtain $\cos\psi = 0$ or $F_1 = \text{const}$. Substituting f_2 in the second equation of (5.1), we have $\cos\psi(-\partial_{x^2}F_2) + \sin\psi = 0$. Combining this relation with $\cos\psi = 0$ we obtain a contradiction. In the other case, when $F_1 = \text{const}$, we obtain the equations which determine new totally real surface

$$M \ldots \begin{cases} f_1 = x^1 - a = 0, \\ f_2 = x^3 - \int \tan\psi(x^2)dx^2 = 0. \end{cases}$$

This totally real surface is a cylindrical surface over a curve $f_2(x^2, x^3) = 0$ in Euclidean 3-space $M(2,3,4;a)$, exhibited in Example 6.4.

Remark 3. Notice that for $\sin\psi = 0$, we obtain a totally real surface $M(2,4;a,c)$ (see Example 5.3). If $\psi(x^2) = x^2$, we obtain a cylindrical surface

$$M \ldots \begin{cases} x^1 = a \text{ (const.)}, \\ x^3 = \ln|\sin x^2| + \text{const.} \end{cases}$$

which lies in the Euclidean 3-space $M(2,3,4;a)$ (see Figure 2).

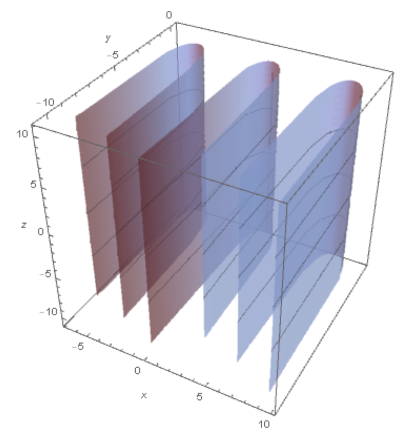

FIGURE 2. Totally real surface $(u, \ln|\sin u|, v)$ in $\mathbb{E}^3$

Thus we proved the following theorem.

Theorem 5.1. *Let M be a totally real surface in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$, normal to the Lee field. Then M is a cylindrical surface which lies in the Euclidean 3-space $M(2,3,4;a)$ for some $a \in \mathbb{R}^+$ and it is parameterized as*

$$M_1 \ldots \begin{cases} x^1 = a, \\ x^2 = \int \cot\psi(x^3)dx^3, \end{cases} \quad \textit{or} \quad M_2 \ldots \begin{cases} x^1 = a, \\ x^3 = \int \tan\psi(x^2)dx^2, \end{cases}$$

where ψ is an arbitrary smooth function.

Remark 4. The unitary orthogonal vectors related to the cases of the Theorem 5.1 are

$$M_1: \ E_1 = e_4, \ E_2 = \cos\psi(x^3)e_2 + \sin\psi(x^3)e_3,$$
$$M_2: \ E_1 = e_4, \ E_2 = \cos\psi(x^2)e_2 + \sin\psi(x^2)e_3.$$

The unitary orthogonal vector fields which generate the normal bundle $T^\perp M$ of M are

$$M_1: \ \nu_1 = e_1, \ \nu_2 = -\sin\psi(x^3)e_2 + \cos\psi(x^3)e_3,$$
$$M_2: \ \nu_1 = e_1, \ \nu_2 = -\sin\psi(x^2)e_2 + \cos\psi(x^2)e_3.$$

Now let us look for minimal ones.

Proposition 5.1. *There is no minimal totally real surface M in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$ normal to the Lee field.*

Proof. In the case of the surface M_1, we have

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_2 = \nu_1 + x^1 \partial_{x^3} \psi \sin \psi(x^3) \nu_2.$$

In the case of the surface M_2, we have

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_2} E_2 = \nu_1 + x^1 \partial_{x^2} \psi \cos \psi(x^2) \nu_2.$$

Obviously, in both cases $H \neq 0$. □

5.2.2. B ***is tangent and*** A ***is normal to*** M. Let

$$E_2 = -\frac{1}{2} B = e_1$$

be the normalized vector field of Lee field B. We are looking for E_1 unitary, orthogonal to E_1 and tangent to M. From the relations $g(E_1, E_1) = 1$, $g(E_1, E_2) = 0$, $g(E_1, A) = 0$, we have

$$E_1 = \cos \psi e_2 + \sin \psi e_3.$$

The integrability condition for the existence of submanifold tangent to E_1 and E_2 is $[E_1, E_2] \in \operatorname{span}\{E_1, E_2\}$. Using Levi-Civita connection we obtain

$$\begin{aligned} [E_1, E_2] &= \nabla_{E_1} E_2 - \nabla_{E_2} E_1 \\ &= -\left(\cos \psi + x^1 \partial_{x^1} \psi \sin \psi\right) e_2 - \left(\sin \psi - x^1 \partial_{x^1} \psi \cos \psi\right) e_3. \end{aligned}$$

Thus the only possibility is $[E_1, E_2] \| E_1$. If we assume $[E_1, E_2] \| E_1$, then we obtain $\partial_{x^1} \psi = 0$. Hence, we conclude, $\psi = \psi(x^2, x^3, x^4)$ and $[E_1, E_2] = -E_1$.

Let us consider a totally real surface M given by $f_i(x^1, x^2, x^3, x^4) = 0$, $i = 1,2$, where f_1, f_2 are smooth functions on $\mathbb{H}^3 \times \mathbb{E}^1$ verifying $\operatorname{rank}\left(D(f_1, f_2)/D(x^1, x^2, x^3, x^4)\right) = 2$ at every point of M.

Since E_1 and E_2 are tangent to M, they belong to $\operatorname{Ker}(df_i)$, $i = 1,2$. Hence, we have

$$\frac{\partial f_i}{\partial x^1} = 0 \quad \text{and} \quad \cos \psi \frac{\partial f_i}{\partial x^2} + \sin \psi \frac{\partial f_i}{\partial x^3} = 0, \ i = 1, 2.$$

Now, similarly to the previous subsection we should consider three cases:

- *Case B1*: $\frac{\partial f_1}{\partial x^4} = \frac{\partial f_2}{\partial x^4} = 0$,
- *Case B2*: $\frac{\partial f_1}{\partial x^4} \neq 0$ and $\frac{\partial f_2}{\partial x^4} \neq 0$,
- *Case B3*: $\frac{\partial f_1}{\partial x^4} \neq 0$ and $\frac{\partial f_2}{\partial x^4} = 0$.

But the complete consideration will be quit analogue to the examination given in previous subsection, except the fact that we have to substitute the coordinate x^1 with the coordinate x^4.

Thus we have the following result.

Theorem 5.2. *Let M be a totally real surface in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$, normal to the anti-Lee field. Then M is a cylindrical surface which lies in the hyperbolic 3-space $M(1, 2, 3; d)$ for some $d \in \mathbb{R}$ and it is parameterized as*

$$M_1 \ldots \begin{cases} x^4 = d, \\ x^2 = \int \cot \psi(x^3) dx^3, \end{cases} \quad \text{or} \quad M_2 \ldots \begin{cases} x^4 = d, \\ x^3 = \int \tan \psi(x^2) dx^2, \end{cases}$$

where ψ is an arbitrary smooth function.

In view of Example 5.6, Theorem 5.2 implies the following corollary.

Corollary 5.1. *Totally real surfaces normal to the anti-Lee field are locally congruent to $M_H(\gamma) \subset \mathbb{H}^3 = M(1,2,3;0)$ for some curve γ.*

Remark 5. The unitary orthogonal vectors related to the cases of the Theorem 5.2 are

$$M_1: \ E_1 = \cos\psi(x^3)e_2 + \sin\psi(x^3)e_3, \ E_2 = e_1,$$
$$M_2: \ E_1 = \cos\psi(x^2)e_2 + \sin\psi(x^2)e_3, \ E_2 = e_1.$$

The unitary orthogonal vector fields which generate the normal bundle $T^{\perp}M$ of M are

$$M_1: \ \nu_1 = e_4, \ \nu_2 = -\sin\psi(x^3)e_2 + \cos\psi(x^3)e_3,$$
$$M_2: \ \nu_1 = e_4, \ \nu_2 = -\sin\psi(x^2)e_2 + \cos\psi(x^2)e_3.$$

Let us classify minimal ones.

Proposition 5.2. *Let M be a minimal totally real surface in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$ normal to the anti-Lee field. Then M is a totally geodesic and isometric to hyperbolic 2-space $\mathbb{H}^2$ and parameterized as*

$$M_1(c_1,c_2,c_3)\ldots\begin{cases} x^4 = c_1, \\ x^2 = c_2x^3 + c_3, \end{cases} \quad \text{or} \quad M_2(c_4,c_5,c_6)\ldots\begin{cases} x^4 = c_4, \\ x^3 = c_5x^2 + c_6, \end{cases}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{R}$.

Proof. In the case of the surface M_1, we have

$$\nabla_{E_1}E_1 = E_2 + x^1(\partial_{x^3}\psi)\sin\psi\nu_2, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_1}E_2 = -E_1, \quad \nabla_{E_2}E_1 = 0.$$

Assuming $H = 0$, we have $\sin\psi = 0$ or $\partial_{x^3}\psi = 0$. The second case implies $\psi = \text{const}$, and covers the first case. The induced metric of M_1 is given by

$$\frac{(dx^1)^2 + (1+c_2^2)(dx^3)^2}{(x^1)^2}.$$

Hence M_1 is isometric to $\mathbb{H}^2$ and totally geodesic.

In the case of the surface M_2, we have

$$\nabla_{E_1}E_1 = E_2 + x^1(\partial_{x^2}\psi)\cos\psi\nu_2, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_1}E_2 = -E_1, \quad \nabla_{E_2}E_1 = 0.$$

Again, if we assume $H = 0$, then we have $\cos\psi = 0$ or $\partial_{x^2}\psi = 0$. The second case implies $\psi = \text{const.}$, and covers the first case. Analogously M_2 is isometric to $\mathbb{H}^2$. □

Remark 6. Sakaki [21] studied surfaces of the form

$$\left(\frac{1}{e^{\psi(u)}\sqrt{1+u^2}}, \frac{u\cos v}{e^{\psi(u)}\sqrt{1+u^2}}, \frac{u\sin v}{e^{\psi(u)}\sqrt{1+u^2}}, kv\right)$$

where $k > 0$ is a constant. This surface is minimal if and only if

$$\psi(z) = \pm\int\frac{b\,dz}{(1+z^2)\sqrt{(1+z^2)(z^2+k^2) - b^2}}.$$

6. CR-submanifolds

Let J be one of the complex structures $J_\pm$ on $\mathbb{H}^3 \times \mathbb{E}^1$.

Definition 6.1. A *CR-submanifold* M in $(\mathbb{H}^3 \times \mathbb{E}^1, J, g)$ is a submanifold endowed with a distribution $\mathfrak{D}$ such that

- $\mathfrak{D}$ is invariant (or holomorphic), i.e.,
$$J_x \mathfrak{D}_x = \mathfrak{D}_x, \quad \forall x \in M.$$
- the orthogonal complement $\mathfrak{D}^\perp$ is anti-invariant (or totally real), i.e.,
$$J_x \mathfrak{D}_x^\perp \subseteq T_x^\perp M, \quad \forall x \in M,$$
where $T_x^\perp M$ is the normal space at x.

Notice that a CR-submanifold M with $\mathfrak{D}^\perp = \{0\}$ is nothing but an invariant submanifold. On the other hand, a CR-submanifold M with $\mathfrak{D} = \{0\}$ is a totally real submanifold.

In this section, we consider *proper* CR-submanifolds in $\mathbb{H}^3 \times \mathbb{E}^1$, i.e.,
$$\dim \mathfrak{D}_x = 2 \quad \text{and} \quad \dim \mathfrak{D}_x^\perp = 1, \quad \forall x \in M.$$

6.1. **Riemannian invariants.** Let M be an orientable hypersurface in $(\mathbb{H}^3 \times \mathbb{E}^1, J, g)$ with unit normal vector field ν. Here J is one of $J_\pm$. Then by the Gauss formula, we have the decomposition
$$\nabla_X Y = \nabla^M_X Y + h(X,Y)\nu,$$
where
$$h(X,Y)\nu = \mathrm{nor}(\nabla_X Y) = \sigma(X,Y)$$
and ∇^M is the Levi-Civita connection of M as before. The symmetric tensor field h is called the *second fundamental form* derived from ν. Next, the *shape operator* $\mathcal{A}$ derived from ν is a self-adjoint tensor field of type $(1,1)$ defined by the Weingarten formula:
$$\nabla_X \nu = -\mathcal{A}X$$
for any vector field X tangent to M. The shape operator is metrically equivalent to h, i.e.,
$$h(X,Y) = g(\mathcal{A}X, Y).$$
The mean curvature vector field H is rewritten as
$$H = \frac{1}{3}(\mathrm{tr}h)\nu = \frac{1}{3}(\mathrm{tr}\mathcal{A})\nu.$$
Hence M is a minimal hypersurface if and only if $\mathrm{tr}\mathcal{A} = 0$.

Let T be the projection of e_4 onto the tangent bundle TM. The *angle function* θ is introduced by $\cos\theta = g(\nu, e_4)$. Then we have
$$e_4 = T + \cos\theta\nu.$$
Then the shape operator satisfies
$$\nabla^M_X T = \cos\theta\mathcal{A}X, \quad X(\cos\theta) = -g(\mathcal{A}X, T).$$
Note that
$$T = -\frac{1}{2}\tan A, \quad \mathrm{nor}A = -2\cos\theta\nu.$$

In literature (e.g., [3, 4, 5, 16, 22]), hypersurface geometry is developed in terms of the behaviour of $T = -(\tan A)/2$ and θ. In the present study our primal interest is the behaviour of the Lee field B.

6.2. **Almost contact structures.** For any vector field X tangent to M, we have the decomposition

$$JX = \varphi X + \eta(X)\nu,$$

where $\varphi X = \tan(JX)$. Let us introduce a vector field ξ by $\xi = -J\nu$, then one can see that $\eta = g(\xi, \cdot)$. The structure (φ, ξ, η) is an almost contact structure on M compatible to the induced metric. On the real hypersurface M, we define a distribution

$$\mathfrak{D}_x = \{X \in T_xM : \eta_x(X) = 0\}, \quad x \in M.$$

Then $\mathfrak{D}$ is J-invariant. The orthogonal complement $\mathfrak{D}_x^{\perp}$ is expressed as $\mathfrak{D}_x^{\perp} = \mathbb{R}\xi_x$. One can see that $\mathfrak{D}^{\perp}$ is anti-invariant under J. Thus M is a proper CR-submanifold.

Conversely, let M be a proper CR-submanifold with holomorphic distribution $\mathfrak{D}$ and global unit normal vector. If the distribution $\mathfrak{D}^{\perp}$ admits a global unit section ξ, then $\nu = J\xi$ is a global unit normal vector field. Thus M is an orientable hypersurface.

Even if $\mathfrak{D}^{\perp}$ does not have a global section, one can take a local section ξ and hence one obtains a local unit normal vector field $\nu = J\xi$. Since our discussion in this section is local, the orientability is not a restriction. If necessarily, one can take a double covering $\tilde{M}$ of M and working on $\tilde{M}$.

An orientable hypersurface M of $\mathbb{H}^3 \times \mathbb{E}^1$ is said to be a *Hopf hypersurface* if ξ is an eigenvector field of $\mathcal{A}$. On the other hand, if the shape operator $\mathcal{A}$ has the form $\mathcal{A} = \alpha_1 \mathrm{I} + \alpha_2 \eta \otimes \xi$ for some constants α_1 and α_2, then M is said to be η-*umbilical*.

Definition 6.2. A CR-submanifold M of an LCK surface N is said to be a *CR-product submanifold* if it is locally isometric to a direct product $M_C \times M_R$, where M_C is an invariant submanifold and M_R is a totally real submanifold.

More precisely, M_C is an integral submanifold of the holomorphic distribution $\mathfrak{D}$ and M_R is an integral submanifold of the anti-holomorphic distribution $\mathfrak{D}^{\perp}$, respectively.

If we assume that M is a CR-product in $\mathbb{H}^3 \times \mathbb{E}^1$ and M is a proper CR-submanifold, then M_R is a curve. An example of CR-product submanifold is given in Example 6.4.

6.3. **Typical hypersurfaces.** Here we exhibit some typical examples of hypersurfaces in $\mathbb{H}^3 \times \mathbb{E}^1$.

Example 6.1 ((−1)-Kenmotsu hypersurface)**.** For any $d \in \mathbb{R}$, the leaf

$$M(1,2,3;d) = \{(x^1, x^2, x^3, d) \in \mathbb{H}^3 \times \mathbb{E}^1\} = \mathbb{H}^3 \times \{d\}$$

has the induced metric

$$\frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(x^1)^2}.$$

Thus, it is isometric to $\mathbb{H}^3$. In particular $\mathbb{H}^3 = M(1,2,3;0)$ is a Lie subgroup of $\mathbb{H}^3 \times \mathbb{E}^1$. Let us take $\nu = e_4$, then the induced almost contact Riemannian structure on $M(1,2,3;d)$ is (-1)-Kenmotsu with $\xi = \xi_{\pm} = e_1$. This hypersurface is a totally geodesic and tangents to the

Lee field $B = -2\xi \in \mathfrak{D}^\perp$ and normal to the anti-Lee field $A = -2e_4$. Note that $T = 0$ and $\theta = 0$. The holomorphic distribution is spanned by e_2 and e_3. This fact is consistent with Theorem 1.1. Indeed, $M(1, 2, 3; d)$ is an integral hypersurface of the distribution $\mathcal{D}_1$ defined by the Pfaff equation $\omega \circ J = 0$ discused in Theorem 1.1.

Example 6.2 ($\mathbb{H}^2 \times \mathbb{E}^1$)**.** For any $c \in \mathbb{R}$, the hypersurface

$$M(1, 2, 4; c) = \{(x^1, x^2, c, x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

has the induced metric

$$\frac{(dx^1)^2 + (dx^2)^2}{(x^1)^2} + (dx^4)^2.$$

Thus $M(1, 2, 4; c)$ is isometric to $\mathbb{H}^2 \times \mathbb{E}^1$. Let us take the unit normal $\nu = e_3$. Hence

$$\xi_\pm = -J_\pm \nu = -J_\pm e_3 = \pm e_2.$$

Then one can see that $M(1, 2, 4; c)$ is a totally geodesic, $T = e_4$ and $\theta = \pi/2$. This hypersurface tangents to both A and B. The holomorphic distribution is spanned by A and B. The hypersurface is regarded as a Riemannian product of a totally real surface $M(1, 2; c, 0)$ given in Example 5.1 and $\mathbb{R}$.

Example 6.3 ($\mathbb{H}^2 \times \mathbb{E}^1$)**.** For any $b \in \mathbb{R}$, the hypersurface

$$M(1, 3, 4; b) = \{(x^1, b, x^3, x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

has the induced metric

$$\frac{(dx^1)^2 + (dx^3)^2}{(x^1)^2} + (dx^4)^2.$$

Thus $M(1, 3, 4; b)$ is isometric to $\mathbb{H}^2 \times \mathbb{E}^1$. Let us take the unit normal $\nu = e_2$. Hence

$$\xi_\pm = -J_\pm \nu = -J_\pm e_2 = \mp e_3.$$

Then one can see that $M(1, 3, 4; b)$ is totally geodesic, $T = e_4$, $\theta = \pi/2$ and tangents to both A and B. The holomorphic distribution is spanned by A and B. The hypersurface is regarded as a Riemannian product of a totally real surface $M(1, 3; b, 0)$ given in Example 5.2 and $\mathbb{R}$.

Here we exhibited three classes of totally geodesic surfaces. All of these exhaust the class of totally geodesic surfaces. Indeed, totally geodesic hypersurfaces were classified by Calvaruso, Kowalczyk and Van der Veken.

Theorem 6.1 ([3])**.** *Let M be a totally geodesic hypersurface of $\mathbb{H}^3 \times \mathbb{E}^1$. Then M is an open portion of a leaf $\mathbb{H}^3 \times \{t_0\}$ for some $t_0 \in \mathbb{R}$ or of a totally geodesic $\mathbb{H}^2 \times \mathbb{E}^1$. More strongly all the parallel hypersurfaces are totally geodesic ones.*

Note that all the totally umbilical hypersurfaces are classified in [3] as well.

Example 6.4 (Euclidean 3-space)**.** For any $a > 0$, the hypersurface

$$M(2, 3, 4; a) = \{(a, x^2, x^3, x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}$$

has the induced metric

$$\frac{(dx^2)^2 + (dx^3)^2}{a^2} + (dx^4)^2.$$

Thus $M(2, 3, 4; a)$ is isometric to $\mathbb{E}^3$.

Let us take the unit normal $\nu = e_1 = -B/2$. Then the mean curvature vector field is $H = 2e_1/3$ which in non-parallel. The shape operator is given by

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover $T = e_4$ and $\theta = \pi/2$. The principal curvature 0 is of multiplicity 1 and the corresponding eigenspace is spanned by T. On the other hand, the principal curvature 1 is of multiplicity 2 and the corresponding eigenspace is spanned by e_2 and e_3.

Here we study the almost contact Riemannian structure. $\xi = -J_{\pm}\nu = -J_{\pm}e_1 = -e_4 = -T$. Hence $\eta = -dx^4$. The structure is cosymplectic, i.e., $\nabla\xi = 0$. Hence we have the formula

$$\mathcal{A} = \mathrm{I} - \eta \otimes \xi.$$

Thus $M(2, 3, 4; a)$ is η-umbilical. The holomorphic distribution is spanned by e_2 and e_3. The integral surface of $\mathfrak{D}$ through $(a, 0, 0, 0)$ is the horosphere $M(2, 3; a, 0)$ exhibited in Example 4.2. On the other hand, the integral curve of $\mathfrak{D}^{\perp}$ is the x^4-axis. Thus $M(2, 3, 4; a)$ is a CR-product of $M(2, 3; a, 0)$ and the x^4-axis.

In [4], by using the cylinder model of $\mathbb{H}^3 \times \mathbb{E}^1$ (embedded in Minkowski 5-space $\mathbb{L}^5$) the notion of rotational hypersurface in $\mathbb{H}^3 \times \mathbb{E}^1$ was introduced. Starting with a curve $\alpha(s)$ in the hyperbolic cylinder $\mathbb{H}^1 \times \mathbb{E}^1$ and a plane P_2 containing the axis of cylinder, one obtains a rotational hypersurface in $\mathbb{H}^3 \times \mathbb{E}^1 \subset \mathbb{L}^5$.

In our half-space model of $\mathbb{H}^3 \times \mathbb{E}^1$, rotational hypersurfaces are parameterized

- When P_2 is timelike,
$$\iota(s, u, v) = \left(\frac{1}{\cosh s + \sinh s \sin u}, \frac{\sinh s \cos u \cos v}{\cosh s + \sinh s \sin u}, \frac{\sinh s \cos u \sin v}{\cosh s + \sinh s \sin u}, \mu(s)\right).$$
- When P_2 is spacelike,
$$\iota(s, u, v) = \left(\frac{1}{\cosh s \cosh u + \sinh s}, \frac{\cosh s \sinh u \cos v}{\cosh s \cosh u + \sinh s}, \frac{\cosh s \sinh u \sin v}{\cosh s \cosh u + \sinh s}, \mu(s)\right).$$
- When P_2 is lightlike,
$$\iota(s, u, v) = \left(\frac{s}{\sqrt{2}}, \frac{u}{\sqrt{2}}, \frac{v}{\sqrt{2}}, \mu(s)\right), \quad s > 0.$$

  Here $\mu(s)$ is a smooth function of s.

One can see that $M(2, 3, 4; a)$ is a rotational hypersurface of lightlike type whose generating curve $\alpha(s)$ is a vertical line in $\mathbb{H}^1 \times \mathbb{E}^1$. Flat rotational hypersurfaces are congruent to $M(2, 3, 4; a)$ or rotational hypersurface of timelike type with $\mu(s) = \pm\cosh s$ (see [4, p. 51]).

On the other hand, the following criterion for rotational hypersurfaces were obtained.

Lemma 6.1 ([4, 16])**.** *Let M be an orientable hypersurface in $\mathbb{H}^3 \times \mathbb{E}^1$ with non-vanishing T. Assume that T is a principal vector field with principal curvature λ and the shape operator has principal curvatures λ of multiplicity 1 and μ of multiplicity 2. If λ is constant along the leaves of the eigenbundle of μ, then M is an open portion of a rotational hypersurface.*

Minimal rotational hypersurfaces are classified in [4, p. 53].

Remark 7. Manfio and Tojeiro gave the following example of non-rotational hypersurface:

$$\iota(u,v) = \left(\frac{1}{e^u \cosh s}, \frac{\sinh s \cos v}{e^u \cosh s}, \frac{\sinh s \sin v}{e^u \cosh s}, Bs \right), \quad B \in \mathbb{R}^{\times}.$$

The resulting hypersurface is the only constant curvature hypersurfaces with three distinct principal curvatures and 0 as principal curvature in the T-direction and whose two remaining principal curvatures are constant along the eigenbundles (see [16]).

Example 6.5 ($\mathbb{R}^+ \times \gamma \times \mathbb{R}$)**.** Take an arc length parameterized curve $\gamma(s) = (x^2(s), x^3(s))$ (defined on an interval I) in the totally umbilical Euclidean plane $M(2,3;1,0)$ defined in Example 4.2. Then we define a hypersurface by

$$M^3_\gamma = \{(x^1, x^2(s), x^3(s), x^4) : s \in I, x^1 > 0, x^4 \in \mathbb{R}\}.$$

Note that $M(1,2,4;c)$ in Example 6.2 and $M(1,3,4;b)$ in Example 6.3 are particular cases of these hypersurfaces.

Then the tangent bundle TM^3_γ is spanned by

$$E_1 = e_1 = -\frac{1}{2}B, \quad E_2 = \dot{x}^2(s)e_2 + \dot{x}^3(s)e_3, \quad E_3 = e_4 = -\frac{1}{2}A = T$$

and $\cos\theta = 0$. We can take a unit normal vector field $\nu = -\dot{x}^3(s)e_2 + \dot{x}^2(s)e_3$. Then we get $\xi_\pm = -J_\pm \nu = \pm E_2$. Hence the holomorphic distribution is spanned by A and B. Direct computation shows that

$$\nabla_{E_1}\nu = 0, \quad \nabla_{E_2}\nu = x^1(-\ddot{x}^3 e_2 + \ddot{x}^2 e_3), \quad \nabla_{E_3}\nu = 0.$$

Then the shape operator is given by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x^1\kappa_{\mathtt{E}}(s) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\kappa_{\mathtt{E}}(s) = \dot{x}^2(s)\ddot{x}^3(s) - \ddot{x}^2(s)\dot{x}^3(s)$ is the Euclidean curvature of $\gamma(s)$. In particular T is a principal vector field with principal curvature 0. The hypersurface M is minimal if and only if $\gamma(s)$ is a line, namely M has the parametrization

$$(x^1, a_2 s + b_2, a_3 s + b_3, x^4)$$

for some constants a_2, a_3, b_2, b_3. In this case M^3_γ is a totally geodesic. The induced metric is

$$\frac{(dx^1)^2 + (ds)^2}{(x^1)^2} + (dx^4)^2.$$

Hence M^3_γ is isometric to $\mathbb{H}^2 \times \mathbb{E}^1$.

6.4. **Some proper CR-submanifolds.** We are interested in finding proper 3-dimensional CR-submanifolds M for which the Lee field B is tangent or normal to M. So, we will study the following cases:

- I. B is tangent to M
  - a. B belongs to the holomorphic distribution $\mathfrak{D}$
  - b. B belongs to the distribution $\mathfrak{D}^\perp$
  - c. B has components in both $\mathfrak{D}$ and $\mathfrak{D}^\perp$
- II. B is normal to M.

6.4.1. B ***is tangent to*** M. We have the three following cases.

Case I.a. B belongs to the holomorphic distribution $\mathfrak{D}$

Since B belongs to $\mathfrak{D}$, it follows that the anti-Lee field A also belongs to $\mathfrak{D}$. Thus $\mathfrak{D}$ is spanned by $A = -2T$ and B, and $[A, B] = 0$.

Let us consider a CR-submanifold M given by the implicit equation $f(x^1, x^2, x^3, x^4) = 0$, where f is a smooth function on $\mathbb{H}^3 \times \mathbb{E}^1$ such that $\mathrm{grad} f \neq 0$ on M.

Since $A = -2e_4$ and $B = -2e_1$ are tangent to M, they belong to $\mathrm{Ker}(df)$. Hence, we have

$$\frac{\partial f}{\partial x^1} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x^4} = 0.$$

Subcase I.a.1: $\frac{\partial f}{\partial x^2} = \frac{\partial f}{\partial x^3} = 0$.

This implies a contradiction to the fact that $\mathrm{grad} f \neq 0$ on M.

Subcase I.a.2: $\frac{\partial f}{\partial x^2} \neq 0$ and $\frac{\partial f}{\partial x^3} = 0$.

In this case, by using the implicit function theorem we have $f(x^2) = x^2 - b = 0$. Thus we obtain the hypersurface

$$M(1, 3, 4; b) = \{(x^1, b, x^3, x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}.$$

Subcase I.a.3: $\frac{\partial f}{\partial x^2} = 0$ and $\frac{\partial f}{\partial x^3} \neq 0$.

In this case, by using the implicit function theorem we assume $f(x^3) = x^3 - c = 0$. Thus we obtain the hypersurface

$$M(1, 2, 4; c) = \{(x^1, x^2, c, x^4) \in \mathbb{H}^3 \times \mathbb{E}^1\}.$$

Subcase I.a.4: $\frac{\partial f}{\partial x^2} \neq 0$ and $\frac{\partial f}{\partial x^3} \neq 0$.

In this case, the implicit function theorem implies $f(x^2, x^3) = x^2 - F_1(x^3)$ or $f(x^2, x^3) = x^3 - F_2(x^2)$, where $F_i \in C^\infty(\mathbb{R})$, $i = 1, 2$. Thus we proved the following theorem.

Theorem 6.2. *Let M be a proper CR-submanifold in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$, such that the Lee field belongs to the holomorphic distribution. Then M is locally given by the implicit equation*

$$M_1 \cdots f(x^2, x^3) = x^2 - F_1(x^3) = 0 \quad \text{or} \quad M_2 \cdots f(x^2, x^3) = x^3 - F_2(x^2) = 0,$$

where F_1 and F_2 are arbitrary smooth functions.

Comparing this with Example 6.5, Theorem 6.2 is reformulated as follows:

Corollary 6.1. *Let M be a proper CR-submanifold in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$, such that the Lee field belongs to the holomorphic distribution. Then M is locally congruent to M^3_γ given in Example 6.5. In particular M is minimal if and only if it is congruent to the totally geodesic M^3_γ, where γ is a line.*

Case I.b. B belongs to distribution $\mathfrak{D}^\perp$

In this case, A is normal to M and parallel to grad f. Thus we have grad $f=e_4(f)e_4=df(A)A/4$. Hence we have $f=f(x^4)$. Thus e_1, e_2 and e_3 are tangent to M. The holomorphic distribution $\mathfrak{D}$ is spanned by e_2 and e_3. The orthogonal complement $\mathfrak{D}^\perp$ is spanned by B. From these M is locally congruent to a totally geodesic hyperbolic 3-space $M(1,2,3;d)$ for some d. The implicit equation $f=0$ is equivalent to $f(x^4)=x^4-d=0$.

We proved the following theorem.

Theorem 6.3. *Let M be a proper CR-submanifold in $(\mathbb{H}^3\times\mathbb{E}^1,J_\pm,g)$ such that the Lee field belongs to $\mathfrak{D}^\perp$. Then M is locally congruent to the hyperbolic 3-space $M(1,2,3;0)$.*

Case I.c. B has components in $\mathfrak{D}$ and $\mathfrak{D}^\perp$

Let us decompose B as $B=B_1+B_2$ such that $B_1\in\mathfrak{D}$ and $B_2\in\mathfrak{D}^\perp$. Then A is oblique and $\tan(A)=J_\pm B_1$ and $\mathrm{nor}(A)=J_\pm B_2$. By the definition of φ, we get

$$A=J_\pm B=J_\pm B_1+J_\pm B_2=\varphi B_1+\eta(B_2)\nu.$$

On the other hand, we know that

$$A=-2T-2\cos\theta\nu.$$

Hence we have

$$\varphi B_1=-2T,\quad \eta(B_2)=-2\cos\theta.$$

Hence

$$B=B_1-2\cos\theta\xi,\quad B_1=2\varphi T.$$

Since $B=-2x^1\partial_{x^1}$, $A=J_\pm B=-2\partial_{x^4}$. Represent $\tan(A)=J_\pm B_1$ as

$$\tan(A)=a\partial_{x^1}+b\partial_{x^2}+c\partial_{x^3}+d\partial_{x^4}=\frac{a}{x^1}e_1+\frac{b}{x^1}e_2+\frac{c}{x^1}e_3+de_4,$$

then by $\tan(A)=J_\pm B_1$, (2.1) and (2.2), we have

$$\begin{aligned}
B_1=&de_1\pm\frac{c}{x^1}e_2\mp\frac{b}{x^1}e_3-\frac{a}{x^1}e_4=dx^1\partial_{x^1}\pm c\partial_{x^2}\mp b\partial_{x^3}-a\partial_{x^4},\\
B_2=&B-B_1=-(2+d)e_1\mp\frac{c}{x^1}e_2\pm\frac{b}{x^1}e_3+\frac{a}{x^1}e_4=\\
=&-(2+d)x^1\partial_{x^1}\mp c\partial_{x^2}\pm b\partial_{x^3}+a\partial_{x^4},\\
J_\pm B_2=&-\frac{a}{x^1}e_1-\frac{b}{x^1}e_2-\frac{c}{x^1}e_3-(d+2)e_4=\\
=&-a\partial_{x^1}-b\partial_{x^2}-c\partial_{x^3}-(d+2)x^1\partial_{x^4}.
\end{aligned}$$

From the orthogonality conditions,

$$0=g(B_1,J_\pm B_2)=2\frac{a}{x^1},$$

Hence we have $a=0$. Next we have

$$0=g(B_1,B_2)=-d(d+2)-\frac{1}{(x^1)^2}(b^2+c^2).$$

Thus

$$0 \leq \frac{1}{(x^1)^2}(b^2+c^2) = -d(d+2).$$

Since $J_{\pm}B_2$ is a normal vector field, we may choose the unit normal ν by

$$\nu = \frac{1}{|B_2|}J_{\pm}B_2 = \frac{-1}{\sqrt{2(d+2)}}\left(\frac{b}{x^1}e_2 + \frac{c}{x^1}e_3 + (d+2)e_4\right).$$

On the other hand, the tangent bundle TM is spanned by $\{B_1, J_{\pm}B_1, B_2\}$, thus we know

$$B(f) = 0 \Longleftrightarrow \partial_{x^1} f = 0,$$
$$B_1(f) = 0 \Longleftrightarrow dx^1\partial_{x^1} f \pm c\partial_{x^2} f \mp b\partial_{x^3} f - a\partial_{x^4} f = 0,$$
$$J_{\pm}B_1(f) = 0 \Longleftrightarrow a\partial_{x^1} f + b\partial_{x^2} f + c\partial_{x^3} f + d\partial_{x^4} f = 0,$$
$$B_2(f) = 0 \Longleftrightarrow -(d+2)x^1\partial_{x^1} f \mp c\partial_{x^2} f \pm b\partial_{x^3} f + a\partial_{x^4} f = 0.$$

Hence we obtain the system

$$c\partial_{x^2} f - b\partial_{x^3} f = 0, \quad b\partial_{x^2} f + c\partial_{x^3} f + d\partial_{x^4} f = 0, \quad \frac{1}{(x^1)^2}(b^2+c^2) = -d(d+2) \tag{6.1}$$

for $f = f(x^2, x^3, x^4)$.

When $d = 0$, by the third equation of (6.1), we have $b = c = 0$. Thus $\tan(A) = 0$, i.e., $B = B_2 \in \mathfrak{D}^{\perp}$. This is a contradiction. On the other hand, if $d = -2$, then $\mathrm{nor}(A) = 0$ and $B_2 = 0$. Hence, $B = B_1 \in \mathfrak{D}$ and we again get a contradiction. Thus we may assume that $a^2 + b^2 \neq 0$, i.e., $-d(d+2) \neq 0$. Next, if $\partial_{x^4} f = 0$ under the assumption $a^2 + b^2 \neq 0$, we have $\partial_{x^2} f = \partial_{x^3} f = 0$. This means that $df = 0$. This is a contradiction. On the other hand, $\partial_{x^2} f = \partial_{x^3} f = 0$ implies also $df = 0$. Thus we only have to investigate the case $\partial_{x^4} f \neq 0$.

Having in mind $d(d+2) \neq 0$, solving the system (6.1), we get

$$\partial_{x^2} f = \frac{b}{(x^1)^2(d+2)}\partial_{x^4} f, \quad \partial_{x^3} f = \frac{c}{(x^1)^2(d+2)}\partial_{x^4} f.$$

Since $\partial_{x^4} f \neq 0$, then by the implicit function theorem, $f = 0$ is written as $f(x^2, x^3, x^4) = x^4 - F(x^2, x^3) = 0$ for some smooth function F.

Thus the CR-submanifold M is locally expressed as

$$\{(t, x^2, x^3, F(x^2, x^3)) | (x^2, x^3) \in \mathcal{D}, 0 < t \in I\},$$

where $\mathcal{D}$ is a region of $\mathbb{R}^2(x^2, x^3)$ and I is an open interval.

The function F is determined by the tangential part $\tan(A)$ of the anti-Lee field A. More precisely, F is determined by the system:

$$F_{x^2} = -\frac{b}{(x^1)^2(d+2)}, \quad F_{x^3} = -\frac{c}{(x^1)^2(d+2)},$$

where the coefficient functions b, c and d are derived from

$$\tan A = b\partial_{x^2} + c\partial_{x^3} + d\partial_{x^4}.$$

The compatibility condition $(F_{x^2})_{x^3} = (F_{x^3})_{x^2}$ is given by

$$(c_{x^2} - b_{x^3})(d+2) = cd_{x^2} - bd_{x^3}. \tag{6.2}$$

The investigation we have done is summarized as follows:

Proposition 6.1. *Let M be a proper CR-submanifold in $(\mathbb{H}^3 \times \mathbb{E}^1, J_\pm, g)$ tangent to the Lee field B. If $B = B_1 + B_2$ has both non-vanishing components $B_1 \in \mathfrak{D}$ and $B_2 \in \mathfrak{D}^\perp$, then M is locally expressed as*:

$$\{(t, x^2, x^3, F(x^2, x^3)) | (x^2, x^3) \in \mathcal{D}, 0 < t \in I\}, \tag{6.3}$$

where $\mathcal{D}$ is a region of $\mathbb{R}^2(x^2, x^3)$, $F : \mathcal{D} \to \mathbb{R}$ is a non-constant smooth function and I is an open interval. The function $F(x^2, x^3)$ is determined by the tangential part $\tan(A)$ *of the anti-Lee field A. More precisely,* $\tan(A)$ *is represented as*

$$\tan A = b\partial_{x^2} + c\partial_{x^3} + d\partial_{x^4}.$$

Then F is determined by the system

$$F_{x^2} = -\frac{b}{(x^1)^2(d+2)}, \quad F_{x^3} = -\frac{c}{(x^1)^2(d+2)}. \tag{6.4}$$

From the assumption $B_1 \neq 0$ and $B_2 \neq 0$, the coefficients b, c and d satisfies

$$0 \leq \frac{1}{(x^1)^2}(b^2 + c^2) = -d(d+2).$$

It should be remarked that the system (6.4) satisfies the the compatibility condition $(F_{x^2})_{x^3} = (F_{x^3})_{x^2}$.

In Proposition 6.1 we showed that a proper CR-submanifold M satisfying the condition $B = B_1 + B_2$, where $B_1 \in \mathfrak{D}$, $B_2 \in \mathfrak{D}^\perp$ and $B_1 \neq 0$ and $B_2 \neq 0$ is locally represented as the form (6.3). The function $F(x^2, x^3)$ is determined by $\tan(A)$.

Let us consider the following inverse problem:

How the tangential part $\tan(A)$ *is determined from the function F?*

To answer this, we investigate the Lee field B and anti-Lee field A of the hypersurface given by (6.3).

Example 6.6. Let us consider a hypersurface of the form

$$M = \{(t, x^2, x^3, F(x^2, x^3)) | (x^2, x^3) \in \mathcal{D},\ t \in \mathbb{R}^+\},$$

where $F : \mathcal{D} \to \mathbb{R}$ is a non-constant smooth function. Take a frame field

$$\hat{E}_1 = \partial_{x^2} + F_{x^2}\partial_{x^4} = \frac{1}{x^1}e_2 + F_{x^2}e_4,$$
$$\hat{E}_2 = \partial_{x^3} + F_{x^3}\partial_{x^4} = \frac{1}{x^1}e_3 + F_{x^3}e_4, \quad \hat{E}_3 = x^1\partial_{x^1} = e_1.$$

Then the first fundamental form is

$$g = \begin{pmatrix} \frac{1}{(x^1)^2} + (F_{x^2})^2 & F_{x^2}F_{x^3} & 0 \\ F_{x^3}F_{x^2} & \frac{1}{(x^1)^2} + (F_{x^3})^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can take a unit normal

$$\nu = \frac{1}{\sqrt{1 + (x^1)^2\left((F_{x^2})^2 + (F_{x^3})^2\right)}}(x^1F_{x^2}e_2 + x^1F_{x^3}e_3 - e_4).$$

Then

$$\begin{aligned}\xi_\pm =&\frac{1}{\sqrt{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}}(\mp x^1F_{x^2}e_3 \pm x^1F_{x^3}e_2 - e_1)\\ =&\frac{1}{\sqrt{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}}(\pm x^1F_{x^3}e_2 \mp x^1F_{x^2}e_3 + \frac{1}{2}B).\end{aligned}$$

$$J_\pm\hat{E}_1 = -F_{x^2}e_1 \pm \frac{1}{x^1}e_3, \quad J_\pm\hat{E}_2 = -F_{x^3}e_1 \mp \frac{1}{x^1}e_2, \quad J_\pm\hat{E}_3 = e_4 = -\frac{1}{2}A.$$

The Lee field $B=-2e_1$ is tangent to M.

We decompose B as $B=B_1+B_2$, where $B_1\in\mathfrak{D}$ and $B_2\in\mathfrak{D}^\perp$.

$$\begin{aligned}B_2 = g(B,\xi_\pm)\xi_\pm = &\frac{2}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}(\pm x^1F_{x^3}e_2 \mp x^1F_{x^2}e_3)\\ &-\frac{2}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}e_1\in\mathfrak{D}^\perp.\\ B_1 = B - B_2 =&\frac{-2}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}(\pm x^1F_{x^3}e_2 \mp x^1F_{x^2}e_3)\\ &-\frac{2(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}e_1\in\mathfrak{D}.\end{aligned}$$

Since F is non-constant, $B_1\neq 0$ and $B_2\neq 0$. Thus M is a proper CR-submanifold which belongs to the Case I.c.

Now let us consider the inverse problem posed above.

By comparing B_1 with

$$B_1 = de_1 \pm \frac{c}{x^1}e_2 \mp \frac{b}{x^1}e_3$$

from Subsection 6.4 (Case I.c.), we obtain

$$\begin{aligned}b =&\frac{-2(x^1)^2}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}F_{x^2},\\ c =&\frac{-2(x^1)^2}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}F_{x^3},\\ d =&\frac{-2(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}.\end{aligned}\tag{6.5}$$

Thus $\tan(A)$ is determined by F.

From (6.5), we have, respectively

$$F_{x^2} = \frac{-b}{2(x^1)^2}\left(1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)\right),\tag{6.6}$$

$$F_{x^3} = \frac{-c}{2(x^1)^2}\left(1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)\right),\tag{6.7}$$

$$(F_{x^2})^2+(F_{x^3})^2 = \frac{-d}{(x^1)^2(d+2)}.\tag{6.8}$$

Substituting (6.8) to (6.6) and (6.7), we get the system (6.4). Thus M satisfies equations of the system (6.4) and the compatibility condition (6.2).

Moreover, substituting (6.4) in (6.8), we have the third equation of the system (6.1),

$$\frac{1}{(x^1)^2}(b^2+c^2) = -d(d+2).$$

It is easy to see that the first and the second equation of (6.1) are also fulfilled. Hence, the hypersurface M satisfies equations of the system (6.1), too.

Let us look for minimal CR-submanifolds of the form

$$M = \{(t, x^2, x^3, F(x^2, x^3)) | (x^2, x^3) \in \mathcal{D},\ t \in \mathbb{R}^+\}.$$

Direct computation gives

$$\nabla_{\hat{E}_1}\hat{E}_1 = \frac{1}{(x^1)^2}e_1 + F_{x^2x^2}e_4, \quad \nabla_{\hat{E}_1}\hat{E}_2 = \nabla_{\hat{E}_2}\hat{E}_1 = F_{x^2x^3}e_4,$$

$$\nabla_{\hat{E}_1}\hat{E}_3 = \nabla_{\hat{E}_3}\hat{E}_1 = -\frac{1}{x^1}e_2, \quad \nabla_{\hat{E}_2}\hat{E}_2 = \frac{1}{(x^1)^2}e_1 + F_{x^3x^3}e_4,$$

$$\nabla_{\hat{E}_2}\hat{E}_3 = \nabla_{\hat{E}_3}\hat{E}_2 = -\frac{1}{x^1}e_3, \quad \nabla_{\hat{E}_3}\hat{E}_3 = 0.$$

Hence the second fundamental form h has the components $h_{ij} = h(\hat{E}_i, \hat{E}_j) = g(\nabla_{\hat{E}_i}\hat{E}_j, \nu)$ as

$$(h_{ij}) = \frac{-1}{\sqrt{1+(x^1)^2\left((F_{x^2})^2+(F_{x^3})^2\right)}}\begin{pmatrix} F_{x^2x^2} & F_{x^2x^3} & F_{x^2} \\ F_{x^3x^2} & F_{x^3x^3} & F_{x^3} \\ F_{x^2} & F_{x^3} & 0 \end{pmatrix}.$$

Since we assume that $(F_{x^2})^2 + (F_{x^3})^2 \neq 0$, M is non-totally geodesic.

The mean curvature function is computed as $\mathrm{tr}\{(g_{ij})^{-1}(h_{ij})\}/3$. Thus M is minimal if and only if

$$(F_{x^2x^2} + F_{x^3x^3}) + (x^1)^2\left((F_{x^2})^2F_{x^3x^3} - 2F_{x^2}F_{x^3}F_{x^2x^3} + (F_{x^3})^2F_{x^2x^2}\right) = 0.$$

Since F does not depends on x^1, the minimal hypersurface equation is the system

$$F_{x^2x^2} + F_{x^3x^3} = 0, \quad F_{x^2x^2}(F_{x^3})^2 + F_{x^3x^3}(F_{x^2})^2 - 2F_{x^2}F_{x^3}F_{x^2x^3} = 0. \tag{6.9}$$

Proposition 6.2. *For a smooth function F defined on a region $\mathcal{D}$ of $\mathbb{R}^2(x^2, x^3)$, the hypersurface*

$$M = \{(t, x^2, x^3, F(x^2, x^3)) | (x^2, x^3) \in \mathcal{D},\ t \in \mathbb{R}^+\}.$$

is minimal in $\mathbb{H}^3 \times \mathbb{E}^1$ if and only if F satisfies the system (6.9).

Here we give two examples.

Example 6.7 ($\mathbb{R}^+ \times$ affine plane)**.** Let us consider a linear function $F(x^2, x^3) = bx^2 + cx^3 + d$ (with $b^2 + c^2 \neq 0$). Obviously $F(x^2, x^3)$ satisfies the system (6.9). The resulting hypersurface

$$M = \{(t, x^2, x^3, bx^2 + cx^3 + d) | 0 < t, x^2, x^3 \in \mathbb{R}\}$$

is diffeomorphic to $\mathbb{R}^+ \times \mathbb{R}^2$. Note that when $b = c = 0$, then M is a hyperbolic 3-space $M(1, 2, 3; d)$ and $B = B_2 \in \mathfrak{D}^\perp$. We already examined this hypersurface in Subsection 6.4.1 (Case I.b).

Example 6.8 ($\mathbb{R}^+\times$ helicoid)**.** On $\mathcal{D}=\mathbb{R}^2\smallsetminus\{(0,0)\}$, the function $F_{\mathtt{hel}}(x^2,x^3)=\tan^{-1}(x^3/x^2)$ is a solution to the system (6.9). It should be remarked that

$$\overline{M}=\{(x^2,x^3,\tan^{-1}(x^3/x^2))|(x^2,x^3)\in\mathcal{D}\}\subset\mathbb{E}^3$$

is a helicoid in the Euclidean 3-space. As is well known, $\overline{M}$ is a minimal surface in $\mathbb{E}^3$. Thus the resulting hypersurface

$$M=\{(t,x^2,x^3,\tan^{-1}(x^3/x^2))|(x^2,x^3)\in\mathcal{D},t\in\mathbb{R}^+\}\subset\mathbb{H}^3\times\mathbb{E}^1$$

is a product manifold of the real half-line and a helicoid.

Let us investigate the solutions of (6.9). First we notice that the system (6.9) is equivalent to the system

$$F_{x^2x^2}+F_{x^3x^3}=0,\quad F_{x^2x^2}(F_{x^2})^2+F_{x^3x^3}(F_{x^3})^2+2F_{x^2}F_{x^3}F_{x^2x^3}=0 \tag{6.10}$$

defined on $\mathbb{R}^2(x^2,x^3)$. Aronsson completely solved the system (6.10) as follows:

Theorem 6.4 ([2])**.** *Let $F(x^2,x^3)$ be a solution to the system* (6.10) *defined on a region $\mathcal{D}\subset\mathbb{R}^2$, then F is either a linear function or*

$$F(x^2,x^3)=A\tan^{-1}\frac{x^3-x^3_0}{x^2-x^2_0}+c$$

for some point (x^2_0,x^3_0) and constants A and c such that $\tan^{-1}\{(x^3-x^3_0)/(x^2-x^2_0)\}$ *is single-valued ans continuous in $\mathcal{D}$.*

Hence we obtain the following classification.

Theorem 6.5. *A proper minimal CR-submanifold in $(\mathbb{H}^3\times\mathbb{E}^1,J_\pm,g)$ such that the Lee field B has non-vanishing components in both $\mathfrak{D}$ and $\mathfrak{D}^\perp$ is locally congruent to either a product manifold exhibited in Example 6.7 or a product manifold exhibited in Example 6.8.*

6.4.2. B ***is normal to*** M. It follows that A belongs to anti-holomorphic distribution $\mathfrak{D}^\perp$. If M is given by $f(x^1,x^2,x^3,x^4)=0$ (with $\mathrm{grad} f\neq 0$), we have $A(f)=0$ and $B\ ||\ \mathrm{grad}\ f$. Thus, it follows $f=f(x^1)$. This implies that M is locally congruent to $M(2,3,4;a)$ for some a. We confirm this fact alternatively as follows:

Let M be be a CR-submanifold in $(\mathbb{H}^3\times\mathbb{E}^1,J_\pm,g)$ normal to the Lee field. Then its tangent bundle is spanned by $\{e_2,e_3,e_4\}$. The distribution spanned by $\{e_2,e_3,e_4\}$ is integrable. The integral manifold is nothing but the Euclidean 3-space $M(2,3,4;a)$.

We proved the following theorem.

Theorem 6.6. *Let M be a proper CR-submanifold in $(\mathbb{H}^3\times\mathbb{E}^1,J_\pm,g)$, normal to the Lee field. Then M is locally congruent to the Euclidean* 3*-space $M(2,3,4;a)$.*

Corollary 6.2. *The only minimal proper CR-submanifold in $(\mathbb{H}^3\times\mathbb{E}^1,J_\pm,g)$ normal to the Lee field is the Euclidean* 3*-space $M(2,3,4;a)$.*

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