THE COMPLEX LANDSLIDE FLOW AND THE METHOD OF INTEGRABLE SYSTEMS

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Abstract. We investigate a connection between the complex landslide flow, defined on a pair of Teichmüller spaces, and the integrable system approach to harmonic maps into a symmetric space. We will prove that the holonomy of the complex landslide flow can be derived from the holonomy of the family of flat connections determined by a harmonic map into the hyperbolic two-space.

1. Introduction and the main result

(A) McMullen's complex earthquake flow [10] extends Thurston's earthquake flow holomorphically, while the *complex landslide flow* [2] provides its cyclic complex extension. This flow arises from the composition of two geometric maps, as described below. Let M denote a differentiable closed oriented surface of genus greater than or equal to two. Let $\mathcal{T}$ denote the Teichmüller space of hyperbolic metrics on M, and $\mathcal{P}$ denote the space of complex projective structures on M. The complex landslide flow P is defined by the following map:

$$P : \overline{\mathbb{D}}^{\times} \times \mathcal{T} \times \mathcal{T} \to \mathcal{P}, \quad (q, h, h^{\star}) \mapsto SGr_s(L_\theta(h, h^{\star})), \tag{1.1}$$

where $s = -\log|q|$ and $\theta = \arg q$, and moreover, $L : [0, 2\pi) \times \mathcal{T} \times \mathcal{T} \to \mathcal{T} \times \mathcal{T}$ and $SGr : [0, \infty) \times \mathcal{T} \times \mathcal{T} \to \mathcal{P}$ are the *landslide flow* and the *smooth grafting* defined below.

The landslide: Let $h, h^{\star} \in \mathcal{T}$ be hyperbolic metrics on M. Then there exists a unique bundle morphism $b : TM \to TM$ such that

(1) It is self-adjoint with respect to h.
(2) The determinant is 1.
(3) It satisfies the Codazzi equation $\mathrm{d}^{\nabla} b = 0$, that is, for the Levi-Civita connection ∇ of h and vector fields u, v on M, $\mathrm{d}^{\nabla} b$ is defined by $(\mathrm{d}^{\nabla} b)(u, v) := \nabla_u(bv) - \nabla_v(bu) - b[u, v]$.
(4) The metric given by $h(b\bullet, b\bullet)$ is isometric to $h^{\star}$ by a diffeomorphism isotopic to the identity.

The above bundle morphism b has been called the *Labourie operator* [9]. Then the landslide L_θ, where $\theta \in [0, 2\pi)$, on $\mathcal{T} \times \mathcal{T}$ is defined by

$$L_\theta(h, h^{\star}) := (h(\beta_\theta \bullet, \beta_\theta \bullet), h(\beta_{\theta+\pi} \bullet, \beta_{\theta+\pi} \bullet)), \quad \beta_\theta := \cos(\theta/2)E + \sin(\theta/2)Jb, \tag{1.2}$$

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where β_θ is a family of bundle morphisms with the identity $E : TM \to TM$ and the complex structure J with respect to h. Clearly $\det \beta_\theta = 1$, and β_θ satisfies the Codazzi equation [2, Lemma 3.2]. Additionally, the metric $h_\theta(u, v) = h(\beta_\theta u, \beta_\theta v)$ is a hyperbolic metric on M for all $\theta \in \mathbb{R}$ and L defines an S^1-flow on $\mathcal{T} \times \mathcal{T}$ [2, Theorem 1.8], that is, $L_{\theta'}(L_\theta(h, h^\star)) = L_{\theta'+\theta}(h, h^\star)$ holds. Define L^1 as the composition of L with the projection onto the first factor. Then $L^1_{e^{i\theta}}$ is analogous to Thurston's earthquake map $E : \mathcal{T} \times \mathcal{ML} \to \mathcal{T}$, where $\mathcal{ML}$ is the space of measured laminations on M.

The smooth grafting: For a given pair $(h, h^\star) \in \mathcal{T} \times \mathcal{T}$ and a positive number $s > 0$, the *smooth grafting* SGr_s is defined using a constant Gaussian curvature $-1 < K < 0$ (CGC) surface in hyperbolic 3-space $\mathbb{H}^3$ and its Gauss map, [2, Section 5]. Let b be the Labourie operator such that $h^\star = h(b\bullet, b\bullet)$. Define the metric $\mathrm{I} = \cosh^2(s/2)h$ and the operator $B = -\tanh(s/2)b$. Then the properties (1) through (4) of the Labourie operator imply that there exists a unique CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface f in $\mathbb{H}^3$ whose first fundamental form and shape operator are given by I and B, respectively. unique geodesic starting from a point $f(p) \in \mathbb{H}^3 (p \in M)$, with the unit normal $n(p)$ of $f(p)$ as initial velocity. Then by considering the point the geodesic intersecting with the boundary of $\mathbb{H}^3 \cong \mathbb{C}P^1$, a developing map $\mathrm{dev}_s : \widetilde{M} \to \mathbb{C}P^1$ and a complex projective structure on M are given, where $\widetilde{M}$ denotes the universal cover of M. In this way, the smooth grafting $SGr_s : \mathcal{T} \times \mathcal{T} \to \mathcal{P}$ is defined.

The complex landslide flow $P_q = SGr_s \circ L_\theta$, $(s = -\log|q|,\ \theta = \arg q)$ shares many properties with the complex earthquake flow, and in particular, it converges to the complex earthquake flow, [2, 3]. In particular, in [2, Theorem 5.1], it was shown that the complex landslide P was actually a flow, and it was holomorphic with respect to $q \in \mathbb{D}^\times$ by showing the holomorphicity of the holonomy of the projective structure. See Section 2.4 below, [2, Section 5] and [2, Proposition 5.14] or [1] for more details.

(B) In [8], weakly complete CGC surfaces with $-1 < K < 0$ in $\mathbb{H}^3$ were classified using integrable system methods, in particular the *loop group method*. The heart of the classification was based on harmonic maps from a Riemann surface M into a symmetric space, the hyperbolic 2-space $\mathbb{H}^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$ and the spectral parameter deformation family in $\mathbb{C}^\times$.

Harmonic maps: Recall that a harmonic map from a Riemann surface M into a symmetric space G/K is characterized as follows, for example see [7]. Let us consider a smooth map g from a Riemann surface M into a symmetric space G/K. Denote the universal cover of M by $\widetilde{M}$, take a lift $F : \widetilde{M} \to G$, and let $\alpha = F^{-1}\mathrm{d}F$ be the Maurer-Cartan form. Moreover, define a family of connections parameterized by $\lambda \in S^1 \subset \mathbb{C}$

$$\nabla^\lambda = \mathrm{d} + \alpha^\lambda, \quad \text{with} \quad \alpha^\lambda = \lambda^{-1}\alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda\alpha''_{\mathfrak{p}}, \tag{1.3}$$

where $\alpha_{\mathfrak{p}}$ and $\alpha_{\mathfrak{k}}$ denote the $\mathfrak{p}$- and the $\mathfrak{k}$-parts of the direct sum decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathrm{Lie}(K) = \mathfrak{k}$, the Cartan decomposition of $\mathfrak{g}$. Moreover, $'$ and $''$ denote the $(1,0)$- and $(0,1)$-part with respect to the complex structure of the Riemann surface M. Then ∇^λ defines a family of connections parameterized by $\lambda \in \mathbb{C}^\times$ and $\mathrm{d} + \alpha^\lambda|_{\lambda=1}$ gives a flat connection from the Maurer-Cartan equation $(\mathrm{d}\alpha^\lambda + \alpha^\lambda \wedge \alpha^\lambda)|_{\lambda=1} = 0$ for the frame F. It is well known that $g : M \to G/K$ is harmonic if and only if ∇^λ gives a family of *flat connections* for any $\lambda \in S^1$, that is, $\mathrm{d}\alpha^\lambda + \alpha^\lambda \wedge \alpha^\lambda = 0$ holds for all $\lambda \in S^1$. Then there exists a family of frames

$$F^\lambda : \widetilde{M} \to \Lambda G_\tau$$

such that $\alpha^\lambda = (F^\lambda)^{-1} \mathrm{d} F^\lambda$ and it is called the *extended frame* of the harmonic map g. Here ΛG_τ denotes the loop group of G:

$$\Lambda G_\tau = \{\gamma : S^1 \to G \mid \gamma \text{ is smooth and } \gamma(-\lambda) = \tau\gamma(\lambda)\},$$

where τ is the involution corresponding to the symmetric space G/K, that is, $(\operatorname{fix}\tau)_0 \subset K \subset \operatorname{fix}\tau$, where $\operatorname{fix}\tau$ denotes the fixed point set of τ in G, and the subscript zero denotes the identity component. Moreover, introducing a suitable topology to ΛG_τ, it becomes an infinite-dimensional Banach Lie group, the so-called *loop group* [12]. It has been known as the Weierstrass type representation for such harmonic maps through the loop group decomposition of ΛG_τ, see [7]. Note that the harmonic map from a Riemann surface M into G/K depends on the complex structure on M and the metric of G/K, which can be induced by the Killing form of G. However the complex structure on M for the harmonic map is not directly related to a pair of points $\mathcal{T} \times \mathcal{T}$ to define the complex landslide in (A).

The spectral parameter deformation: In our case, G/K is the hyperbolic two space $\mathbb{H}^2$ which can be represented by the degree 2 indefinite special unitary group $G = \mathrm{SU}(1,1)$ and the isotropy subgroup $K = \mathrm{U}(1) = \{\operatorname{diag}(e^{it}, e^{-it}) \mid t \in \mathbb{R}\} \subset \mathrm{SU}(1,1)$. Note that the involution τ is explicitly given by $\tau F = \operatorname{Ad}\operatorname{diag}(1,-1)F$ for $F \in \mathrm{SU}(1,1)$. Let us consider a harmonic map from M into $\mathbb{H}^2$ and take the extended frame $F^\lambda : \widetilde{M} \to \Lambda\mathrm{SU}(1,1)_\tau$ as above. If we evaluate the extended frame F^λ at $\lambda \in \mathbb{C}^\times \setminus S^1$ then it takes values in $\Lambda\mathrm{SL}(2,\mathbb{C})_\tau$ not in $\Lambda\mathrm{SU}(1,1)_\tau$. In [8, Theorem 2.1], it was shown that the extended frame F^λ gave a family of CGC surfaces in $\mathbb{H}^3$ as

$$\hat{f}_q = F^\lambda (F^\lambda)^*|_{\lambda=q}, \quad (q \in \mathbb{D}^\times),$$

where the upper subscript $*$ denotes a composition of the complex conjugation and the transpose, and the constant Gaussian curvature K of $\hat{f}_q$ is given by

$$K = -\left(\frac{2|q|}{|q|^2+1}\right)^2 \in (-1,0), \quad (|q| \in (0,1)).$$

Note here that the hyperbolic three-space $\mathbb{H}^3$ is realized by the quadric in the complex Hermitian 2-by-2 matrices, see Section 2.1. The evaluation at $q \in \mathbb{D}^\times$ has been called the *spectral parameter deformation* of the extended frame [8, Section 2.4]. This construction was the heart of the classification of weakly complete CGC $-1 < K < 0$ surfaces in $\mathbb{H}^3$ Moreover, for a fixed $|q| \in (0,1)$, $\hat{f}_q$ gives the S^1-family of CGC surfaces with a fixed Gaussian curvature in $-1 < K = -4|q|^2/(|q|^2+1)^2 < 0$ in $\mathbb{H}^3$ and it becomes the *associated family* of a CGC $-1 < K < 0$ surface in $\mathbb{H}^3$. The scaled first and third fundamental forms of $\hat{f}_q$ are hyperbolic metrics. In this way, we obtain a point in $\mathcal{T} \times \mathcal{T}$ from the harmonic map from M into $\mathbb{H}^2$.

(C) From the discussions of (A) and (B), it is natural to expect a certain relation between the above constructions. In both constructions, the S^1-families and the scaling families naturally appear, and in particular the landslide and the smooth grafting look similar to the associated family and the spectral parameter deformation, respectively. The following theorem is the main result of the paper.

Theorem 1.1. *Let $P_q(h, h^\star)$ ($q \in \overline{\mathbb{D}}^\times, (h, h^\star) \in \mathcal{T} \times \mathcal{T}$) be the complex landslide flow. Then there exists a family of flat connections ∇^λ of a harmonic map into $\mathbb{H}^2$ corresponding to*

$(h, h^\star)$ *such that the holonomy of* $P_q(h, h^\star)$ *is given by the holonomy of* ∇^λ *evaluated at* $\lambda = \sqrt{q} \in \overline{\mathbb{D}}^\times$.

We will give a proof in Section 3. Note that the extended frame F^λ has a holomorphic dependence on the spectral parameter $\lambda \in \mathbb{C}^\times$. In particular, it is defined on $\overline{\mathbb{D}}^\times$. Then, by noting Remark 1.1 (1), the holomorphic dependence of the holonomy of the complex landslide flow follows, and the following statement holds.

Corollary 1.1. *The map* $q \mapsto P_q(h, h^\star)$*, from* $\mathbb{D}^\times$ *to* $\mathcal{P}$*, is holomorphic.*

Remark 1.1.

(1) *Because we utilized the twisted loop group formulation of a harmonic map into a symmetric space, the square root* $\lambda = \sqrt{q}$ *in* Theorem 1.1 *arises. However, using the untwisted loop group formulation as demonstrated in* [4]*, it can be expressed by the evaluation as* $\lambda = q$*, see the proof in* Section 3*. Then* Corollary 1.1 *follows immediately.*
(2) Corollary 1.1 *has been proved in* [2, Theorem 5.1] *by a direct computation of derivatives of the holonomy and by showing that the Cauchy-Riemann equation holds, however,* Theorem 1.1 *gives a natural correspondence between the complex landslide flow and the family of flat connections.*
(3) *A relation between the harmonic map and the complex landslide has been also discussed in* [2]*, and the spectral parameter deformation family in this paper will give a natural understanding of such relation through the integrable systems approach. The key observation is a relation in* Lemma 3.1 *of two complex structures on* M*, namely, the* J *and* $i = \sqrt{-1}$ *induced from the first and second fundamental forms of a CGC surface, respectively.*

The paper is organized as follows. Preliminaries will be given in Section 2 and the proof of Theorem 1.1 will be given in Section 3.

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2. Preliminaries

In this section, according to [8] we will recall surfaces in the hyperbolic three-space $\mathbb{H}^3$ and two Gauss maps, that is, the Legendrian and the Lagrangian Gauss maps, respectively. Then by the spectral parameter deformation of the Lagrangian Gauss map, we will relate the extended frame of a harmonic map into hyperbolic two-space $\mathbb{H}^2$ and a constant Gaussian curvature $-1 < K < 0$ surface in $\mathbb{H}^3$. Moreover, we will review the complex landslide and a CGC $-1 < K < 0$ surface in $\mathbb{H}^3$ according to [2].

2.1. **Surfaces in the hyperbolic three-space.** Let

$$\boldsymbol{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \boldsymbol{e}_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The 4-dimensional Minkowski space $\mathbb{E}^{1,3}$ can be identified with the complex Hermitian 2-by-2 matrices

$$\mathrm{Her}(2,\mathbb{C}) = \left\{ \boldsymbol{\xi} = \sum_{j=0}^{3} \xi_j \boldsymbol{e}_j \mid \xi_j \in \mathbb{R} \right\} \longleftrightarrow \mathbb{E}^{1,3} = \{(\xi_0, \xi_1, \xi_2, \xi_3) \mid \xi_0, \xi_1, \xi_2, \xi_3 \in \mathbb{R}\}\,.$$

Then the standard inner product of $\mathbb{E}^{1,3}$ is given by $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\xi} \boldsymbol{e}_2 {}^t\boldsymbol{\xi} \boldsymbol{e}_2)$ and $\langle \boldsymbol{\xi}, \boldsymbol{\xi} \rangle = -\det \boldsymbol{\xi}$ under the identification of $\mathbb{E}^{1,3} \cong \mathrm{Her}(2,\mathbb{C})$. The *hyperbolic* 3*-space* $\mathbb{H}^3$ is defined by the unit central hyperquadrics in $\mathbb{E}^{1,3} \cong \mathrm{Her}(2,\mathbb{C})$:

$$\mathbb{H}^3 = \{\boldsymbol{\xi} \in \mathrm{Her}(2,\mathbb{C}) \mid \det \boldsymbol{\xi} = 1,\ \operatorname{tr} \boldsymbol{\xi} > 0\}\,.$$

The special linear group of degree two $\mathrm{SL}(2,\mathbb{C})$ acts isometrically and transitively on $\mathbb{H}^3$ via the action $(g, \boldsymbol{\xi})$ as $g\boldsymbol{\xi}g^*$, where the subscript $*$ denotes $g^* = {}^t\bar{g}$ for $g \in \mathrm{SL}(2,\mathbb{C})$. The isotropy subgroup of this action at $\boldsymbol{e}_0 = (1,0,0,0) \cong \mathrm{id}$ is the special unitary group $\mathrm{SU}(2)$ and hence $\mathbb{H}^3$ can be represented as a homogeneous Riemannian symmetric space $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. The natural projection $\pi : \mathrm{SL}(2,\mathbb{C}) \to \mathbb{H}^3$ is given by $\pi(g) = gg^*$ and thus the hyperbolic 3-space can be represented as $\mathbb{H}^3 = \{gg^* \mid g \in \mathrm{SL}(2,\mathbb{C})\}$.

Let $f : M \to \mathbb{H}^3$ be a oriented surface and assume that its Gaussian curvature satisfies $K > -1$. Then by the formula $K = -1 + \det B$, where B is the shape operator of f, $\det B > 0$ follows. Thus the second fundamental form $\mathrm{I\!I}$ defines a Riemannian metric which induces a complex structure on f. Let us denote the complex coordinate induced from the second fundamental form by $z = x + yi$. Then the fundamental forms can be represented as follows; Set $\sigma = \det B$ and the unit normal of f as n. The first, second and third fundamental forms can be computed by

$$\mathrm{I} = \langle \mathrm{d}f, \mathrm{d}f \rangle = Q\, \mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\, \mathrm{d}z\mathrm{d}\bar{z} + \bar{Q}\, \mathrm{d}\bar{z}^2, \tag{2.1}$$

$$\mathrm{I\!I} = \langle \mathrm{d}f, \mathrm{d}n \rangle = \sigma(e^u - |Q|^2 e^{-u})\, \mathrm{d}z\mathrm{d}\bar{z}, \tag{2.2}$$

$$\mathrm{I\!I\!I} = \langle \mathrm{d}n, \mathrm{d}n \rangle = \sigma^2 \left(-Q\, \mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\, \mathrm{d}z\mathrm{d}\bar{z} - \bar{Q}\, \mathrm{d}\bar{z}^2\right), \tag{2.3}$$

where $u : M \to \mathbb{R}$ is a smooth function and $Q\,\mathrm{d}z^2$ is the $(2,0)$-part of the first fundamental form with respect to the conformal structure of the second fundamental form, which will be called the *Klotz differential*. Note that the mean curvature H and the Gaussian curvature K of the surface f can be represented by

$$H = \frac{\sigma}{2} \frac{e^{2u} + |Q|^2}{e^{2u} - |Q|^2} \quad \text{and} \quad K = -1 + \sigma^2, \tag{2.4}$$

see [8, Section 1.1] for more details.

2.2. **Natural projections, two Gauss maps and a Ruh-Vilms type theorem.** We have several natural projections from the unit tangent sphere bundle of $\mathbb{H}^3$

$$\mathrm{U}\mathbb{H}^3 = \{(\boldsymbol{x}, \boldsymbol{v}) \in \mathrm{Her}_2\mathbb{C} \times \mathrm{Her}_2\mathbb{C} \mid \det \boldsymbol{x} = 1, \operatorname{tr} \boldsymbol{x} > 0,\ \det \boldsymbol{v} = -1,\ \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0\},$$

see [6] for details. A natural projection $\pi_+ : \mathrm{U}\mathbb{H}^3 \to \mathbb{H}^3$ is given by $\pi_+(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x}$. Let $\mathrm{Geo}(\mathbb{H}^3)$ be the space of all oriented geodesics in $\mathbb{H}^3$ which is identified with the Grassmannian manifold $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,3})$ of all oriented timelike planes in $\mathbb{E}^{1,3} = \mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C}$. $\mathrm{Geo}(\mathbb{H}^3)$ can be represented as

$$\mathrm{Geo}(\mathbb{H}^3) \cong \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}) = \mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C} \cong \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \mathrm{diag}\,. \tag{2.5}$$

Then we have a natural projection $\pi_0 : \mathrm{U}\mathbb{H}^3 \to \mathrm{Geo}(\mathbb{H}^3)$ by $\pi_0(\boldsymbol{x}, \boldsymbol{v}) = \boldsymbol{x} \wedge \boldsymbol{v}$. Finally there is also a natural projection from $\pi_* : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{U}\mathbb{H}^3$ by $\pi_*(g) = (gg^*, g\boldsymbol{e}_1 g^*)$. Note that the projection $\pi : \mathrm{SL}(2, \mathbb{C}) \to \mathbb{H}^3 = \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$ is given by the composition $\pi = \pi_+ \circ \pi_*$.

Using these projections, we will recall two Gauss maps for a surface in $\mathbb{H}^3$, which are crucial in this paper. Note that several natural Gauss maps were already defined in see Appendix of the archive version [6].

Definition 2.1. *Let $f : M \to \mathbb{H}^3$ be a surface with unit normal n, and let* $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}) = \mathrm{Geo}(\mathbb{H}^3)$. *The* Legendrian Gauss map G_{Le} *and the* Lagrangian Gauss map G_{La} *of f are respective defined by*

$$G_{Le} = (f, n) : M \to \mathrm{U}\mathbb{H}^3 \quad \textit{and} \quad G_{La} = f \wedge n : M \to \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}).$$

In [8, Theorem A], the following theorem has been proved.

Theorem 2.1. *Let $f : M \to \mathbb{H}^3$ be a surface with Gaussian curvature $K > -1$ and let n be the unit normal of f. Moreover let $G_{Le} = (f, n) : M \to \mathrm{U}\mathbb{H}^3$ and $G_{La} = f \wedge n : M \to \mathrm{Geo}(\mathbb{H}^3)$ be the Legendrian and Lagrangian Gauss maps, respectively. Then the following statements are mutually equivalent*:

(1) *The Gaussian curvature of f is constant.*
(2) *The Legendrian Gauss map is harmonic.*
(3) *The Lagrangian Gauss map is harmonic.*
(4) *The Klotz differential is holomorphic.*

2.3. **The extended frames and CGC surfaces.** Let us consider the structure equations for a CGC $K > -1$ surface:

$$\bar{\partial}\partial u + \frac{K}{2}(e^u - |Q|^2 e^{-u}) = 0 \quad \text{and} \quad \bar{\partial} Q = 0, \tag{2.6}$$

where we set

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Note that, under a normalization $Q = 1$, the elliptic PDE (2.6) for u becomes the famous *sinh-Gordon* equation [11].

If u and Q are solutions to (2.6), then u and $\lambda^{-2}Q$ with any constant $\lambda \in S^1$ are also solutions to (2.6). Thus there exists a family of CGC $K > -1$ surfaces $\{f^\lambda\}_{\lambda \in S^1}$ in $\mathbb{H}^3$. Accordingly, there exists a family of unit normal vectors $\{n^\lambda\}_{\lambda \in S^1}$ such that

$$G^\lambda_{Le} = (f^\lambda, n^\lambda) : M \to \mathrm{U}\mathbb{H}^3 \quad \text{and} \quad G^\lambda_{La} = f^\lambda \wedge n^\lambda : M \to \mathrm{Geo}(\mathbb{H}^3)$$

are a family of Legendrian and Lagrangian harmonic Gauss maps of f^λ, respectively. Then the family of CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surfaces in $\mathbb{H}^3$ has the following fundamental forms:

$$\mathrm{I}^\lambda = \langle \mathrm{d}f^\lambda, \mathrm{d}f^\lambda\rangle = \lambda^{-2} Q\,\mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar{z} + \lambda^2 \bar{Q}\,\mathrm{d}\bar{z}^2, \tag{2.7}$$

$$\mathrm{II}^\lambda = \langle \mathrm{d}f^\lambda, \mathrm{d}n^\lambda\rangle = \sigma(e^u - |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar{z}, \tag{2.8}$$

$$\mathrm{III}^\lambda = \langle \mathrm{d}n^\lambda, \mathrm{d}n^\lambda\rangle = \sigma^2\left(-\lambda^{-2} Q\,\mathrm{d}z^2 + (e^u + |Q|^2 e^{-u})\,\mathrm{d}z\mathrm{d}\bar{z} - \lambda^2 \bar{Q}\,\mathrm{d}\bar{z}^2\right), \tag{2.9}$$

where

$$\sigma = \tanh(s/2), \quad (s > 0), \tag{2.10}$$

and the Gaussian curvature K of f^λ is

$$-1 < K = -1 + \sigma^2 = -\frac{1}{\cosh^2(s/2)} < 0.$$

The S^1-parameter λ has been called the *spectral parameter*, and the family of CGC $-1 < K < 0$ surfaces $\{f^\lambda\}_{\lambda\in S^1}$ in the above has been called the *associated family* of f. Note that II^λ is the same for any $\lambda \in S^1$, that is $\mathrm{II}^\lambda = \mathrm{II}$.

On the other hand, for a fixed positive number $s > 0$, define two metrics parameterized by $\theta = \arg\lambda$, $(\lambda \in S^1)$ by

$$h_\theta := \frac{1}{\cosh^2(s/2)} \mathrm{I}^\lambda \quad \text{and} \quad h^\star_\theta := \frac{1}{\sinh^2(s/2)} \mathrm{III}^\lambda.$$

Then $(h_\theta, h^\star_\theta)$ is a pair of hyperbolic metrics. Moreover by using the shape operator $B^\lambda = \left(\mathrm{I}^\lambda\right)^{-1}\mathrm{II}^\lambda$,

$$b_\theta = -\coth(s/2)\, B^\lambda$$

is the Labourie operator for $(h_\theta, h^\star_\theta)$.

In [8, Theorem B (1)], it has been proved that the family $\{G^\lambda_{La}\}_{\lambda\in S^1}$ of Lagrangian Gauss maps of $\{f^\lambda\}_{\lambda\in S^1}$ can be characterized as follows:

Theorem 2.2 (Theorem B (1) in [8])**.** *Let f^λ be the family of CGC $-1 < K < 0$ surfaces in $\mathbb{H}^3$ defined as above, and set $q \in \mathbb{D}^\times$ such that $|q| = \exp(-\operatorname{arcosh}\sqrt{-1/K})$ holds. Then the map $G^\lambda_{La}|_{\lambda=q}$ is a harmonic local diffeomorphism into $\mathbb{H}^2$.*

Conversely, let $g : M \to \mathbb{H}^2$ be a harmonic map and consider the extended frame

$$F^\lambda : \widetilde{M} \to \Lambda\mathrm{SU}(1,1)_\tau,$$

as explained in the introduction. The extended frame is defined not only on S^1 but also on $\mathbb{C}^\times$. Then $F^\lambda|_{\lambda\in\mathbb{C}^\times}$ is a map into $\Lambda\mathrm{SL}(2,\mathbb{C})_\tau$ not $\Lambda\mathrm{SU}(1,1)_\tau$.

Let $\pi : \mathrm{SU}(1,1) \to \mathbb{H}^2$ be a natural projection. For the extended frame F^λ, $\pi \circ F^\lambda|_{\lambda=1}$ is the original harmonic map g up to isometry and moreover, $g^\lambda = \pi \circ F^\lambda|_{\lambda\in S^1}$ gives a family of harmonic maps in $\mathbb{H}^2$, the so-called associated family of g. If $\hat{Q}\,\mathrm{d}z^2$ is the Hopf differential of g, then the Hopf differential of g^λ can be realized as $\lambda^{-2}\hat{Q}\,\mathrm{d}z^2$.

Theorem 2.3 (Theorem 2.1 in [8])**.** *Let F^λ be the extended frame of a harmonic map $g : M \to \mathbb{H}^2 = \mathrm{SU}(1,1)/\mathrm{U}(1)$ and let $q = \exp(-t + i\theta) \in \mathbb{D}^\times$. Then*

$$\hat{f}_q = F^\lambda (F^\lambda)^*|_{\lambda=q}, \quad \hat{n}_q = F^\lambda \boldsymbol{e}_1 (F^\lambda)^*|_{\lambda=q},$$

is a family of CGC surfaces in $\mathbb{H}^3$ with the constant Gaussian curvature

$$K = -\left(\frac{2|q|}{|q|^2+1}\right)^2 = -\frac{1}{\cosh^2 t} \in (-1,0) \tag{2.11}$$

and a family of unit normals $\hat{n}_q$. The Klotz differential of $\hat{f}_q$ is given by

$$Q^q \,\mathrm{d}z^2 = e^{-2i\theta} \left(\frac{|q|^2+1}{2|q|}\right)^2 \hat{Q}\,\mathrm{d}z^2. \tag{2.12}$$

By abuse of notation, the CGC $K = -1/\cosh^2 t$ surface $\hat{f}_q$ will be called the *spectral parameter deformation*, and the S^1-family $\{\hat{f}_q\}_{q=|q|e^{i\theta}}$ with fixed $|q|$ becomes the associated family. Therefore there exists the associated family $\{f^\lambda\}_{\lambda \in S^1}$ of some CGC $K = -1/\cosh^2(s/2)$ with $s = 2t$ and $Q = Q^q$ surface f such that

$$f^\lambda = \hat{f}_q$$

holds up to rigid motion.

Moreover, the Lagrangian Gauss map can be represented by F^λ as

$$G^\lambda_{La} = \hat{f}_q \wedge \hat{n}_q = F^\lambda \boldsymbol{e}_1 (F^\lambda)^{-1}|_{\lambda=q\in\mathbb{D}^\times}. \tag{2.13}$$

Remark 2.1. *From* (2.11) *and* (2.12)*, the family of CGC surfaces $\hat{f}_q$ in* Theorem 2.3 *has a symmetry, that is, $\hat{f}_q$ and $\hat{f}_{1/q}$ are the same CGC surface up to rigid motion.*

2.4. **The complex landslides flow and CGC surfaces.** Recall that the landslide (1.2) is given by

$$L_\theta(h, h^\star) := (h(\beta_\theta, \beta_\theta), h(\beta_{\theta+\pi}, \beta_{\theta+\pi})),$$

where $\beta_\theta := \cos(\theta/2)E + \sin(\theta/2)Jb$, and b is the Labourie operator for $(h, h^\star)$, see the introduction. Note that J is the complex structure compatible with h and it is not related to the complex structure given in Section 2.1. We recall the following fundamental lemma, see [2].

Lemma 2.1. *Let $(h, h^\star) \in \mathcal{T} \times \mathcal{T}$ and a positive constant $s > 0$. Then there exists a unique CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface f in $\mathbb{H}^3$ such that the first and third fundamental forms are respectively given as*

$$\mathrm{I} = \cosh^2(s/2)h \quad \textit{and} \quad \mathrm{I\!I\!I} = \sinh^2(s/2)h^\star. \tag{2.14}$$

Conversely, for a given CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface f in $\mathbb{H}^3$, define h and $h^\star$ by (2.14)*. Then h and $h^\star$ are hyperbolic metrics.*

Proof. For a given $(h, h^\star)$, there exists a unique Labourie operator b such that $h(b\bullet, b\bullet) = h^\star$, see the introduction. Define

$$\mathrm{I} = \cosh^2(s/2)h, \quad B = -\tanh(s/2)b.$$

Note that $\det B = \tanh^2(s/2)$ holds.

Then from the properties (2), (3) of the Labourie operator b in the introduction (A), the pair (I, B) satisfies

$$\mathrm{d}^{\nabla} B = 0, \quad K = -1 + \det B, \tag{2.15}$$

where ∇ is the Levi-Civita connection of I and $K = -1/\cosh^2(s/2)$ is the curvature of I. Since $(\mathrm{d}^{\nabla} B)(u, v) = 0$ is

$$\nabla_u(Bv) - \nabla_v(Bu) = B[u, v],$$

which is the Codazzi equation for a surface in $\mathbb{H}^3$, see [5, p.138]. Moreover, B is self-adjoint with respect to h by (1) in the introduction (A). Since the second equation in (2.15) is just the Gauss equation, thus there exists a unique surface f up to rigid motion such that the first fundamental form is I and the second fundamental form II is $\mathrm{II}(u, v) := \mathrm{I}(u, Bv)$. Then, the third fundamental form III can be computed as

$$\begin{aligned} \mathrm{III}(u, v) &= \mathrm{I}(Bu, Bv) \\ &= \tanh^2(s/2)\mathrm{I}(bu, bv) \\ &= \sinh^2(s/2)h(bu, bv) \\ &= \sinh^2(s/2)h^\star(u, v), \end{aligned}$$

and the claim follows. The converse statement is clear and this completes the proof. $\square$

The complex landslide flow $P : \overline{\mathbb{D}}^{\times} \times \mathcal{T} \times \mathcal{T} \to \mathcal{P}$ is defined by $P(q, h, h^\star) = SGr_s(L_\theta(h, h^\star))$ with $s = -\log|q|$ and $\theta = -\arg q$, where SGr_s is the smooth grafting, which has been explained in the introduction:

(1) Begin by considering the pair of hyperbolic metrics by the landslide $L_\theta(h, h^\star)$.
(2) By Lemma 2.1, there exists a CGC surface f with Gaussian curvature in $-1 < K = -1/\cosh^2(s/2) < 0$.
(3) Consider the geodesic whose initial velocity is the unit normal $n(p)$ of $f(p)$, and take the point where it intersects the boundary of $\mathbb{H}^3 \cong \mathbb{C}P^1$.
(4) It gives a developing map

$$\mathrm{dev}_s : \widetilde{M} \to \mathbb{C}P^1, \tag{2.16}$$

which defines a complex projective structure on M.

It is proved that the complex landslide flow P is holomorphic with respect to $\overline{\mathbb{D}}^{\times}$ by showing that the holonomy of the developing map is holomorphic, [2, Theorem 5.1].

3. Proof of the main theorem

We now prove Theorem 1.1 in the introduction. The crucial observation is the following lemma about a relation of the complex structure J of the first fundamental form I and the complex structure, which is denoted by $z = x + yi$, in the conformal class of the second fundamental form II.

Lemma 3.1. *Let f be a CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surface in $\mathbb{H}^3$ with the fundamental forms* I, II *and* III *in* (2.1), (2.2) *and* (2.3), *respectively. Define an endomorphism $J : TM \to TM$ by*

$$\begin{cases} J\partial_z := i\coth(s/2)B\partial_z \\ J\partial_{\bar{z}} := -i\coth(s/2)B\partial_{\bar{z}} \end{cases}, \tag{3.1}$$

where B is the shape operator of f. Then J is the unique complex structure on M compatible with the first fundamental form I.

Proof. Since $\mathrm{II}(\bullet,\bullet) = \mathrm{I}(\bullet, B\bullet)$, $\mathrm{III}(\bullet,\bullet) = \mathrm{I}(B\bullet, B\bullet)$, we have

$$\mathrm{I}(J\partial_z, J\partial_z) = -\coth^2(s/2)\mathrm{I}(B\partial_z, B\partial_z) = -\coth^2(s/2)\mathrm{III}(\partial_z,\partial_z) = Q = \mathrm{I}(\partial_z,\partial_z),$$
$$\mathrm{I}(J\partial_{\bar{z}}, J\partial_{\bar{z}}) = -\coth^2(s/2)\mathrm{I}(B\partial_{\bar{z}}, B\partial_{\bar{z}}) = -\coth^2(s/2)\mathrm{III}(\partial_{\bar{z}},\partial_{\bar{z}}) = \bar{Q} = \mathrm{I}(\partial_{\bar{z}},\partial_{\bar{z}}),$$

and

$$\mathrm{I}(J\partial_z, J\partial_{\bar{z}}) = \coth^2(s/2)\mathrm{I}(B\partial_z, B\partial_{\bar{z}}) = \coth^2(s/2)\mathrm{III}(\partial_z,\partial_{\bar{z}}) = \frac{1}{2}(e^u + |Q|^2e^{-u}) = \mathrm{I}(\partial_z,\partial_{\bar{z}})$$

hold, where $Q\,\mathrm{d}z^2$ is the Klotz differential and H is the mean curvature in (2.4). Thus J is compatible with I. Moreover, by using the above equations, we have explicit formulas of $B\partial_z$ and $B\partial_{\bar{z}}$, and

$$\begin{aligned} J\partial_z &= \frac{2iH}{\tanh(s/2)}\partial_z - \frac{2iQ}{e^u - |Q|^2e^{-u}}\partial_{\bar{z}}, \\ J\partial_{\bar{z}} &= \frac{2i\bar{Q}}{e^u - |Q|^2e^{-u}}\partial_z - \frac{2iH}{\tanh(s/2)}\partial_{\bar{z}} \end{aligned} \tag{3.2}$$

follow. From the form J in (3.2), it is easy to see that

$$J^2\partial_z = -\partial_z \quad \text{and} \quad J^2\partial_{\bar{z}} = -\partial_{\bar{z}}.$$

It is also easy to see that J preserves the orientation, and thus J is the unique complex structure on M compatible with I. This completes the proof. □

Corollary 3.1. *Let β_θ be the family of operators by $\beta_\theta = \cos(\theta/2)E + \sin(\theta/2)Jb$, $(\theta \in [0, 2\pi))$ of the landslide flow in* (1.2). *Moreover, set $\lambda^{1/2} = \cos(\theta/2) + i\sin(\theta/2) \in S^1$. Then the following relations hold*:

$$\beta_\theta\partial_z = \lambda^{-1/2}\partial_z \quad \textit{and} \quad \beta_\theta\partial_{\bar{z}} = \lambda^{1/2}\partial_{\bar{z}}. \tag{3.3}$$

Proof. A straightforward computation by using (3.1) shows that

$$\beta_\theta\partial_z = \cos(\theta/2)\partial_z + \sin(\theta/2)Jb\partial_z = \cos(\theta/2)\partial_z - i\sin(\theta/2)\partial_z = \lambda^{-1/2}\partial_z$$

holds. Similarly $\beta_\theta\partial_{\bar{z}} = \lambda^{1/2}\partial_{\bar{z}}$ holds. □

Combining the landslide flow and Corollary 3.1, we have the following proposition.

Proposition 3.1. *For a given landslide flow $(h_\theta, h^\star_\theta) = L_\theta(h, h^\star) \in \mathcal{T} \times \mathcal{T}$, $(\theta \in [0, 2\pi))$ and a positive constant $s > 0$, let $\tilde{f}_q$ $(q = \exp(-s + i\theta))$ be the corresponding CGC surface with $K = -1/\cosh^2(s/2)$ in $\mathbb{H}^3$ given by* Lemma 2.1. *Then there exists a CGC surface f with*

$K = -1/\cosh^2(s/2)$ *and its associated family* $\{f^\lambda\}_{\lambda \in S^1}$ *(where the conformal structure is given by the second fundamental form of* f*) in* $\mathbb{H}^3$ *such that*

$$\tilde{f}_q = f^{\sqrt{\lambda}} \quad (\lambda = \exp(i\theta) \in S^1)$$

holds up to rigid motion.

Proof. First recall that $(h_\theta, h_\theta^\star) = L_\theta(h, h^\star)$ is given by

$$(h_\theta, h_\theta^\star) = (h(\beta_\theta, \beta_\theta), h^\star(\beta_{\theta+\pi}, \beta_{\theta+\pi})), \quad \text{where } \beta_\theta = \cos(\theta/2)E + \sin(\theta/2)Jb,$$

and b is the Labourie operator between h and $h^\star$. The first and third fundamental forms of $\tilde{f}_q$ are given by

$$(\mathrm{I}_q, \mathrm{III}_q) = (\cosh^2(s/2)h_\theta, \sinh^2(s/2)h_\theta^\star).$$

Moreover, the second fundamental form is $\mathrm{II}_\theta = \mathrm{I}_q(\bullet, -\tanh(s/2)b_\theta\bullet)$, where b_θ is the Labourie operator between h_θ and $h_\theta^\star$, that is, $h_\theta^\star = h_\theta(b_\theta\bullet, b_\theta\bullet)$ holds, see Lemma 2.1. For $q_0 = \exp(-s) \in (0,1)$, we introduce the coordinates $z = x + yi$ such that the first, second and third fundamental forms $\mathrm{I}_{q_0}, \mathrm{II}_{q_0}$ and III_{q_0} of $\tilde{f}_{q_0}$ can be represented by (2.1), (2.2) and (2.3), that is,

$$(\mathrm{I}_{q_0}, \mathrm{II}_{q_0}, \mathrm{III}_{q_0}) = \cosh^2(s/2)(h,\ h(\bullet, B\bullet),\ h(B\bullet, B\bullet))$$

holds, where we set $B = -\tanh(s/2)b$ with $b = b_0$. It has been shown [2, Lemma 3.5] that the Labourie operator b_θ is explicitly given by $b_\theta = \beta_{-\theta} \circ b \circ \beta_\theta$.

To show that a CGC surface $\tilde{f}_q$ $(q = \exp(-s+i\theta))$ constructed by Lemma 2.1 is the associated family $f^{\sqrt{\lambda}}$ $(\lambda \in S^1)$ in Section 2.3, we compare $\mathrm{I}_q, \mathrm{II}_q$ and III_q and I^λ in (2.7), II^λ in (2.8) and III^λ in (2.9), respectively; Set $\lambda = e^{i\theta} \in S^1$. By Corollary 3.1,

$$\begin{aligned}
\mathrm{I}_\theta(\partial_z, \partial_z) &:= \cosh^2(s/2)h(\beta_\theta\partial_z, \beta_\theta\partial_z) \\
&= \cosh^2(s/2)h(\lambda^{-1/2}\partial_z, \lambda^{-1/2}\partial_z) \\
&= \lambda^{-1}\cosh^2(s/2)h(\partial_z, \partial_z) \\
&= \mathrm{I}^{\sqrt{\lambda}}(\partial_z, \partial_z)
\end{aligned}$$

holds. Similarly, $\mathrm{I}_\theta(\partial_{\bar{z}}, \partial_{\bar{z}}) = \mathrm{I}^{\sqrt{\lambda}}(\partial_{\bar{z}}, \partial_{\bar{z}})$ follows. Moreover,

$$\begin{aligned}
\mathrm{I}_\theta(\partial_z, \partial_{\bar{z}}) &:= \cosh^2(s/2)h(\beta_\theta\partial_z, \beta_\theta\partial_{\bar{z}}) \\
&= \cosh^2(s/2)h(\lambda^{-1/2}\partial_z, \lambda^{1/2}\partial_{\bar{z}}) \\
&= \cosh^2(s/2)h(\partial_z, \partial_{\bar{z}}) \\
&= \mathrm{I}^{\sqrt{\lambda}}(\partial_z, \partial_{\bar{z}})
\end{aligned}$$

holds, therefore $\mathrm{I}_\theta = \mathrm{I}^{\sqrt{\lambda}}$. The same argument can be applied to III_θ, and it follows $\mathrm{III}_\theta = \mathrm{III}^{\sqrt{\lambda}}$. Next the second fundamental form II_θ can be computed as

$$\mathrm{II}_\theta(\partial_z, \partial_z) = \mathrm{I}_\theta(\partial_z, B_\theta\partial_z).$$

Since $b_\theta = \beta_{-\theta} \circ b \circ \beta_\theta$, the shape operator B_θ is given by $\beta_{-\theta} \circ B \circ \beta_\theta$, and thus

$$\begin{aligned} \mathrm{I\!I}_\theta(\partial_z, \partial_z) &= \mathrm{I}_\theta(\partial_z, \beta_{-\theta} \circ B \circ \beta_\theta \partial_z) \\ &= \mathrm{I}(\beta_\theta \partial_z, B \circ \beta_\theta \partial_z) \\ &= \lambda^{-1} \mathrm{I}(\partial_z, B\partial_z) \\ &= 0 = \mathrm{I\!I}^{\sqrt{\lambda}}(\partial_z, \partial_z) \end{aligned}$$

holds. Similarly $\mathrm{I\!I}_\theta(\partial_{\bar z}, \partial_{\bar z}) = 0 = \mathrm{I\!I}^{\sqrt{\lambda}}(\partial_{\bar z}, \partial_{\bar z})$ holds. Finally

$$\begin{aligned} \mathrm{I\!I}_\theta(\partial_z, \partial_{\bar z}) &= \mathrm{I}_\theta(\partial_z, \beta_{-\theta} \circ B \circ \beta_\theta \partial_{\bar z}) \\ &= \mathrm{I}(\beta_\theta \partial_z, B \circ \beta_\theta \partial_{\bar z}) \\ &= \mathrm{I}(\partial_z, B\partial_{\bar z}) \\ &= \mathrm{I\!I}^{\sqrt{\lambda}}(\partial_z, \partial_{\bar z}). \end{aligned}$$

Thus $\mathrm{I\!I}_\theta = \mathrm{I\!I}^{\sqrt{\lambda}}$ also follows, and therefore the two families of CGC surfaces $\{\tilde f_q\}_{q=|q|e^{i\theta}}$ ($\theta \in [0, 2\pi)$) and $\{f^{\sqrt{\lambda}}\}_{\lambda \in S^1}$ are the same up to isometry. This completes the proof. □

Remark 3.1. *In the proof of* Proposition 3.1*, we showed the equivalence of the first, second and third fundamental forms, respectively. In fact the equivalence of two of them are sufficient, since surfaces are convex.*

We now look at a relation between the complex landslide and the spectral parameter deformation in Theorem 2.3.

Corollary 3.2. *Retain the assumptions in* Proposition 3.1*. Moreover let $\hat f_q$ be the spectral parameter deformation in* Theorem 2.3 *corresponding to f^λ. Then*

$$\tilde f_q = \hat f_{\sqrt{q}}, \quad (q = \exp(-s + i\theta) \in \mathbb{D}^\times),$$

holds. Moreover, the complex landslide $P_q (= SGr \circ L)$, $(q = \exp(-s + i\theta))$ is given by the associated family of the Lagrangian Gauss map of $\hat f_{\sqrt{q}}$ and vice versa.

Proof. The first statement follows from Theorem 2.3. Let us consider the smooth grafting SGr_s of the landslide $L_\theta(h, h^\star)$. By Proposition 3.1, there exists the family of corresponding CGC $-1 < K = -1/\cosh^2(s/2) < 0$ surfaces, which is the associated family $\{f^{\sqrt{\lambda}}\}_{\lambda \in S^1}$ with $\lambda = e^{i\theta}$. Then the developing map $\mathrm{dev}_s : \widetilde{M} \to \mathbb{C}P^1$ of SGr_s is given as in (2.16).

The Lagrangian Gauss map $G^\lambda_{La} = f^\lambda \wedge n^\lambda$ takes values in

$$\mathrm{Geo}(\mathbb{H}^3) \cong \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}) = \mathrm{SL}_2\mathbb{C}/\mathrm{GL}_1\mathbb{C} \cong \mathbb{C}P^1 \times \mathbb{C}P^1 \setminus \mathrm{diag},$$

which is a pair of intersecting points of the geodesic from the point $f^{\sqrt{\lambda}}(p)$ with initial velocity $\pm n(p)$ to the pair of boundaries of $\mathbb{H}^3 \cong \mathbb{C}P^1$, and thus dev_s is given by $\mathrm{pr}_1 \circ G^\lambda_{La}$, where pr_1 denotes the projection to the first component. Moreover, G^λ_{La} can be represented by

$$G^\lambda_{La} = \hat f_{\sqrt{q}} \wedge \hat n_{\sqrt{q}} = (F^\lambda)^{-1} \boldsymbol{e}_1 F^\lambda|_{\lambda=\sqrt{q}}, \quad (q = \exp(-s + i\theta) \in \mathbb{D}^\times). \tag{3.4}$$

Therefore the Lagrangian Gauss map gives the developing map dev_s and thus the complex landslide. The converse statement is also clear. This completes the proof. □

By Corollary 3.2, the holonomy of the complex landslide $P_q = SGr_s \circ L_\theta$ with $s = -\log|q|, \theta = \arg q$ is given by the holonomy of the extended frame F^λ evaluated at $\lambda = \sqrt{q}$, and the statement of Theorem 1.1 follows. We are now ready to prove Corollary 1.1 in the introduction.

Proof of Corollary 1.1. To see the holomorphic dependence with respect to λ, we consider the gauged extended frame:

$$F^\lambda \boldsymbol{e}_1 (F^\lambda)^{-1}|_{\lambda=\sqrt{q}} = F^\lambda D^\lambda \boldsymbol{e}_1 (F^\lambda D^\lambda)^{-1}|_{\lambda=\sqrt{q}} = D^{\sqrt{\lambda}} \hat{F}^\lambda \boldsymbol{e}_1 (\hat{F}^\lambda)^{-1} D^{1/\sqrt{\lambda}}\big|_{\lambda=q},$$

where $D^\lambda = \operatorname{diag}(\lambda^{-1/2}, \lambda^{1/2})$ and we set $\hat{F}^\lambda = (D^{1/\lambda} F^\lambda D^\lambda)|_{\lambda=\sqrt{\lambda}}$. Recalling that $\alpha^\lambda = (F^\lambda)^{-1} \mathrm{d} F^\lambda = \lambda^{-1} \alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \lambda \alpha''_{\mathfrak{p}}$, it is straightforward to see that

$$\hat{\alpha}^\lambda := (\hat{F}^\lambda)^{-1} \mathrm{d} \hat{F}^\lambda = \lambda^{-1} \alpha'_{\mathfrak{p}_{21}} + \alpha'_{\mathfrak{p}_{12}} + \alpha_{\mathfrak{k}} + \lambda \alpha''_{\mathfrak{p}_{12}} + \alpha''_{\mathfrak{p}_{21}}, \tag{3.5}$$

where $x_{ij} (i, j \in \{1, 2\})$ denotes the (ij)-entry for a 2 by 2 matrix valued 1-form x. Therefore $\hat{F}^\lambda$ is holomorphic with respect to $\lambda \in \mathbb{D}^\times$, and moreover

$$\widehat{\mathrm{dev}}_s = \mathrm{pr}_1 \circ \left(\hat{F}^\lambda \boldsymbol{e}_1 (\hat{F}^\lambda)^{-1} \right) \Big|_{\lambda=q}$$

defines the same complex projective structure given by dev_s. The gauged frame $\hat{F}^\lambda$ will be called the *untwisted extended frame*, and we set $\widehat{\nabla}^\lambda = \mathrm{d} + \hat{\alpha}^\lambda$.

Then the holonomy of the untwisted extended frame of $\widehat{\nabla}^\lambda$ is given by

$$\hat{H}^\lambda : \pi_1(M) \to \Lambda \mathrm{SL}(2, \mathbb{C})$$

with $\gamma^* \hat{F}^\lambda = \hat{H}^\lambda \hat{F}^\lambda$ for $\gamma \in \pi_1(M)$ and $\lambda \in \mathbb{D}^\times$. Clearly the holonomy $\hat{H}^\lambda$ is holomorphic with respect to the spectral parameter $\lambda \in \mathbb{D}^\times$, and thus the holonomy of the developing map $\widehat{\mathrm{dev}}_s$ is holomorphic with respect to λ in $\mathbb{D}^\times$. The limit $s \to 0$ to the holonomy of the developing map $\widehat{\mathrm{dev}}_s$ gives the holonomy at $\lambda \in S^1$. This completes the proof. $\square$

Remark 3.2. *It is well known* [4] *that the relation*

$$F^\lambda \longleftrightarrow \hat{F}^\lambda = (D^{1/\lambda} F^\lambda D^\lambda)|_{\lambda=\sqrt{\lambda}} \quad \textit{with} \quad D^\lambda = \operatorname{diag}(\lambda^{1/2}, \lambda^{-1/2})$$

is an isomorphism between twisted and untwisted loop groups. Note that, F^λ belongs to the twisted loop group $\Lambda \mathrm{SL}(2, \mathbb{C})_\tau$ with respect to τ, that is, $F^{-\lambda} = \tau F^\lambda$ holds, while $\hat{F}^\lambda$ belongs to the untwisted loop group $\Lambda \mathrm{SL}(2, \mathbb{C})$ and it does not satisfy such a relation.

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