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CODAZZI AND TOTALLY UMBILICAL HYPERSURFACES IN Sol_1^4

ZLATKO ERJAVEC AND JUN-ICHI INOGUCHI

1. Introduction

Totally geodesic submanifolds are the simplest and fundamental submanifolds of Riemannian manifolds. The existence of totally geodesic hypersurfaces imposes strong restrictions for Riemannian manifolds. For example, the only 3-dimensional simply connected homogeneous Riemannian spaces admitting totally geodesic surfaces are Riemannian symmetric spaces and $\widetilde{\mathrm{SL}}_2\mathbb{R}$, the universal covering of the special linear group equipped with a specific left invariant metric (see [28]).

Chen and Nagano [8] proved that a simply connected, irreducible symmetric space admits a totally geodesic hypersurface if and only if it is of constant curvature. Tsukada [27] extended this result to the class of simply connected irreducible naturally reductive homogeneous spaces. It should be remarked that the assumption of irreducibility of Chen-Nagano's theorem can not be removed. Indeed, the reducible Riemannian symmetric spaces $\mathbb{S}^n \times \mathbb{E}^1$ and $\mathbb{H}^n \times \mathbb{E}^1$ have totally geodesic hypersurfaces. Moreover $\mathbb{S}^2 \times \mathbb{S}^2$ admits totally geodesic surfaces (see [8, 21]).

As generalizations of totally geodesic hypersurfaces, parallel hypersurfaces as well as Codazzi hypersurfaces of homogeneous Riemannian spaces have been paid much attention of differential geometers.

Let us turn our attention to hypersurfaces in model spaces of 4-dimensional Thurston geometries. According to Filipkiewicz [18], there are nineteen kinds of simply conncted model spaces. In the list of model spaces of four dimensional geometries [18, 31], there are fourteen model spaces which admit compatible complex structure [31, Theorem 2.1]:

Complex space form	Hermitian symmetric	Kähler	Vaisman	non-Vaisman
$\mathbb{C}P^2$, $\mathbb{C}H^2$, $\mathbb{E}^4$	$\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$ $\mathbb{E}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$	F^4	$\mathbb{S}^3 \times \mathbb{E}^1$ $\mathrm{Nil}_3 \times \mathbb{E}^1$ $\widetilde{\mathrm{SL}}_2R \times \mathbb{E}^1$	Sol_0^4 Sol_1^4

Chen, Ogiue and Verheyen [3, 4, 5, 6, 7, 9, 10] studied totally umbilical submanifolds in Kähler manifolds, especially in Hermitian symmetric spaces.

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Recently, submanifold geometry of non-Hermitian symmetric model spaces are developed. For instance, De Leo and Van der Veken proved the non-existence of totally geodesic hypersurfaces in the model space F^4 [11] (for further details on F^4, see also [17]). Some minimal submanifolds in the model spaces Sol_0^4 and Sol_1^4 are investigated in [15] and [16, 23], respectively.

In [13], D'haene, Van der Veken and the second named author of the present article classified Codazzi hypersurfaces and totally umbilical hypersurfaces in the model space Sol_0^4. In particular, they confirmed the existence of totally geodesic hypersurfaces and totally umbilical hypsersurfaces of Sol_0^4 (see also [24]).

Both the model spaces Sol_0^4 and Sol_1^4 are universal coverings of Inoue surfaces equipped with Tricerri metric [20, 26] as well as solvable Lie groups. However, submanifold geometries of these two spaces are quite different. The purpose of this paper is to prove the non-existence of Codazzi and totally umbilical hypersurfaces in Sol_1^4.

2. Preliminaries

2.1. **Hypersurface geometry.** Let M be an immersed hypersurface of a Riemannian manifold $(\widetilde{M}, g)$. Since we will work locally, we may assume that M is orientable. Take a unit normal vector field ν of M in $\widetilde{M}$. We denote the Riemannian metric on M induced from g by the same letter g and call it the *first fundamental form*. We denote by TM and $T^\perp M$ the tangent bundle and normal bundle of M, respectively.

Then the Levi-Civita connection ∇^M of the first fundamental form is described by the *Gauss formula*:

$$\nabla_X Y = \nabla^M_X Y + h(X,Y)\nu, \quad X, Y \in \mathfrak{X}(M),$$

where

$$\nabla^M_X Y = \tan(\nabla_X Y), \quad \text{and} \quad h(X,Y)\nu = \mathrm{nor}(\nabla_X Y).$$

The symmetric tensor field h is called the *second fundamental form* derived from ν.

For a vector field ξ normal to M, we have the decomposition (*Weingarten formula*):

$$\nabla_X \xi = -A_\xi X + \nabla^\perp_X \xi, \quad X \in \mathrm{X}(M),$$

where

$$-A_\xi X = \tan(\nabla_X \xi), \quad \text{and} \quad \nabla^\perp_X \xi = \mathrm{nor}(\nabla_X \xi).$$

The Weingaten formula induces a connection $\nabla^\perp$ of the normal bundle $T^\perp M$ (called the *normal connection*). Next, A_ξ is an endomorphism field which is called the *Weingarten operator* corresponding to ξ. In particular, for the unit normal vector field ν, Weingarten formula is simplified as

$$\nabla_X \nu = -AX, \quad X \in \mathfrak{X}(M).$$

The endomorphism field $A := A_\nu$ is called the *shape operator* derived from ν. The shape operator is metrically equivalent to h, i.e.,

$$g(AX, Y) = h(X, Y)$$

holds. The mean curvature function H is defined by $H = (\mathrm{tr}_g A)/\dim M$.

It is known that M is minimal if and only if $H = 0$. The covariant derivative ∇h of h is defined by

$$(\nabla h)(Y, Z; X) = (\nabla_X h)(Y, Z) = Xh(Y, Z) - h(\nabla^M_X Y, Z) - h(Y, \nabla^M_X Z).$$

The *Gauss equation* and *Codazzi equation* are given by

$$\begin{aligned} \tan(R(X, Y)Z) &= R^M(X, Y)Z - A_{h(Y,Z)\nu}X + A_{h(Z,X)\nu}Y, \\ g(R(X, Y)Z, \nu) &= (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \end{aligned} \tag{2.1}$$

where R^M is the Riemannian curvature of M.

Here, we recall some definitions:

Definition 2.1. A hypersurface $M \subset \widetilde{M}$ is said to be

- *totally geodesic* if $h = 0$,
- *parallel* if $\nabla h = 0$,
- *Codazzi* if ∇h is totally symmetric,
- *totally umbilical* if $h = H\, g$.

Obviously, we have the following inclusion relations.

$$\text{totally geodesic} \Longrightarrow \text{parallel} \Longrightarrow \text{Codazzi}$$

A totally umbilical hypersurface M is called an *extrinsic sphere* if its mean curvature function is a non-zero constant. It should be remarked that M is a Codazzi hypersurface if and only if $R(X, Y)Z$ is tangent to M for all vector fields X, Y and Z on M.

3. The model space Sol_1^4

The model space Sol_1^4 is the Cartesian 4-space $\mathbb{R}^4(x, y, z, t)$ equipped with *Tricerri metric* [26]:

$$g = e^{-2t}dx^2 + e^{2t}dy^2 + (dz - xdy)^2 + dt^2.$$

The Riemannian metric g is left invariant with respect to the solvable Lie group structure

$$(x_1, y_1, z_1, t_1)(x_2, y_2, z_2, t_2) = (x_1 + e^{t_1}x_2, y_1 + e^{-t_1}y_2, z_1 + z_2 + e^{-t_1}x_1y_2, t_1 + t_2).$$

Note that the underlying manifold of the model space Sol_1^4 is the solvable Lie group G_8 given in [18, p. 101].

Here, we explain the reason for our interest in the Tricerri metric. Since Sol_1^4 is 4-dimensional, we look for left-invariant Riemannian metric from both a Riemannian and a complex geometric viewpoint.

The moduli space of left invariant metrics on Sol_1^4 is 5-dimensional [30, Theorem 5.1]. Lee and Van Thuong [22] proved that a maximal compact subgroup of $\mathrm{Aut}(\mathrm{Sol}_1^4)$ is the dihedral group D_4 of 8 elements [30, Lemma 4.3]. Furthermore, it is unique up to conjugation. Thus, among all left invariant Riemannian metrics on the underlying solvable Lie group of Sol_1^4, we first investigate those metrics whose isometry groups contains D_4 (see e.g., [29]). There are two-parameter family of left invariant metrics with full isometry group $\mathrm{Sol}_1^4 \rtimes \mathrm{D}_4$ (see [1, 12]).

Wall [31] proved that Sol_1^4 admits a compatible complex structure J which will be introduced below. Thus we may look for left invariant metrics which is compatible to the complex structure J in the 2-parameter family mentioned above. By this requirement, we get the Tricerri metric g.

One can take a left invariant orthonormal frame field with respect to the Tricerri metric g:

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^{-t}\left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right), \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}.$$

The Levi-Civita connection ∇ is described as

$$\begin{array}{llll}
\nabla_{e_1}e_1 = e_4, & \nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}e_2, & \nabla_{e_1}e_4 = -e_1,\\
\nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_2 = -e_4, & \nabla_{e_2}e_3 = \frac{1}{2}e_1, & \nabla_{e_2}e_4 = e_2,\\
\nabla_{e_3}e_1 = -\frac{1}{2}e_2, & \nabla_{e_3}e_2 = \frac{1}{2}e_1, & \nabla_{e_3}e_3 = 0, & \nabla_{e_3}e_4 = 0,\\
\nabla_{e_4}e_1 = 0, & \nabla_{e_4}e_2 = 0, & \nabla_{e_4}e_3 = 0, & \nabla_{e_4}e_4 = 0.
\end{array}$$

Note that e_1, e_2, e_3 and e_4 are divergence free.

The commutation relations of the $\{e_1, e_2, e_3, e_4\}$ are

$$[e_1, e_2] = e_3, \quad [e_1, e_4] = -e_1, \quad [e_2, e_4] = e_2.$$

The non-zero components of Riemannian curvature $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, up to symmetries $R(e_i, e_j)e_k = -R(e_j, e_i)e_k$ and $g(R(e_i, e_j)e_k, e_l) = -g(R(e_i, e_j)e_l, e_k)$, are identified as follows

$$\begin{array}{lll}
R(e_1,e_2)e_2 = \frac{1}{4}e_1, & R(e_1,e_3)e_2 = \frac{1}{2}e_4, & R(e_1,e_3)e_3 = \frac{1}{4}e_1,\\
R(e_1,e_4)e_1 = e_4, & R(e_1,e_4)e_2 = \frac{1}{2}e_3, & R(e_2,e_3)e_1 = \frac{1}{2}e_4,\\
R(e_2,e_3)e_3 = \frac{1}{4}e_2, & R(e_2,e_4)e_1 = \frac{1}{2}e_3, & R(e_2,e_4)e_2 = e_4.
\end{array} \tag{3.1}$$

The sectional curvatures $K_{ij} = K(e_i \wedge e_j)$ of a tangent plane spanned by e_i and e_j are given by

$$K_{12} = K_{13} = K_{23} = \frac{1}{4}, \quad K_{24} = K_{14} = -1, \quad K_{34} = 0.$$

The scalar curvature is $-\frac{5}{2}$.

Remark 1. The formulas (3.1) correct those formulas given in [16].

If a left-invariant complex structure J is introduced by

$$Je_1 = e_3, \quad Je_2 = e_4, \quad Je_3 = -e_1, \quad Je_4 = -e_2,$$

then the Hermitian surface (Sol_1^4, g, J) is the universal covering of the Inoue surfaces of type $S^{\pm}$ (see [20]). Sawai [25] gave a characterization of Sol_1^4 and Inoue surface of type S^+ among 4-dimensional LCK solvmanifolds. Note that Sol_0^4 is the universal covering of Inoue surfaces of type S^0.

4. Codazzi hypersurfaces of Sol_1^4

We prove the following non-existence theorem of Codazzi hypersurfaces.

Theorem 4.1. *There exists no Codazzi hypersurface in* Sol_1^4.

Proof. Let M be a Codazzi hypersurface of Sol_1^4 and assume that its unit normal vector field takes the form $\nu = ae_1 + be_2 + ce_3 + de_4$ for some local smooth functions a, b, c and d on M, satisfying $a^2+b^2+c^2+d^2 = 1$. Then the following three vector fields form a local orthonormal frame field on M:

$$\begin{aligned} T_1 =& be_1 - ae_2 + de_3 - ce_4, \\ T_2 =& ce_1 - de_2 - ae_3 + be_4, \\ T_3 =& de_1 + ce_2 - be_3 - ae_4. \end{aligned} \tag{4.1}$$

The Codazzi equation (2.1) for Codazzi hypersurfaces reduces to

$$g(R(X,Y)Z,\nu) = 0 \tag{4.2}$$

for $X, Y, Z \in \mathfrak{X}(M)$. In particular, taking $X = T_1, Y = T_2$ and $Z = T_3$, by use of (3.1), we obtain

$$\frac{1}{2}\left[(a-b)^2 + (c-d)^2\right]\left[(a+b)^2 + (c+d)^2\right] = 0.$$

Hence, $b = \pm a$ and $d = \pm c$. Using these equalities by applying again (2.1) to $X = T_1, Y = T_2$ and $Z = T_2$, we have

$$\frac{5}{2}d^2\left(b^2 + d^2\right) = 0.$$

Hence, $c = d = 0$ and $|a| = |b| = \frac{1}{\sqrt{2}}$.

Therefore, the unit normal vector field ν is chosen as $\nu = \frac{1}{\sqrt{2}}(e_1 \pm e_2)$ and the tangent bundle TM is generated by the distribution spanned by $E_1 = \frac{1}{\sqrt{2}}(e_1 \mp e_2)$, $E_2 = e_3$ and $E_3 = e_4$.

However, this distribution is not integrable, indeed we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = -\nu, \quad [E_2, E_3] = 0.$$

□

We now present the following corollary.

Corollary 4.1. *There is no parallel hypersurface in* Sol_1^4. *In particular, there is no totally geodesic hypersurface in* Sol_1^4.

Remark 2. The 4-dimensional model spaces without compatible complex structures are listed as:

SO(4)-isotropy	SO(3)-isotropy	trivial isotropy
$\mathbb{S}^4$, $\mathbb{H}^4$	$\mathbb{H}^3 \times \mathbb{E}^1$	Nil_4, $\mathrm{Sol}_{m,n}^4$

In [14], the authors studied Codazzi hypersurfaces in the model space Nil_4. They proved the existence of such hypersurfaces and showed that these hypersurfaces are parallel. Furthermore, they demonstrated that there are no totally geodesic hypersurfaces in Nil_4.

5. Totally umbilical hypersurfaces in Sol_1^4

The existence of proper totally umbilical hypersurface is a strong restriction for Riemannian symmetric spaces. Chen [2] proved that if a Riemannian symmetric space of dimension ≥ 4 admits non-totally geodesic totally umbilical hypersurfaces then it is conformally flat. In this section, we prove the non-existence of totally umbilical hypersurfaces in Sol_1^4.

Let M be a totally umbilical hypersurface in Sol_1^4. We define its unit normal vector field ν and tangent vector fields T_1, T_2 and T_3 as in the previous section (see (4.1)). The Codazzi equation (2.1) for totally umbilical hypersurfaces is

$$g(R(X,Y)Z,\nu) = g(Y,Z)X(H) - g(Z,X)Y(H), \qquad X,Y,Z \in \mathfrak{X}(M). \tag{5.1}$$

In particular, substituting $X = T_1, Y = T_2$ and $Z = T_3$ into (5.1), we have again

$$\frac{1}{2}\left[(a-b)^2 + (c-d)^2\right]\left[(a+b)^2 + (c+d)^2\right] = 0.$$

Hence, $b = \pm a$ and $d = \pm c$. Using these equalities by applying again (5.1) to $X = T_1, Y = T_2, Z = T_1$, (and also $X = T_2, Y = T_3, Z = T_3$), we have

$$\frac{5}{2}ac\left(a^2 + c^2\right) = -T_2(H)$$

Applying (5.1) to $X = T_1, Y = T_2, Z = T_2$, (and $X = T_3, Y = T_1, Z = T_3$, too) we obtain

$$\frac{5}{2}c^2\left(a^2 + c^2\right) = T_1(H)$$

If we apply (5.1) to $X = T_3, Y = T_1, Z = T_1$, (and also $X = T_2, Y = T_3, Z = T_2$), we obtain the following.

$$\frac{1}{2}ac\left(a^2 + c^2\right) = T_3(H).$$

Next, we check the integrability conditions. First, we express the orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ of the ambient space by local orthonormal frame field $\{T_1, T_2, T_3\}$ along M and the unit normal vector field ν, by using (4.1), $b = \pm a$ and $d = \pm c$. It suffices to investigate the case $b = a$ and $d = c$. We have

$$\begin{aligned} e_1 =& a\nu + aT_1 + cT_2 + cT_3, \\ e_2 =& a\nu - aT_1 - cT_2 + cT_3, \\ e_3 =& c\nu + cT_1 - aT_2 - aT_3, \\ e_4 =& c\nu - cT_1 + aT_2 - aT_3. \end{aligned} \tag{5.2}$$

Computing $[T_1, T_2]$ for $b = a$ and $d = c$, we obtain

$$\begin{aligned} [T_1,T_2] = &\left(T_1(c) - T_2(a) - (a^2+c^2)\right)e_1 - \left(T_1(c) - T_2(a) + (a^2+c^2)\right)e_2 + \\ &- \left(T_1(a) + T_2(c)\right)e_3 + \left(T_1(a) + T_2(c)\right)e_4. \end{aligned}$$

Substituting (5.2) into this expression, using $2(a^2 + c^2) = 1$, we obtain that the normal component of $[T_1, T_2]$ is $-a\nu$. Hence, it follows $a = 0$.

Analogously, computing $[T_1, T_3]$ for $b = a$ and $d = c$, we obtain

$$\begin{aligned} [T_1,T_3] = &\quad \left(T_1(c) - T_3(a) + a^2 - c^2\right)e_1 + \left(T_1(c) + T_3(a) + a^2 + c^2\right)e_2 \\ &+ \left(-T_1(a) - T_3(c) + 2ac\right)e_3 + \left(-T_1(a) + T_3(c)\right)e_4. \end{aligned}$$

Substituting (5.2) into this expression, we find that the normal component of $[T_1, T_3]$ is $(a + 2aT_1(c) - 2cT_1(a))\,\nu$. Since it is known that $[T_1, T_3]$ has no normal component, it follows that $a = 0$.

Finally, we have

$$\begin{aligned}[T_2, T_3] = &\left(2ac + T_2(c) - T_3(c)\right) e_1 + \left(T_2(c) + T_3(c)\right) e_2 + \\ &+ \left(2c^2 - T_2(a) + T_3(a)\right) e_3 + \left(-T_2(a) - T_3(a)\right) e_4,\end{aligned}$$

and the normal component of $[T_2, T_3]$ is $(c + 2aT_2(c) - 2cT_2(a))\,\nu$. Hence, it follows $c = 0$. This contradicts to $a^2 + c^2 = 1/2$. Therefore, the distribution spanned by $\{T_1, T_2, T_3\}$ is not integrable and we proved the following theorem.

Theorem 5.1. *There exists no totally umbilical hypersurface in* Sol_1^4.

Our results give a complete answer to the problem posed in [19, p. 126, Problem 1].

As we mentioned, the moduli space of left-invariant metrics on Sol_1^4 is 5-dimensional. In particular, there exists a 2-parameter family of left-invariant metrics with full isometry group $\mathrm{Sol}_1^4 \rtimes \mathrm{D}_4$. There are three possible cases for the full isometry group: $\mathrm{Sol}_1^4 \rtimes \mathrm{D}_4$, $\mathrm{Sol}_1^4 \rtimes \mathbb{Z}_2$ or Sol_1^4 (see [1]).

As a first step in the study of hypersurface geometry of Sol_1^4, we focus our attention on the Tricerri metric in this article. In a forthcoming article, we well investigate Codazzi and totally umbilical hypersurfaces in Sol_1^4 with respect to general left-invariant metric.

We conclude this article with the following problem: *Determine the left-invariant metrics on* Sol_1^4 *that admit totally geodesic hypersurfaces.*

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