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THE EVOLUTION OF A CURVE INDUCED BY THE POHLMAYER-LUND-REGGE EQUATION

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ABSTRACT. This paper investigates the evolution of space curves governed by the Pohlmeyer-Lund-Regge (PLR) equation, an integrable extension of the sine-Gordon equation. We examine a specific type of curve evolution, known as the Lund-Regge evolution, and derive its representation in the Frenet frame. We show the Frenet frame evolution aligns with the Lax system of the PLR equation and develop a construction method for curve families via the Sym formula. In conclusion, we describe the Lund-Regge evolution corresponding to the Date multi-soliton solutions to the PLR equation, with illustrations of curves and surfaces.

1. INTRODUCTION

The study of space curve evolution contributes to differential geometry, with applications extending to fluid dynamics and integrable systems. One example is the *vortex filament*, which has a notable impact on fluid dynamics [19, 22, 23, 6]. In classical fluid dynamics, the vortex filament model serves as a fundamental idealization [10, 14]. Representing the vortex filament by γ , the associated system of partial differential equations governing its evolution is given by $\dot{\gamma} = \gamma'' \times \gamma'$, where the derivatives with respect to the curve and time parameters are denoted by $'$ and $\cdot$, respectively, and $\times$ denotes the cross product in $\mathbb{R}^3$. Using the Frenet frame, this equation can be reformulated with curvature κ and the binormal vector B , resulting in $\dot{\gamma} = \kappa B$, also known as the *binormal motion* of the filament [21].

Hasimoto [17] introduced a transformation, which effectively reduced the nonlinear PDEs for curvature κ and torsion τ of the vortex filament to the nonlinear Schrödinger equation, $i\dot{q} + q'' + 2|q|^2q = 0$, where $i = \sqrt{-1}$ is the imaginary unit, and the complex-valued function q is defined by $q = \kappa \exp(i \int^s \tau ds)$ (the *Hasimoto transformation*). This nonlinear Schrödinger equation was widely recognized as an integrable system, and both its soliton and quasi-periodic solutions have been extensively studied [5]. Calini and Ivey [2] have rigorously explored the differential geometric properties of vortex filaments.

Parallel to the study of vortex filaments, another significant equation arises in the context of space curve evolution, namely, the sine-Gordon equation [9]: This equation, well-established in the study of constant negative curvature surfaces in three-dimensional Euclidean space [18, 1] and recognized in quantum field theory as the sine-Gordon model [7], constitutes a

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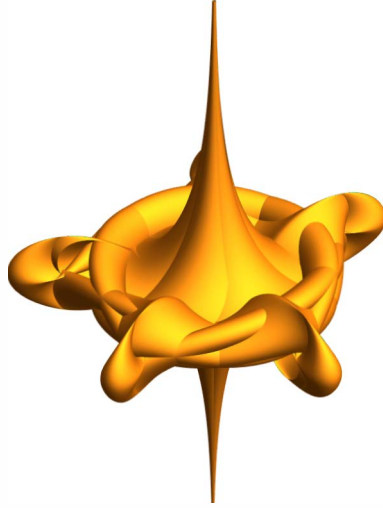


FIGURE 1. A surface induced by a 2-soliton solution of the Pohlmeyer-Lund-Regge equation, see Section 3.3 for details.

foundational integrable system. In 1978, Lund and Regge [8] discovered a generalization of the sine-Gordon equation and formulated it via the time evolution of a space curve γ as the following system of nonlinear partial differential equations

$$\dot{\gamma}' = \gamma' \times \dot{\gamma}.$$

We henceforth refer to this equation as the *Lund-Regge evolution*. When a curve γ moves according to the Lund-Regge evolution, its curvature and torsion satisfy a special system of partial differential equations that extends the sine-Gordon equation. This system, also derived by Pohlmeyer [13] from a viewpoint of integrable systems theory, is now known as the *Pohlmeyer-Lund-Regge (PLR for short) equation*. In 1982, Date [4] applied an extension of Krichever's method to construct soliton and quasi-periodic solutions to the PLR equation, introducing an approach to finding solutions by directly solving a special linear equation through Cramer's rule, a technique subsequently referred to as the *Date's direct method* (see Section 2.3 for further details). Figure 1 describes a surface induced by a 2-soliton solution of the PLR equation.

This paper establishes the framework for space curve evolution using the Frenet frame, alongside the partial differential equations that describe the evolution of curvature and torsion (see Section 2.1). We further reformulate these representations using the 2×2 matrix model through the identification $\mathbb{R}^3 \cong \mathfrak{su}(2)$. Additionally, we summarize the PLR equation and its associated Lax system [20], with a particular emphasis on the Date construction of N -soliton solutions (Sections 2.2 and 2.3).

The core of this paper is developed in Section 3. In that section, we focus on the 2×2 matrix-valued Frenet frame associated with the Lund-Regge evolution, and derive the system of partial differential equations induced by its time evolution (Theorem 3.1). We also define the complex-valued function $q = \kappa \exp(i \int^s (\tau - 1) ds)$ as an analogue to the Hasimoto transformation. By utilizing an appropriate gauge transformation, we demonstrate that the time evolution of the Frenet frame aligns precisely with the Lax system of the PLR equation, with the spectral parameter set to 1 (see Corollary 3.1). This correspondence reveals that

the function q satisfies the following partial differential equation:

$$\dot{q}' + \frac{1}{2}q \int^s (|q|^2) \cdot ds = 0,$$

an equation analogous to the nonlinear Schrödinger equation, and we refer to this equation as also the *PLR equation*. Moreover, we show that by taking the logarithmic derivative of the Lax equation solutions F with respect to the spectral parameter

$$\gamma = \lambda(\partial_\lambda F)F^{-1} \in \mathfrak{su}(2) \cong \mathbb{R}^3,$$

a family of space curves meeting the Lund-Regge evolution can be constructed (Theorem 3.2, also known as the *Sym formula*). When the torsion is identically $\tau = 1$, the dependent variable q in the PLR equation degenerates to a real-valued function, and in this case the PLR equation itself is reduced to the sine-Gordon equation (Remark 3.2 (2)). Additionally, we describe the Lund-Regge evolution corresponding to the Date N -soliton solutions, while presenting illustrations of curves and surfaces as visual aids to support our explanation (Proposition 3.2 and Figures 2, 3, 4). To our knowledge, this paper is the first to present a rigorous geometric derivation of the PLR curve evolution by combining the Frenet frame and the Lax system and to provide an analogue of the Hasimoto transformation. Moreover, the space-curve evolutions corresponding to the Date N -soliton solutions are computed explicitly and visualized for the first time.

2. PRELIMINARIES

In this section, we first collect basic results on a curve evolution in the Euclidean three-space. We then recall the Pohlmeyer-Lund-Regge equation in terms of the Lax pair and its N -soliton solutions [4].

2.1. Evolutions of space curves. For a given regular space curve γ in the Euclidean three-space, let us consider the Frenet frame

$$T = \frac{\gamma'}{|\gamma'|}, \quad N = B \times T, \quad B = \frac{\gamma' \times \gamma''}{|\gamma' \times \gamma''|},$$

where $'$ denotes the derivative with respect to the parameter of the curve γ and $\times$ denotes the Euclidean cross-product. Then it is well known [16] that the curvature κ and the torsion τ of γ can be represented by

$$(2.1) \quad \kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}, \quad \tau = \frac{\langle \gamma' \times \gamma'', \gamma''' \rangle}{|\gamma' \times \gamma''|^2},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^3$. Note that the Frenet frame¹ field $F = (T, N, B)$ takes values in

$$\mathrm{SO}(3, \mathbb{R}) = \{A \in M(3, \mathbb{R}) \mid AA^\top = \mathrm{id}, \det A = 1\}.$$

Then by the Frenet-Serret equation, F satisfies

$$(2.2) \quad F' = FL, \quad L = \begin{pmatrix} 0 & -|\gamma'|\kappa & 0 \\ |\gamma'|\kappa & 0 & -|\gamma'|\tau \\ 0 & |\gamma'|\tau & 0 \end{pmatrix}.$$

¹Strictly speaking, we need to assume that the curvature κ is positive to define the Frenet frame.

Let us consider an evolution of the curve γ and its derivative is denoted by the dot, and assume γ is regular for the evolution ², and the Frenet frame is defined along γ as $F = (T, N, B)$ and it satisfies (2.2). We denote the evolution law of γ by

$$(2.3) \quad \dot{\gamma} = aT + bN + cB,$$

where a, b, c are functions of two variables with respect to the curve parameter and the evolution parameter. Then it can be verified that the evolution of the Frenet frame F can be represented as

$$(2.4) \quad \dot{F} = FM, \quad M = \begin{pmatrix} 0 & -m_{21} & -m_{31} \\ m_{21} & 0 & -m_{32} \\ m_{31} & m_{32} & 0 \end{pmatrix},$$

where $m_{32} = \left(\frac{c'}{|\gamma'|} + b\tau \right)' \frac{1}{|\gamma'|\kappa} + \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) \frac{\tau}{\kappa}$ and

$$(2.5) \quad m_{21} = \frac{b'}{|\gamma'|} + a\kappa - c\tau, \quad m_{31} = \frac{c'}{|\gamma'|} + b\tau,$$

see Appendix A for details. The compatibility condition $(\dot{F})' = (F')^\cdot$ implies the following proposition.

Proposition 2.1. *Let γ be an evolution of a space curve given in (2.3) and denote the length element of γ by*

$$\ell = |\gamma'| > 0.$$

Then the following system of partial differential equations holds:

$$(2.6) \quad \dot{\ell} = a' - b\ell\kappa,$$

$$(2.7) \quad (\ell\kappa)^\cdot = \left(\frac{b'}{\ell} + a\kappa - c\tau \right)' - \left(\frac{c'}{\ell} + b\tau \right) \ell\tau,$$

$$(2.8) \quad (\ell\tau)^\cdot = \left(\frac{c'}{\ell} + b\tau \right) \ell\kappa + \left\{ \left(\frac{c'}{\ell} + b\tau \right)' \frac{1}{\ell\kappa} + \left(\frac{b'}{\ell} + a\kappa - c\tau \right) \frac{\tau}{\kappa} \right\}'.$$

Conversely, let $a, b, c, \kappa, \tau, \ell$ be functions of two variables such that the equations (2.6), (2.7) and (2.8) hold. Then there exists the evolution of a space curve γ such that the length element, the curvature, and the torsion of γ are given by $\ell = |\gamma'|$ and κ and τ , respectively. Furthermore, the evolution of γ is represented as in (2.3).

Proof. First, we show the equation (2.6). From (2.2) and (2.3), the following equation holds:

$$(2.9) \quad \dot{\gamma}' = (a' - b|\gamma'|\kappa)T + (b' + a|\gamma'|\kappa - c|\gamma'|\tau)N + (c' + b|\gamma'|\tau)B.$$

Thus $\langle \dot{\gamma}', \gamma' \rangle = |\gamma'|(a' - b|\gamma'|\kappa)$ holds, and we have $|\gamma'|^\cdot = a' - b|\gamma'|\kappa$, which is (2.6) since we set $|\gamma'| = \ell$. Next, we calculate the compatibility condition of L and M in (2.2) and (2.4).

²Strictly speaking, we should use a different symbol for the evolution of γ with an evolution parameter $|t| \leq \epsilon$ with sufficiently small $\epsilon > 0$, that is, $\tilde{\gamma}$ such that $\tilde{\gamma}|_{t=0} = \gamma$, however, we denote the evolution of γ by the same symbol.

The compatibility condition $[L, M] + M' - \dot{L} = 0$ is

$$(2.10) \quad \begin{pmatrix} 0 & l_{32}m_{31} - m'_{21} + \dot{l}_{21} & -l_{32}m_{21} + l_{21}m_{32} - m'_{31} \\ -l_{32}m_{31} + m'_{21} - \dot{l}_{21} & 0 & -l_{21}m_{31} - m'_{32} + \dot{l}_{32} \\ l_{32}m_{21} - l_{21}m_{32} + m'_{31} & l_{21}m_{31} + m'_{32} - \dot{l}_{32} & 0 \end{pmatrix} = 0,$$

where $l_{ij}, m_{ij} (i, j \in \{1, 2, 3\})$ denote the (i, j) -entries of L and M , respectively. A straightforward computation shows that the $(3, 1)$ -entry of the left-hand side is zero. Therefore (2.10) reduces to the pair of equations

$$\begin{aligned} -|\gamma'|\tau \left(\frac{c'}{|\gamma'|} + b\tau \right) + \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right)' - (|\gamma'|\kappa)' &= 0, \\ |\gamma'|\kappa \left(\frac{c'}{|\gamma'|} + b\tau \right) + \left(\left(\frac{c'}{|\gamma'|} + b\tau \right)' \frac{1}{|\gamma'|\kappa} + \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) \frac{\tau}{\kappa} \right)' - (|\gamma'|\tau)' &= 0. \end{aligned}$$

These equations are (2.7) and (2.8), respectively. Conversely, let $\tilde{L}$ and $\tilde{M}$ be 4×4 matrices

$$(2.11) \quad \tilde{L} = \begin{pmatrix} & \ell & & \\ & 0 & & \\ L & & & \\ & 0 & & \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} & a & & \\ & b & & \\ M & & & \\ & c & & \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where 3×3 matrices L and M are defined in (2.2) and (2.4) with $|\gamma'| = \ell$, respectively. Note that a, b, c are arbitrary functions of two variables. Then it can be verified that the compatibility conditions of $\tilde{L}$ and $\tilde{M}$ are (2.6), (2.7), and (2.8). Therefore there exists a 4×4 matrix-valued function $\tilde{F} = \tilde{F}(s, t)$ that is a solution to the partial differential equations:

$$(2.12) \quad \tilde{F}' = \tilde{F}\tilde{L}, \quad \dot{\tilde{F}} = \tilde{F}\tilde{M},$$

and by choosing $\tilde{F}(0, 0) = \text{id}_4$, we have

$$(2.13) \quad \tilde{F} = \begin{pmatrix} & & & \gamma_1 \\ & & & \gamma_2 \\ F & & & \gamma_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\gamma(s, t) := (\gamma_1(s, t), \gamma_2(s, t), \gamma_3(s, t))^T$ gives the evolution of a curve by (2.3). $\square$

We now identify $\mathfrak{so}(3, \mathbb{R}) = \{A \in M(3, \mathbb{R}) \mid A + A^T = 0\}$ with

$$\mathfrak{su}(2) = \{X \in \mathfrak{sl}(2, \mathbb{C}) \mid X + \overline{X}^T = 0\}$$

by the following explicit map:

$$(2.14) \quad \begin{pmatrix} 0 & -p & -q \\ p & 0 & -r \\ q & r & 0 \end{pmatrix} \in \mathfrak{so}(3, \mathbb{R}) \longleftrightarrow \frac{1}{2} \begin{pmatrix} ir & -p - iq \\ p - iq & -ir \end{pmatrix} \in \mathfrak{su}(2).$$

Note that, under the above identification, the evolution of the Frenet frame F taking values in $\text{SO}(3, \mathbb{R})$ can be identified with a map $\mathcal{F}$ taking values in $\text{SU}(2)$, see for an explicit correspondence [11]. We now arrive at the following fundamental theorem of an evolution in $\mathfrak{su}(2) \cong \mathbb{R}^3$.

Theorem 2.1. *Let $\mathcal{F}$ be a map taking values in $\text{SU}(2)$ by the correspondence in (2.14) to the evolution of the Frenet frame F given by (2.2) and (2.4). Then the following system of partial differential equations holds:*

$$(2.15) \quad \mathcal{F}' = \mathcal{F}\mathcal{L}, \quad \dot{\mathcal{F}} = \mathcal{F}\mathcal{M},$$

with

$$(2.16) \quad \mathcal{L} = \frac{|\gamma'|}{2} \begin{pmatrix} i\tau & -\kappa \\ \kappa & -i\tau \end{pmatrix}, \quad \mathcal{M} = \frac{1}{2} \begin{pmatrix} \frac{i}{\kappa} \left(\frac{m'_{31}}{|\gamma'|} + m_{21}\tau \right) & -m_{21} - im_{31} \\ m_{21} - im_{31} & -\frac{i}{\kappa} \left(\frac{m'_{31}}{|\gamma'|} + m_{21}\tau \right) \end{pmatrix},$$

and (2.5). The compatibility equations of the system (2.15) are (2.7) and (2.8).

Proof. The compatibility condition $[\mathcal{L}, \mathcal{M}] + \mathcal{M}' - \dot{\mathcal{L}} = 0$ is equivalent with

$$(2.17) \quad \mathcal{L}_{12}\mathcal{M}_{21} - \mathcal{M}_{12}\mathcal{L}_{21} + \mathcal{M}'_{11} - (\mathcal{L}_{11})' = 0, \quad \sum_{i=1}^2 (\mathcal{L}_{2i}\mathcal{M}_{i1} - \mathcal{M}_{2i}\mathcal{L}_{i1}) + \mathcal{M}'_{21} - (\mathcal{L}_{21})' = 0,$$

where $\mathcal{L}_{ij}, \mathcal{M}_{ij} (i, j \in \{1, 2\})$ denote the (i, j) -entries of $\mathcal{L}$ and $\mathcal{M}$ in (2.16). A straightforward computation shows that equations in (2.17) are equivalent to (2.7) and (2.8), respectively. $\square$

2.2. The Pohlmeyer-Lund-Regge equation. We now recall the Pohlmeyer-Lund-Regge equation [8, 13] according to the formulation in [4]. Let Ψ be a 2×2 matrix-valued map satisfying the following system of PDEs, the so-called *Lax pair*:

$$(2.18) \quad \Psi^{-1}\Psi' = \frac{1}{2} \begin{pmatrix} i\lambda & a \\ -\bar{a} & -i\lambda \end{pmatrix}, \quad \Psi^{-1}\dot{\Psi} = \frac{i}{2\lambda} \begin{pmatrix} \cos u & -\exp(i\omega) \sin u \\ -\exp(-i\omega) \sin u & -\cos u \end{pmatrix},$$

where λ is a constant in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, the *spectral parameter*, and u is a real-valued function of two variables³. Moreover, a complex-valued function a and a real-valued function ω of two variables are given by u and another real-valued function v as

$$(2.19) \quad a = -\frac{\{\exp(i\omega) \sin u\}'}{\cos u}, \quad \omega' = \frac{v' \cos u}{2 \cos^2(u/2)}, \quad \dot{\omega} = \frac{\dot{v}}{\cos^2(u/2)},$$

where we assume $0 < u < \pi/2$. Note that ω is defined as the solution to the pair of PDEs in (2.19). Then, the compatibility conditions of the above Lax pair are the so-called *Pohlmeyer-Lund-Regge equation*:

$$(2.20) \quad \ddot{u}' - \frac{v'\dot{v} \sin(u/2)}{2 \cos^3(u/2)} + \sin u = 0, \quad \dot{v}' + \frac{u'\dot{v} + \dot{u}v'}{\sin u} = 0.$$

If the function v is constant, then (2.20) reduces to the sine-Gordon equation:

$$(2.21) \quad \dot{u}' + \sin u = 0.$$

The matrix-valued function Ψ in (2.18) will be called the *wave function*.

³The Lax pair has been transposed and the definition of a is multiplied by $i/2$ in [4].

2.3. The Date N -solitons. We recall the method of Date in [4, Section 3] for constructing N -soliton solutions of Pohlmeyer-Lund-Regge equation (2.20) and the wave function in (2.18). Let us assume that the wave function

$$\Psi(s, t, \lambda) = \begin{pmatrix} \Psi_{11}(s, t, \lambda) & \Psi_{12}(s, t, \lambda) \\ \Psi_{21}(s, t, \lambda) & \Psi_{22}(s, t, \lambda) \end{pmatrix}$$

has the following form:

$$\begin{aligned} \Psi_{11}(s, t, \lambda) &= \left(\lambda^N + \sum_{j=0}^{N-1} \psi_{1j}(s, t) \lambda^j \right) \exp \left\{ \frac{i}{2} (\lambda s + \lambda^{-1} t) \right\}, \\ \Psi_{21}(s, t, \lambda) &= - \left(\sum_{j=0}^{N-1} \psi_{2j}(s, t) \lambda^j \right) \exp \left\{ -\frac{i}{2} (\lambda s + \lambda^{-1} t) \right\}, \\ \Psi_{12}(s, t, \lambda) &= -\overline{\Psi_{12}(s, t, \bar{\lambda})}, \quad \Psi_{22}(s, t, \lambda) = \overline{\Psi_{11}(s, t, \bar{\lambda})}, \end{aligned}$$

where $\psi_{ij}(s, t)$ ($i \in \{1, 2\}, j \in \{0, \dots, N-1\}$) are λ -independent complex-valued functions. For each number $j \in \{1, \dots, N\}$, let α_j be mutually distinct complex numbers such that their imaginary parts $\text{Im } \alpha_j$ have the same sign for all j , and let c_j be arbitrary complex numbers. Then, we consider the solution $\psi = (\psi_{10}, \dots, \psi_{1,N-1}, \psi_{20}, \dots, \psi_{2,N-1})^\top \in \mathbb{C}^{2N}$ of the following linear equation:

$$(2.22) \quad T_0 \psi = b.$$

Here the vector $b \in \mathbb{C}^{2N}$ is defined by

$$b = - \left((\alpha_1)^N e(\alpha_1), \dots, (\alpha_N)^N e(\alpha_N), (\bar{\alpha}_1)^N \bar{c}_1 e(\bar{\alpha}_1), \dots, (\bar{\alpha}_N)^N \bar{c}_N e(\bar{\alpha}_N) \right)^\top$$

with $e(\lambda) = \exp(\frac{i}{2}(\lambda s + \lambda^{-1} t))$, and the $2N \times 2N$ matrix T_0 is defined by

$$T_0 = \begin{pmatrix} EA & -CE^{-1}A \\ \overline{CE^{-1}A} & \overline{EA} \end{pmatrix}$$

with $N \times N$ matrices

$$A = \begin{pmatrix} 1 & (\alpha_1)^1 & \cdots & (\alpha_1)^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (\alpha_N)^1 & \cdots & (\alpha_N)^{N-1} \end{pmatrix}, \quad E = \begin{pmatrix} e(\alpha_1) & & O \\ & \ddots & \\ O & & e(\alpha_N) \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & & O \\ & \ddots & \\ O & & c_N \end{pmatrix},$$

where the bars denote the complex conjugate. Note that under the conditions on α_j the Gram matrix of functions

$$f_j(x) = (1, c_j \exp(-i(\alpha_j s + \alpha_j^{-1} t))) \exp(\pm \alpha_j x)$$

is non-singular for $x \in [0, \infty)$, therefore the coefficient matrix T_0 in (2.22) is non-singular. For the detailed derivation, see Section 3.2 of [4]. In fact, by Cramer's rule, the functions ψ_{ij} are explicitly computed as follows:

$$(2.23) \quad \psi_{1j}(s, t) = \frac{\det T_{j+1}}{\det T_0} = \frac{d_{j+1}}{d_0}, \quad \psi_{2j}(s, t) = \frac{\det T_{N+j+1}}{\det T_0} = \frac{d_{N+j+1}}{d_0}$$

for $j \in \{0, \dots, N-1\}$. Here T_k denotes the matrix formed by replacing the k -th column of T_0 by the vector b in (2.22), and d_j are defined for $j \in \{0, \dots, N\}$ as

$$d_j = \det T_j.$$

Following Date's discussion in [4, section 3], the parameters $\alpha_1, \dots, \alpha_N, c_1, \dots, c_N$ determine a unique wave function $\Psi(s, t, \lambda)$ that satisfies the conditions

$$(1, c_j)\Psi(s, t, \alpha_j) = 0$$

for all $j \in \{1, \dots, N\}$. Furthermore, we can show that the function $\Psi(s, t, \lambda)$ satisfies the linear differential equations

$$(2.24) \quad \begin{cases} \Psi^{-1}\Psi' = \frac{1}{2} \begin{pmatrix} i\lambda & i\overline{d_{2N}/d_0} \\ id_{2N}/d_0 & -i\lambda \end{pmatrix}, \\ \Psi^{-1}\dot{\Psi} = \frac{i}{2\lambda(|d_1|^2 + |d_{N+1}|^2)} \begin{pmatrix} |d_1|^2 - |d_{N+1}|^2 & 2\overline{d_1 d_{N+1}} \\ 2d_1 d_{N+1} & -|d_1|^2 + |d_{N+1}|^2 \end{pmatrix}. \end{cases}$$

The detailed computation follows exactly the argument given in Section 1 and 3.2 of [4]. Comparing (2.24) with (2.18), we have that a solution of (2.20) can be given as follows:

$$(2.25) \quad a = i \overline{\left(\frac{d_{2N}}{d_0}\right)}, \quad u = \arccos \left(\frac{|d_1|^2 - |d_{N+1}|^2}{|d_1|^2 + |d_{N+1}|^2} \right), \quad v = 2 \arg \left(\frac{d_{N+1}}{d_0} \right) + v_0,$$

where $v_0 \in \mathbb{R}$, and moreover $\Psi(s, t, \lambda)$ satisfies (2.18).

For example, to give the N -soliton solutions to the sine-Gordon equation, we choose parameters $\alpha_1, \dots, \alpha_N$ and $c_1, \dots, c_N$ such that there exists a suitable permutation $\sigma \in \mathfrak{S}_N$ of $\{1, \dots, N\}$ satisfying the conditions

$$(2.26) \quad \overline{\alpha_j} = -\alpha_{\sigma(j)}, \quad \overline{c_j} = -c_{\sigma(j)}$$

for all indices j . Then we have for $k \in \{1, \dots, 2N\}$ that

$$(2.27) \quad \frac{d_k}{d_0} = \overline{\left(\frac{d_k}{d_0}\right)}.$$

Now v is constant by the third equation in (2.25), and thus u is a solution of the sine-Gordon equation.

3. SPECIAL CURVE EVOLUTIONS AND THE POHLMAYER-LUND-REGGE EQUATION

In this section, we consider a special curve evolution γ in $\mathbb{R}^3$ according to the Pohlmeyer-Lund-Regge equation, the so-called *Lund-Regge* evolution [8], see Definition 3.1. We show that the Lund-Regge evolution is equivalent to the pair of nonlinear PDEs, see Theorem 3.1. Next we rephrase the Pohlmeyer-Lund-Regge equation in (2.20), the pair of PDEs, in terms of a single PDE with respect to a complex-valued function in Corollaries 3.1 and 3.2, which is an analogue of the *Hasimoto transformation* $q = \kappa \exp(\int^s \tau ds)$ of the nonlinear Schrödinger equation [17]. Then the representation formula of the Lund-Regge evolution in terms of the logarithmic derivative of the Lax pair with respect to the spectral parameter, the so-called *Sym-formula* is given, Theorem 3.2. Finally, we give examples of N -soliton curves.

3.1. The Lund-Regge evolution. Let us start with the following definition:

Definition 3.1. *If a space curve evolution γ satisfies the following law*

$$(3.1) \quad \dot{\gamma}' = \gamma' \times \dot{\gamma},$$

then γ will be called the Lund-Regge evolution. Furthermore, the Lund-Regge evolution will be said to be regular if $\gamma' \times \dot{\gamma} \neq 0$ holds.

Remark 3.1. *The original definition of the Lund-Regge evolution in [8, (3.2), (3.4) and (3.5)] requires additional conditions:*

$$|\gamma'| = \ell, \quad |\dot{\gamma}| = \ell^{-1}$$

for some positive constant $\ell > 0$. However, the following lemma implies that we do not need the constant length conditions.

Lemma 3.1. *For a regular Lund-Regge evolution γ , by appropriate change of coordinates,*

$$|\gamma'| = |\dot{\gamma}|^{-1} = \ell$$

holds.

Proof. By the equation of Lund-Regge evolution (3.1), we have that

$$|\gamma'|' = |\dot{\gamma}|' = 0,$$

thus $|\gamma'|$ and $|\dot{\gamma}|$ depend only on the curve parameter and the evolution parameter, respectively. Therefore, the change of coordinates $\tilde{s}(s) = \ell \int^s |\gamma'| ds$, and $\tilde{t}(t) = \ell^{-1} \int^t |\dot{\gamma}| dt$ implies the claim. $\square$

Proposition 3.1. *Let γ be a Lund-Regge evolution such that $|\gamma'| = 1$. Expressing the direction of the evolution as $\dot{\gamma} = aT + bN + cB$ via the Frenet frame, the defining equation (3.1) is equivalent to the following equations:*

$$(3.2) \quad a' - b\kappa = 0, \quad b' + a\kappa - c\tau = -c, \quad c' + b\tau = b.$$

Moreover, the matrices $\mathcal{L}$ and $\mathcal{M}$ in (2.15) are simplified as

$$(3.3) \quad \mathcal{L} = \frac{1}{2} \begin{pmatrix} i\tau & -\kappa \\ \kappa & -i\tau \end{pmatrix}, \quad \mathcal{M} = \frac{1}{2} \begin{pmatrix} \frac{i}{\kappa}(b' - c\tau) & c - ib \\ -c - ib & -\frac{i}{\kappa}(b' - c\tau) \end{pmatrix}$$

with the compatibility condition

$$(3.4) \quad \dot{\kappa} = -b, \quad \dot{\tau} = -\left(\frac{c}{\kappa}\right)'.$$

Proof. Recalling the general computational result (2.9) on the evolution of a space curve, we see that the defining equation of the Lund-Regge evolution (3.1) is equivalent to the equation

$$(3.5) \quad (a' - b|\gamma'|\kappa)T + (b' + a|\gamma'|\kappa - c|\gamma'|(\tau - 1))N + (c' + b|\gamma'|(\tau - 1))B = 0.$$

Therefore, the equations (3.2) describe the Lund-Regge evolution (3.1) with $|\gamma'| = 1$. By using (3.2), the equations in (2.5) are simplified as

$$(3.6) \quad m_{21} = -c, \quad m_{31} = b.$$

Therefore, the matrices $\mathcal{L}$ and $\mathcal{M}$ in the evolution of the Frenet frame $\mathcal{F}$ in (2.15) can be simplified to (3.3). The compatibility condition between (3.3) is (3.4). $\square$

From (3.4) and the first equation in (3.2), the coefficient functions a, b, c of the Lund-Regge evolution γ can be represented by its curvature κ and torsion τ as

$$(3.7) \quad a = - \int^s \dot{\kappa} \kappa ds \quad b = -\dot{\kappa}, \quad c = -\kappa \int^s \dot{\tau} ds.$$

The second and third equations in (3.2) respectively becomes

$$(3.8) \quad \frac{\dot{\kappa}'}{\kappa} - (\tau - 1) \int^s \dot{\tau} ds + \int^s \kappa \dot{\kappa} ds = 0,$$

$$(3.9) \quad \left(\kappa \int^s \dot{\tau} ds \right)' + \dot{\kappa}(\tau - 1) = 0.$$

We now summarize the discussion to this point as the following theorem.

Theorem 3.1. *Regarding the Lund-Regge evolution, we have the following:*

- (1) *Let γ be the Lund-Regge evolution such that $|\gamma'| = 1$. Then it satisfies (2.3) with the coefficient functions (3.7), and its Frenet frame $\mathcal{F}$ evolves according to (2.15) with the matrices*

$$(3.10) \quad \mathcal{L} = \frac{1}{2} \begin{pmatrix} i\tau & -\kappa \\ \kappa & -i\tau \end{pmatrix}, \quad \mathcal{M} = \frac{1}{2} \begin{pmatrix} \frac{i}{\kappa} (m'_{31} + m_{21}\tau) & -m_{21} - im_{31} \\ m_{21} - im_{31} & -\frac{i}{\kappa} (m'_{31} + m_{21}\tau) \end{pmatrix},$$

where m_{21} and m_{31} are explicitly given by

$$(3.11) \quad m_{21} = \kappa \int^s \dot{\tau} ds, \quad m_{31} = -\dot{\kappa}$$

Moreover, the curvature κ and torsion τ of γ satisfy the pair of PDEs (3.8) and (3.9), which is the compatibility condition between (3.10).

- (2) *Conversely, let κ and τ be solutions to the system (3.8) and (3.9) and let γ be the evolution determined by (2.3) and (3.7), where the integration and derivative are taken with respect to the length element. Then $|\gamma'|$ does not depend on t and thus γ gives an isoperimetric curve flow. Moreover, by choosing the arc-length parametrization $|\gamma'| = 1$ of γ , the curve γ satisfies the Lund-Regge evolution $\dot{\gamma}' = \gamma' \times \dot{\gamma}$ with unit-speed.*

Proof. The first statement is a consequence of Proposition 2.1. Let us consider the second statement. By the Frenet-Serre equation in (2.2) and (3.7), we have

$$\langle \dot{\gamma}', \gamma' \rangle = |\gamma'|^2 \left(\frac{a'}{|\gamma'|} - b\kappa \right) = 0.$$

Therefore, the length element $|\gamma'|$ does not depend on t , and without loss of generality we can assume $|\gamma'| = 1$ for all t . Then the Lund-Regge condition (3.5) can be satisfied by the choices of a, b and c in (3.7) and the equations in (3.8) and (3.9). $\square$

Note that $\mathcal{M}$ in (3.10) can be explicitly represented by

$$(3.12) \quad \mathcal{M} = \frac{1}{2} \begin{pmatrix} i \left(-\frac{\dot{\kappa}'}{\kappa} + \tau \int^s \dot{\tau} ds \right) & -\kappa \int^s \dot{\tau} ds + i\dot{\kappa} \\ \kappa \int^s \dot{\tau} ds + i\dot{\kappa} & -i \left(-\frac{\dot{\kappa}'}{\kappa} + \tau \int^s \dot{\tau} ds \right) \end{pmatrix}.$$

Finally, we arrive at a straightforward formulation of the Lund-Regge evolution as follows:

Corollary 3.1. *By choosing the diagonal gauge $\mathcal{D}$ taking values in $U(1) = \{\text{diag}(e^{ip}, e^{-ip}) \mid p \in \mathbb{R}\}$ as in (3.16) and introducing the complex-valued function*

$$(3.13) \quad q = \kappa \exp \left(i \int^s (\tau - 1) ds \right),$$

the evolution of the gauged Frenet frame $F = \mathcal{F}\mathcal{D}$ with $\mathcal{L}$ in (3.10) and $\mathcal{M}$ in (3.12) can be rephrased as

$$(3.14) \quad \begin{aligned} F' &= FL, \quad L = \frac{1}{2} \begin{pmatrix} i & q \\ -\bar{q} & -i \end{pmatrix}, \\ \dot{F} &= FM, \quad M = \frac{i}{2} \begin{pmatrix} -\text{Re}(\dot{q}/q) & -\dot{q} \\ -\dot{\bar{q}} & \text{Re}(\dot{q}/q) \end{pmatrix}. \end{aligned}$$

The compatibility condition of the above system can be computed as $\{\text{Re}(\dot{q}/q)\}' = -\frac{1}{2}(|q|^2)'$, and $\text{Im}(\dot{q}/q) = 0$, which are equivalent to the following nonlinear PDE:

$$(3.15) \quad \dot{q}' + \frac{1}{2}q \int^s (|q|^2)' ds = 0.$$

Proof. Let

$$(3.16) \quad \mathcal{D} = \begin{pmatrix} i \exp(-\frac{i}{2} \int^s (\tau - 1) ds) & 0 \\ 0 & -i \exp(\frac{i}{2} \int^s (\tau - 1) ds) \end{pmatrix}.$$

The matrices $\mathcal{L}$ in (3.10) and $\mathcal{M}$ in (3.12) are computed as $L = \mathcal{D}^{-1}\mathcal{L}\mathcal{D} + \mathcal{D}^{-1}\mathcal{D}'$ and $M = \mathcal{D}^{-1}\mathcal{M}\mathcal{D} + \mathcal{D}^{-1}\dot{\mathcal{D}}$, respectively. The matrices L and M are explicitly given by

$$(3.17) \quad L = \frac{1}{2} \begin{pmatrix} i & \kappa e^{i \int^s (\tau - 1) ds} \\ -\kappa e^{-i \int^s (\tau - 1) ds} & -i \end{pmatrix},$$

$$(3.18) \quad M = \frac{i}{2} \begin{pmatrix} -\frac{\kappa'}{\kappa} + (\tau - 1) \int \dot{\tau} ds & -(\kappa + i\kappa \int \dot{\tau} ds) e^{i \int^s (\tau - 1) ds} \\ -(\kappa - i\kappa \int \dot{\tau} ds) e^{i \int^s (\tau - 1) ds} & -(-\frac{\kappa'}{\kappa} + (\tau - 1) \int \dot{\tau} ds) \end{pmatrix}.$$

Next, it can be verified that $\dot{q}$ and $\dot{q}/q$ can be computed as follows:

$$(3.19) \quad \dot{q} = \left(\kappa + i\kappa \int^s \dot{\tau} ds \right) e^{i \int^s (\tau - 1) ds},$$

$$(3.20) \quad \frac{\dot{q}'}{q} = \left(\frac{\kappa'}{\kappa} - (\tau - 1) \int^s \dot{\tau} ds + \frac{i}{\kappa} (\kappa \dot{\tau} + \kappa(\tau - 1) + \kappa' \int^s \dot{\tau} ds) \right).$$

We now obtain (3.14) by substituting (3.13), (3.19), and (3.20) into each component of matrices L and M in (3.17) and (3.18), respectively. The compatibility conditions of the matrices L and M in (3.14) are equivalent to

$$\{\text{Re}(\dot{q}/q)\}' = -\frac{1}{2}(|q|^2)', \quad \text{Im}(\dot{q}/q) = 0$$

by a similar computation in (2.17). We finally obtain (3.15) by integrating both sides of $\{\text{Re}(\dot{q}/q)\}' = -\frac{1}{2}(|q|^2)'$ with respect to the curve parameter s and using $\text{Im}(\dot{q}/q) = 0$. $\square$

Corollary 3.2. *The complex-valued function q in (3.13) can be identified with the complex-valued function a in (2.19), that is, q gives a solution of the Pohlmeyer-Lund-Regge equation. Thus the gauged Frenet frame F in (3.14) is the matrix-valued wave function Ψ in (2.18) with $\lambda = 1$ up to multiplication of some constant matrix by left, that is, $F_0 F = \Psi$ for some*

$F_0 \in \text{SU}(2)$. Moreover, there exists a family $\{F^\lambda\}_{\lambda>0}$ such that $F^\lambda|_{\lambda=1} = F_0 F$ and F^λ satisfies the following system of PDEs:

$$(3.21) \quad \begin{aligned} (F^\lambda)' &= F^\lambda L^\lambda, & L^\lambda &= \frac{1}{2} \begin{pmatrix} i\lambda & q \\ -\bar{q} & -i\lambda \end{pmatrix}, \\ (F^\lambda)^\cdot &= F^\lambda M^\lambda, & M^\lambda &= \frac{i}{2\lambda} \begin{pmatrix} -\text{Re}(\dot{q}/q) & -\dot{q} \\ -\bar{\dot{q}} & \text{Re}(\dot{q}/q) \end{pmatrix}. \end{aligned}$$

Proof. Note that we set

$$q = a = -\frac{\{\exp(i\omega) \sin u\}'}{\cos u}.$$

A straightforward computation using (2.19) shows that

$$-\cos u = \text{Re}\left(\frac{\dot{q}}{q}\right), \quad \exp(i\omega) \sin u = \dot{q}.$$

Therefore, we can show the claim by comparing the right-hand side of the first equation in (2.18) with L^λ in (3.21) and the right-hand side of the second equation in (2.18) with M^λ in (3.21), respectively. $\square$

Remark 3.2.

- (1) The integrals $\int^s (|q|^2)^\cdot ds$ and $\int^s (\tau - 1) ds$ represent the indefinite integrals of $(|q|^2)^\cdot$ and $\tau - 1$ with respect to the curve parameter s , allowing arbitrary functions of the evolution parameter t as integration constants.
- (2) If the complex-valued function q reduces to the real-valued function, that is, the torsion τ is identically 1, then (3.15) becomes the sine-Gordon equation, see [12]: let us define $u = \int^s \kappa ds$ and $f(s, t)$ by $\kappa = \sin u + f$, thus $\dot{u}' = \sin u + f$. Then (3.15) becomes $f' + u' \int^s u' f ds = 0$, and it can be written as

$$\frac{df}{du} + \int^u f du = 0.$$

The general solution is given by $f = A \cos u + B \sin u$ and one obtains $\dot{u}' = C \sin(u + u_0)$, and the claim follows. Moreover, the L and M in (3.14) can be rephrased as

$$(3.22) \quad L = \frac{1}{2} \begin{pmatrix} i & u' \\ -u' & -i \end{pmatrix}, \quad M = \frac{i}{2} \begin{pmatrix} \cos u & -\sin u \\ -\sin u & -\cos u \end{pmatrix},$$

since $\kappa = \dot{u}' = \sin u$ holds, which have been known the Lax pair for the sine-Gordon equation [15] under spectral parameter is equal to 1.

- (3) It is well known that the nonlinear Schrödinger equation $i\dot{q} + q'' + 2|q|^2 q = 0$ can be obtained by the evolution of $\dot{\gamma} = \gamma'' \times \gamma'$, see [17]. It is an analogue that the complex nonlinear equation $\dot{q}' + \frac{1}{2}q \int^s (|q|^2)^\cdot ds = 0$ can be obtained by the Lund-Regge evolution $\dot{\gamma}' = \gamma' \times \dot{\gamma}$.
- (4) The Pohlmeyer-Lund-Regge equation (2.20) and the nonlinear PDEs (3.8), (3.9) are equivalent. Moreover, they are equivalent to the single complex nonlinear PDE (3.15). We also call the PDE (3.15) as the Pohlmeyer-Lund-Regge equation. This complex equation was also discussed in [24].

3.2. The representation formula. Recall that the Euclidean three-space $\mathbb{R}^3$ can be identified with $\mathfrak{su}(2)$ by

$$(3.23) \quad (p \quad q \quad r)^\top \in \mathbb{R}^3 \longleftrightarrow \frac{1}{2} \begin{pmatrix} ir & -p - iq \\ p - iq & -ir \end{pmatrix} \in \mathfrak{su}(2).$$

Here we note that the bracket $[\cdot, \cdot]$ and the trace of the matrix product in $\mathfrak{su}(2)$ can be translated to the cross product $\times$ and the inner product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^3$ as

$$(3.24) \quad a \times b = [a, b], \quad \langle a, b \rangle = -2 \operatorname{tr}(ab).$$

Theorem 3.2 (The representation formula). *Let q be a solution of (3.15) and $F(=F^\lambda)$ the solution of the Lax pair in (3.21). Define an $\mathfrak{su}(2)$ -valued map*

$$(3.25) \quad \gamma = \lambda(\partial_\lambda F)F^{-1}|_{\lambda=1},$$

where $\partial_\lambda = \frac{\partial}{\partial \lambda}$. Then under the identification $\mathbb{R}^3 \cong \mathfrak{su}(2)$, γ is the Lund-Regge evolution in (3.1) with $|\gamma'| = 1$. Conversely, all Lund-Regge evolutions can be obtained by this way.

Proof. The derivatives of γ in (3.25) can be computed as

$$\gamma' = F(\lambda \partial_\lambda L)F^{-1}|_{\lambda=1}, \quad \dot{\gamma} = F(\lambda \partial_\lambda M)F^{-1}|_{\lambda=1}$$

and thus $-2 \operatorname{tr}(\gamma' \gamma') = 1$. Furthermore, $-2 \operatorname{tr}(\dot{\gamma} \dot{\gamma}) = \operatorname{Re}(\dot{q}'/q)^2 + \dot{q} \dot{q}$ and it does not depend on the parameter s . Then

$$[\gamma', \dot{\gamma}] = F[\lambda \partial_\lambda L, \lambda \partial_\lambda M]F^{-1}|_{\lambda=1} = F[M, \lambda \partial_\lambda L]F^{-1}|_{\lambda=1}$$

follows. On the one hand,

$$\begin{aligned} (\gamma')^\cdot &= (F(\lambda \partial_\lambda L)F^{-1})^\cdot|_{\lambda=1} \\ &= F(\lambda \partial_\lambda \dot{L} + [M, \lambda \partial_\lambda L])F^{-1}|_{\lambda=1} \\ &= F([M, \lambda \partial_\lambda L])F^{-1}|_{\lambda=1} \end{aligned}$$

holds. By the identification (3.24), γ is the Lund-Regge evolution. Conversely, any Lund-Regge evolution γ satisfies (3.14) and there exists a corresponding solution q in (3.15). Then it can be recovered by the formula in (3.25). This completes the proof. $\square$

Remark 3.3.

- (1) The norm of $\dot{\gamma}$ in (3.25) is a function of the evolution parameter only and it is not constant in general.
- (2) Let

$$(3.26) \quad \tilde{\gamma} = (\lambda \partial_\lambda F)F^{-1}|_{\lambda>0}.$$

Then $\tilde{\gamma}$ also gives the Lund-Regge evolution with $|\tilde{\gamma}'| = \lambda$, and the family $\{\tilde{\gamma}\}_{\lambda>0}$ gives a non congruent family of the Lund-Regge evolutions in general.

3.3. N -soliton curves. We refer to the curves with the Lund-Regge evolution associated with N -soliton solutions of the Pohlmeyer-Lund-Regge equation as the N -soliton curves. We construct N -soliton curves by using the wave function $\Psi(s, t, \lambda)$ in (2.24).

Proposition 3.2. *Let $\Psi(s, t, \lambda)$ be 2×2 matrix-valued wave function in (2.24) and define complex-valued functions $e(s, t, \lambda)$, $f(s, t, \lambda)$ and $g(s, t, \lambda)$ by*

$$(3.27) \quad e(s, t, \lambda) = \exp \left(\frac{i}{2} (\lambda s + \lambda^{-1} t) \right),$$

$$(3.28) \quad f(s, t, \lambda) = \Psi_{11}(s, t, \lambda) e(s, t, -\lambda) = \lambda^N + \sum_{j=0}^{N-1} \frac{d_{j+1}}{d_0} \lambda^j,$$

$$(3.29) \quad g(s, t, \lambda) = \Psi_{21}(s, t, \lambda) e(s, t, \lambda) = - \sum_{j=0}^{N-1} \frac{d_{N+j+1}}{d_0} \lambda^j,$$

where $\Psi_{ij}(s, t, \lambda)$ for $i, j \in \{1, 2\}$ denotes the (i, j) -entry of $\Psi(s, t, \lambda)$ and d_k for $k \in \{0, \dots, 2N\}$ is the function given in (2.23). Then the explicit formula for the N -soliton curves $\gamma(s, t) = (\gamma_1(s, t), \gamma_2(s, t), \gamma_3(s, t))^T$ is given as follows:

$$(3.30) \quad \gamma_1(s, t) = 2 \operatorname{Re} \left[\frac{\bar{f} \partial_\lambda g - g \partial_\lambda \bar{f}}{f \bar{f} + g \bar{g}} e^2 \right] \Big|_{\lambda=1},$$

$$(3.31) \quad \gamma_2(s, t) = -2 \operatorname{Im} \left[\frac{\bar{f} \partial_\lambda g - g \partial_\lambda \bar{f}}{f \bar{f} + g \bar{g}} e^2 \right] \Big|_{\lambda=1},$$

$$(3.32) \quad \gamma_3(s, t) = s - t + 2 \operatorname{Im} \left[\frac{\bar{f} \partial_\lambda f + g \partial_\lambda \bar{g}}{f \bar{f} + g \bar{g}} \right] \Big|_{\lambda=1},$$

where e, f and g denote $e(s, t, \lambda)$, $f(s, t, \lambda)$ and $g(s, t, \lambda)$ in (3.27), (3.28) and (3.29), respectively.

Proof. The spectral parameter λ can take values in $\mathbb{C}^\times$, but here we restrict it to positive, because λ in Lund-Regge evolution should be the length of the curve. We can normalize $\Psi(s, t, \lambda)$ in (2.18) to obtain $F(s, t, \lambda)$ in $\operatorname{SU}(2)$. In fact, we define:

$$(3.33) \quad F(s, t, \lambda) = \frac{1}{\sqrt{\det \Psi(s, t, \lambda)}} \Psi(s, t, \lambda).$$

By using the definitions of f, g and e , we can represent $F(s, t, \lambda)$ as follows:

$$(3.34) \quad F = \frac{1}{\sqrt{f \bar{f} + g \bar{g}}} \begin{pmatrix} f e & -\bar{g} e \\ g \bar{e} & \bar{f} e \end{pmatrix}.$$

We now define a map $\tilde{\gamma}(s, t, \lambda)$ using $F(s, t, \lambda)$ in (3.34) and the Sym formula in (3.26) as $\tilde{\gamma} = (\lambda \partial_\lambda F) F^{-1}|_{\lambda>0}$. A straightforward computation shows that the entries of $\tilde{\gamma}(s, t, \lambda)$ can be computed as

$$\begin{aligned} \tilde{\gamma}_{11}(s, t, \lambda) &= \frac{i}{2} \left(\lambda s - \lambda^{-1} t + 2\lambda \frac{\operatorname{Im}[\bar{f} \partial_\lambda f + g \partial_\lambda \bar{g}]}{f \bar{f} + g \bar{g}} \right) \in i\mathbb{R}, \\ \tilde{\gamma}_{21}(s, t, \lambda) &= \frac{\lambda(\bar{f} \partial_\lambda g - g \partial_\lambda \bar{f}) e^2}{f \bar{f} + g \bar{g}} \in \mathbb{C}, \\ \tilde{\gamma}_{12}(s, t, \lambda) &= -\overline{\tilde{\gamma}_{21}(s, t, \lambda)}, \quad \tilde{\gamma}_{22}(s, t, \lambda) = -\tilde{\gamma}_{11}(s, t, \lambda), \end{aligned}$$

where $f = f(s, t, \lambda)$, $g = g(s, t, \lambda)$ and $e = e(s, t, \lambda)$. By using the identification (3.23) and the representation formula (3.25), we can show that the N -soliton curves $\gamma(s, t) = \tilde{\gamma}(s, t, \lambda = 1)$ are given by

$$(3.35) \quad \gamma_1(s, t) = 2 \operatorname{Re}[\tilde{\gamma}_{21}(s, t, 1)], \quad \gamma_2(s, t) = -2 \operatorname{Im}[\tilde{\gamma}_{21}(s, t, 1)], \quad \gamma_3(s, t) = 2 \operatorname{Im}[\tilde{\gamma}_{11}(s, t, 1)].$$

It can be verified that they are explicitly given in (3.30), (3.31), and (3.32), respectively. $\square$

Next, by using the representation formula for N -soliton curves and taking some parameters $\alpha_1, \dots, \alpha_N$ and $c_1, \dots, c_N$ as in [4], we give the surface formed by the N -soliton curve.

Example 3.1. *We give surfaces induced by 1, 2, and 3-soliton solutions for PLR and sine-Gordon equations in Figure 2. Let us choose parameters for the surfaces as follows:*

- (A) $(c_1, \alpha_1) = (1, e^{i \arccos \frac{1}{4}})$,
- (B) $(c_1, c_2, \alpha_1, \alpha_2) = (1, 1, e^{\frac{i\pi}{4}}, i)$,
- (C) $(c_1, c_2, c_3, \alpha_1, \alpha_2, \alpha_3) = (1, 1, 1, e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{3}}, e^{\frac{5i\pi}{6}})$,
- (D) $(c_1, \alpha_1) = (1, i)$,
- (E) $(c_1, c_2, \alpha_1, \alpha_2) = (1, -1, e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}})$,
- (F) $(c_1, c_2, c_3, \alpha_1, \alpha_2, \alpha_3) = (1, -1, i, e^{\frac{i\pi}{6}}, e^{\frac{5i\pi}{6}}, i)$.

Example 3.2 (A 4-soliton curve by the PLR equation). *Let us choose*

$$(c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{3}}, e^{\frac{5i\pi}{6}}, e^{\frac{2i\pi}{3}}, e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{\frac{5i\pi}{6}}).$$

Then it gives a surface (a family of curves) induced by a 4-soliton solution of the PLR equation in Figure 3.

Example 3.3 (A 4-soliton curve by the sine-Gordon equation). *Let us choose*

$$(c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{\frac{5i\pi}{6}}, e^{\frac{i\pi}{6}}, e^{\frac{i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{\frac{5i\pi}{6}}).$$

Then the parameters satisfy the condition in (2.26), and it gives a surface (a family of curves) induced by a 4-soliton solution of the sine-Gordon equation in Figure 4.

APPENDIX A. THE DERIVATION OF A GENERAL EVOLUTION

We now derive the time-evolution of the Frenet-frame in (2.4). A similar computation in the proof of Proposition 2.1 shows

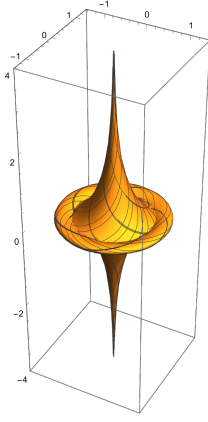
$$(A.1) \quad |\gamma'| \cdot = a' - b|\gamma'| \kappa.$$

Differentiation of both sides of $\gamma' = |\gamma'|T$ with respect to the evolution parameter t yields $\dot{\gamma}' = |\gamma'| \cdot T + |\gamma'| (T) \cdot$. By substituting (A.1) into this equation, we have

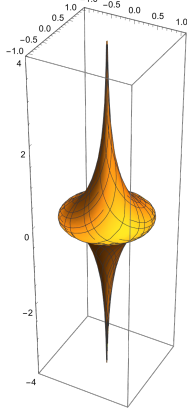
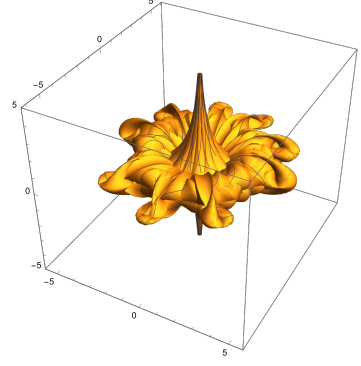
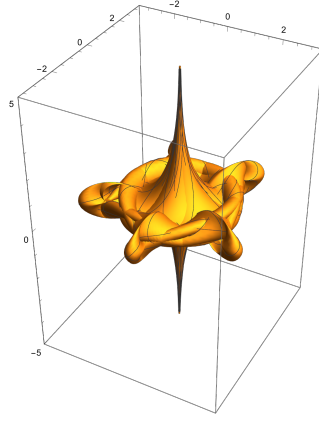
$$(A.2) \quad \dot{T} = \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) N + \left(\frac{c'}{|\gamma'|} + b\tau \right).$$

Differentiation of both sides of (A.2) by the curve parameter s shows

$$(A.3) \quad \begin{aligned} \dot{T}' = & - \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) |\gamma'| \kappa T + \left(\left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right)' - \left(\frac{c'}{|\gamma'|} + b\tau \right) |\gamma'| \tau \right) N \\ & + \left(\left(\frac{c'}{|\gamma'|} + b\tau \right)' + \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) |\gamma'| \tau \right) B. \end{aligned}$$



(A) A PLR 1-soliton surface ; (B) A PLR 2-soliton surface ; (C) A PLR 3-soliton surface ;
 $s, t \in [-10, 10]$ $s, t \in [-10, 10]$ $s, t \in [-40, 40]$



(D) A sine-Gordon 1-soliton surface ; $s, t \in [-10, 10]$ (E) A sine-Gordon 2-soliton surface ; $s, t \in [-10, 10]$ (F) A sine-Gordon 3-soliton surface ; $s, t \in [-25, 25]$

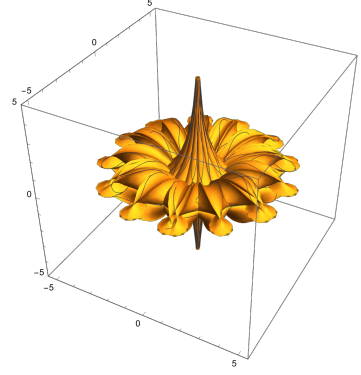
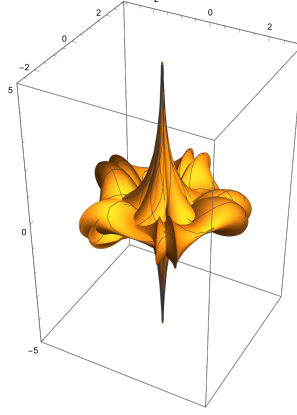


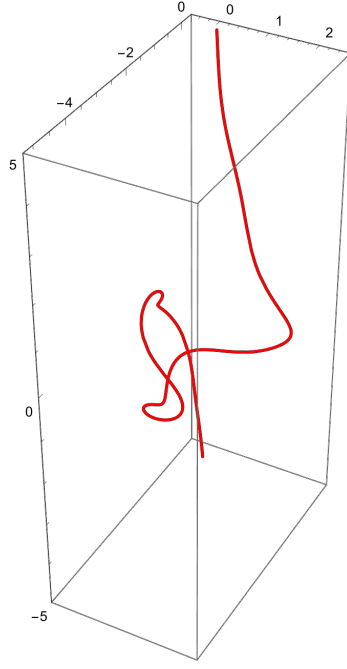
FIGURE 2. Swept surfaces formed by the curves with 1, 2, and 3-soliton solutions for the PLR and the sine-Gordon equations using $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ in (3.30), (3.31), and (3.32)

Differentiation of both sides of $T' = |\gamma'| \kappa N$ by t yields $\dot{T}' = (a' - b|\gamma'| \kappa) \kappa N + |\gamma'| \dot{\kappa} N + |\gamma'| \kappa \dot{N}$. By substituting (A.3) into this equation, we have

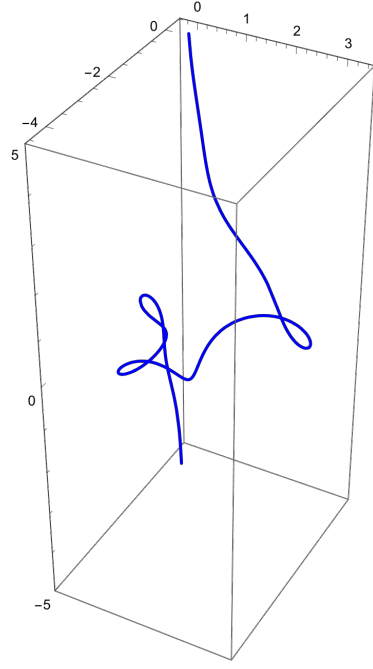
$$\begin{aligned}
 (A.4) \quad \dot{N} = & - \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) T \\
 & + \left(\left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right)' \frac{1}{|\gamma'| \kappa} - \left(\frac{c'}{|\gamma'|} + b\tau \right) \frac{\tau}{\kappa} - \left(\frac{a'}{|\gamma'|} - b\kappa \right) - \frac{\dot{\kappa}}{\kappa} \right) N \\
 & + \left(\left(\frac{c'}{|\gamma'|} + b\tau \right)' \frac{1}{|\gamma'| \kappa} + \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right) |\gamma'| \frac{\tau}{\kappa} \right).
 \end{aligned}$$

We note that $\langle N, N \rangle = 1$. Differentiation of both sides by t yields $\langle \dot{N}, N \rangle = 0$, which implies that $\dot{N}$ and N are orthogonal. From this and (A.4), we have

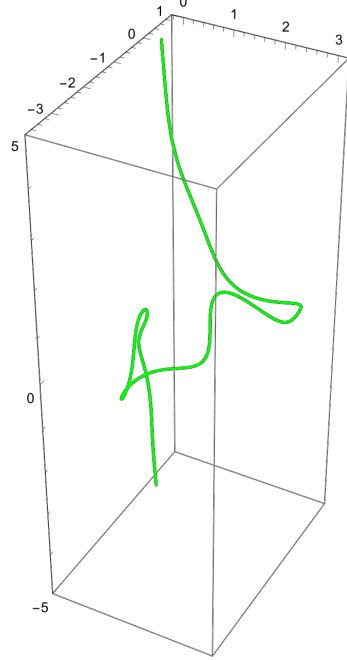
$$\dot{\kappa} = \left(\frac{b'}{|\gamma'|} + a\kappa - c\tau \right)' \frac{1}{|\gamma'|} - \left(\frac{c'}{|\gamma'|} + b\tau \right) \tau - \left(\frac{a'}{|\gamma'|} - b\kappa \right) \kappa.$$



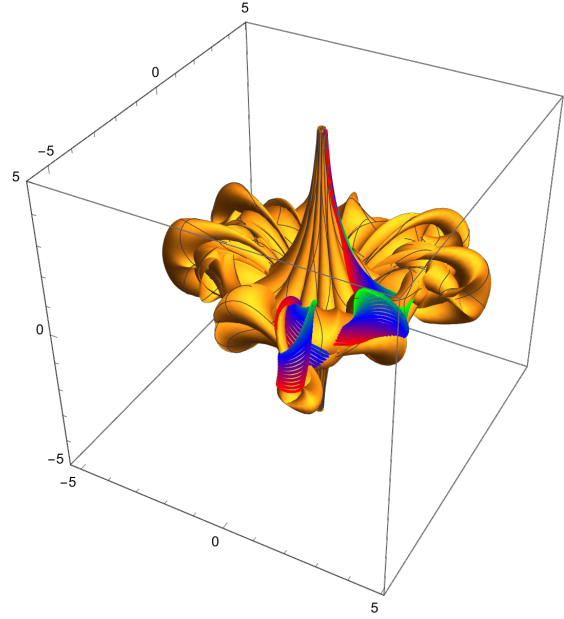
(A) A space curve at time $t = 0$; $s \in [-25, 25]$



(B) A space curve at time $t = 1$; $s \in [-25, 25]$

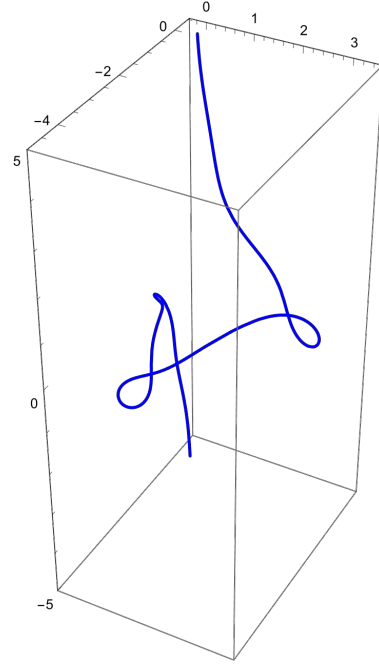
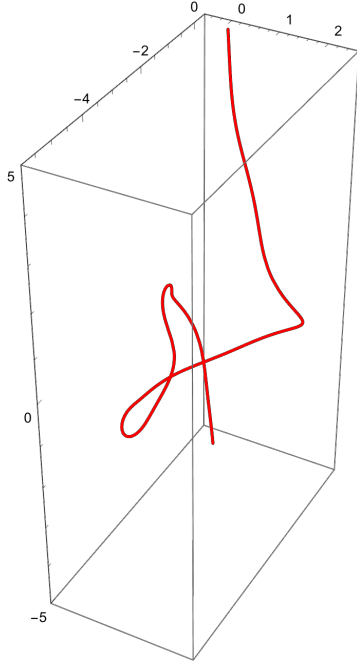


(C) A space curve at time $t = 2$; $s \in [-25, 25]$

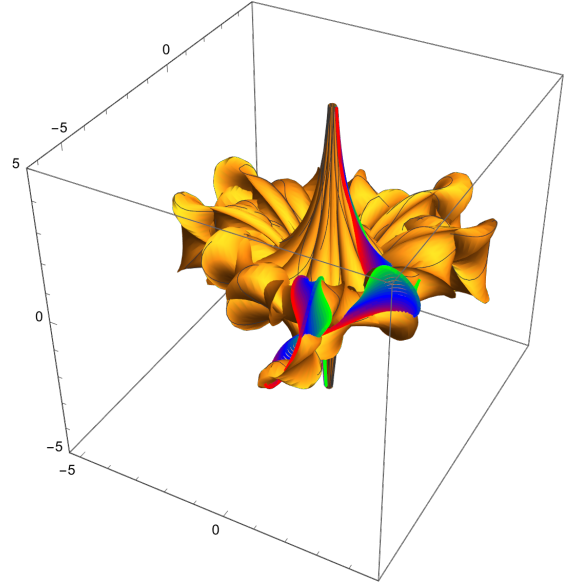
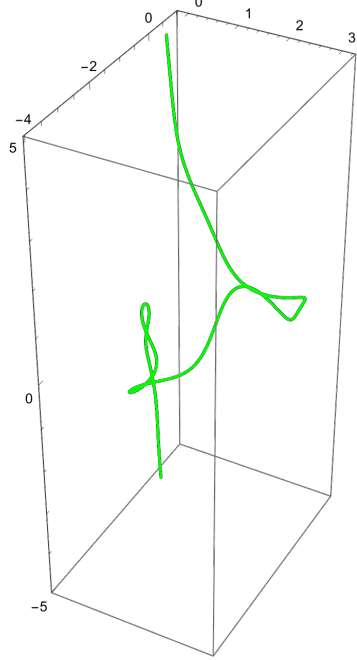


(D) The curves of (A), (B), (C) and its swept surface ; $s, t \in [-20, 20]$

FIGURE 3. A curve evolution determined from a 4-soliton solution of the PLR equation and its swept surface by the curves using $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ in (3.30), (3.31), and (3.32)



(A) The space curve at time $t = 0$; $s \in [-25, 25]$ (B) The space curve at time $t = 1$; $s \in [-25, 25]$



(C) The space curve at time $t = 2$; $s \in [-25, 25]$ (D) The curves of (A), (B), (C) and its swept surface ; $s, t \in [-20, 20]$

FIGURE 4. The curve evolution determined from a 4-soliton solution of the sine-Gordon equation and its swept surface by the curves using $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ in (3.30), (3.31), and (3.32)

This equation is equivalent to (2.7) under (2.6). From the above computation, we show that the time evolution of Frenet-frame can be expressed by (2.4).

AUTHOR CONTRIBUTIONS

S.-P. Kobayashi conceived the research idea. S.-P. Kobayashi, Y. Kogo, and N. Matsuura contributed to the implementation of the research and preparation of the manuscript.

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