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Remarks on Ricci tensor fields of contact metric manifolds*

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Abstract

We show that there is a significant difference in the weak invariance of Ricci tensor fields on contact metric manifolds in the three-dimensional case and in the higher dimensional case.

Keywords: Contact manifolds, Ricci operator, ϕ -invariance, weakly ϕ -invariance.

1 Introduction

As is well known, the Riemannian curvature R and the Ricci tensor field S of an arbitrary Kähler manifold (M, J, g) are J -invariant, *i.e.*,

$$R(X, Y) \circ J = J \circ R(X, Y), \quad S(JX, JY) = S(X, Y)$$

for all vector fields X and Y on M (see *e.g.* [45, p. 130, Proposition 4.1]). Motivated by this fundamental fact, J -invariant curvature-like tensor fields on Kähler manifolds have been studied extensively, see *e.g.*, [29, 40]. More generally, curvature-like tensor fields on almost Hermitian manifolds are investigated by Tricerri and Vanhecke [42]. It should be remarked that the Ricci operator Q of a Kähler manifold (M, J, g)

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commutes with the complex structure J , *i.e.*,

$$QJ = JQ$$

holds on (M, g, J) . Note that the J -invariance of a Ricci tensor fields on almost Kähler manifolds has a variational characterization (see Section 8 or [6, Section 10] and [8]).

On the other hand, the Riemannian curvature R and the Ricci tensor field S of a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n - 1$ satisfies

$$R(X, Y)\phi Z - \phi R(X, Y)Z = g(Z, \phi X)Y - g(Y, Z)\phi X + g(Z, X)\phi Y - g(\phi Y, Z)X$$

and

$$S(\phi X, \phi Y) - S(X, Y) = -2(n - 1)\eta(X)\eta(Y)$$

for all vector fields X and Y on M (see *e.g.* [45, p. 277]). Moreover

$$S(X, \xi) = 2(n - 1)\eta(X)$$

holds for all vector field X on M . Thus both R and S are *never* ϕ -invariant. From this fact it can be inferred that the ϕ -invariance of Ricci tensor field is also a strong restriction on almost contact metric manifolds.

When an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is realized as a real hypersurface in a *non-flat* complex space form $\widetilde{M}_n(c)$, then the ϕ -invariance of S is a too strong restriction. Indeed, the second named author of the present article proved the *non-existence* of Hopf hypersurfaces as well as ruled real hypersurfaces with ϕ -invariant Ricci tensor field in non-flat complex space forms [30].

These discouraging results motivate him to study weak ϕ -invariance of S . According to [30], the Ricci tensor field S of an almost contact metric manifold M is said to be *weakly ϕ -invariant* provided that

$$S(\phi X, \phi Y) = S(X, Y)$$

for all vector fields X and Y on M *orthogonal* to ξ . He classified Hopf hypersurfaces with weakly ϕ -invariant Ricci tensor field in non-flat complex space form.

As pointed out in [30, Remark 3], when the endomorphism field ϕ on an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ commutes with the Ricci operator Q then the Ricci tensor field S is weakly ϕ -invariant. This article addresses the converse statement as well as related topics.

The purpose of this paper is to show that there is a significant difference in the weak invariance of Ricci tensor fields on contact metric manifolds in the three-dimensional case and in the higher dimensional case.

The only example of non-Sasakian contact metric manifolds with ϕ -invariant Ricci tensor field exhibited in [30, §6.1] is the universal covering $\widetilde{E}_2$ of the Euclidean motion group $E_2 = SO_2 \ltimes \mathbb{R}^2$ equipped with standard left invariant contact metric structure. However this example is a trivial one. Because the left invariant metric is *flat* (and hence its Ricci tensor field vanishes). Thus as a Riemannian manifold, $\widetilde{E}_2$ is the

Euclidean 3-space. One may be curious about the existence of non-trivial examples, that is, examples of contact metric manifolds which possess non-vanishing ϕ -invariant Ricci tensor field.

This article is organized as follows. First, we classify contact metric 3-manifolds with ϕ -invariant Ricci tensor field. Such a contact metric 3-manifold is flat and locally isomorphic to $\tilde{E}_2$ (Proposition 11).

Next we show the existence of homogeneous contact metric 3-manifolds whose Ricci tensor field is weakly ϕ -invariant but does not satisfy $Q\phi = \phi Q$ (Example 2 and Example 3). In particular a non-unimodular Lie group with $\gamma = -2$ provides an example of a contact metric 3-manifold with weakly ϕ -invariant Ricci tensor field but it satisfies neither $Q\phi = \phi Q$ nor η -Einstein condition.

In Section 5 (Proposition 10), we prove that the Ricci tensor field S of a contact metric 3-manifold M is weakly ϕ -invariant if and only if g is a critical associated metric, that is, it satisfies the Euler-Lagrange equation (3- τ -condition) $\nabla_\xi h = 0$ of the total scalar curvature functional on the space Riemannian metrics associated to the prescribed contact form η .

From this fundamental fact one may be interested in the relation between the weak ϕ -invariance of S and the criticality of the metric g in higher dimension. We show that there is no strong or weak relation between the weak ϕ -invariance of S and the criticality of the metric g .

For this purpose, we study contact metric hypersurfaces in complex projective space in Section 6. In Example 5 we exhibit an example of contact metric 5-manifold with weakly ϕ -invariant Ricci tensor field but its metric is not critical.

In Section 7, we study tangent sphere bundles. We prove that the tangent sphere bundle of radius $r > 0$ over a Riemannian $(n - 1)$ -manifold of constant curvature has weakly ϕ -invariant Ricci tensor field when and only when $c = 1/r^2$ or $(n - 2)/r^2$ (Theorem 12). From this result we deduce that the metric of the tangent sphere bundle $T^{(r)}\mathbb{E}^{n-1}$ of the Euclidean $(n - 1)$ -space $\mathbb{E}^{n-1}$ is critical, but the Ricci tensor field of $T^{(r)}\mathbb{E}^{n-1}$ is *not* weakly ϕ -invariant.

In the final section (Section 8), we investigate the relationship between contact metric manifolds with weakly ϕ -invariant Ricci tensor fields and their corresponding almost Kähler cones with J -invariant Ricci tensor fields.

2 Contact metric manifolds

2.1 Contact structure

Let M be a manifold of odd dimension $2n - 1 \geq 3$. A *contact form* is a 1-form η which satisfies $(d\eta)^{n-1} \wedge \eta \neq 0$ on M . A hyperplane field $\mathfrak{D} \subset TM$ is said to be a *contact structure* on M if for any point $p \in M$, there exists a contact form η defined on a neighborhood U_p of p which annihilates $\mathfrak{D}$, i.e., $\text{Ker } \eta = \mathfrak{D}$ on U_p .

A $(2n - 1)$ -manifold M together with a contact structure $\mathfrak{D}$ is called a *contact manifold*. In this article we assume that there exists a globally defined contact form η which annihilates $\mathfrak{D}$.

On a contact manifold (M, η) with a fixed contact form η , there exists a unique vector field ξ such that

$$\eta(\xi) = 1, \quad d\eta(\xi, \cdot) = 0.$$

The vector field ξ is called the *Reeb vector field* of (M, η) . Note that ξ is traditionally called the *characteristic vector field* of M in analytical mechanics. Moreover, (M, η) admits a Riemannian metric g and an endomorphism field ϕ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad (1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

and

$$d\eta = \Phi,$$

where Φ is a 2-form defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

and called the *fundamental 2-form* of M . The structure (ϕ, ξ, η, g) is called a *contact metric structure* of M associated to the contact form η . A contact manifold (M, η) together with its associated contact metric structure is called a *contact metric manifold* and denoted by (M, ϕ, ξ, η, g) .

The volume element dv_g of the associated metric g is given by (see [6, Theorem 4.6]):

$$dv_g = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!} \eta \wedge (d\eta)^{n-1}. \quad (3)$$

More generally, an *almost contact metric structure* of a $(2n-1)$ -manifold M is a quintet (ϕ, ξ, η, g) of structure tensor fields which satisfies (1)–(2). A $(2n-1)$ -manifold $M = (M, \phi, \xi, \eta, g)$ equipped with an almost contact metric structure is called an *almost contact metric manifold*. The vector field ξ is called the *structure vector field*. Hereafter we denote by ∇ the Levi-Civita connection of a contact metric manifold $M = (M, \phi, \xi, \eta, g)$. We use the sign convention

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

for the Riemannian curvature R .

Definition 1. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. A tangent plane Π_p at $p \in M$ is said to be *holomorphic* if it is invariant under ϕ_p .

It is easy to see that a tangent plane Π_p is holomorphic if and only if ξ_p is orthogonal to Π_p . The sectional curvature of a holomorphic plane is called a holomorphic sectional curvature. When $\dim M = 3$, holomorphic sectional curvature is a smooth function on M .

We denote by S the Ricci tensor field of a contact metric manifold $M = (M, \phi, \xi, \eta, g)$. The Ricci operator Q is defined by

$$S(X, Y) = g(QX, Y).$$

The scalar curvature is denoted by s .

Definition 2. Let (M, ϕ, ξ, η, g) be an almost contact metric manifold with Ricci tensor field S . Then S is said to be

- ϕ -invariant if

$$S(\phi X, \phi Y) = S(X, Y)$$

holds for any vector fields X and Y on M .

- weakly ϕ -invariant if

$$S(\phi X, \phi Y) = S(X, Y)$$

holds for any vector fields X and Y on M orthogonal to ξ .

Here we recall the notion of weakly η -Einstein almost contact metric manifold.

Definition 3. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be *weakly η -Einstein* if its Ricci tensor field S has the form

$$S = \alpha g + \beta \eta \otimes \eta$$

for some smooth functions α and β . In particular, a weakly η -Einstein almost contact metric manifold is said to be *η -Einstein* if both α and β are constant. Obviously, if M is weakly η -Einstein, then its Ricci tensor field is weakly ϕ -invariant.

2.2 The h -operator and the normality

Now we recall an endomorphism field h which is useful for the study of contact metric manifolds.

$$hX = \frac{1}{2}(\mathcal{L}_\xi \phi)X = \frac{1}{2}\{[\xi, \phi X] - \phi[\xi, X]\},$$

where $\mathcal{L}_\xi$ is the Lie differentiation by ξ . We observe that on a contact metric manifold M , h is self-adjoint with respect to g and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \tag{4}$$

$$\nabla_X \xi = -\phi(I + h)X, \quad X \in \mathfrak{X}(M). \tag{5}$$

From (4)–(5), we see that each trajectory of ξ is a geodesic.

A contact metric manifold M is said to be *K -contact* if ξ is a Killing vector field. It is known that M is K -contact if and only if $h = 0$.

Next, a contact metric manifold M is said to be a *Sasakian manifold* if it is *normal*. Here an almost contact metric manifold M is said to be normal if it satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the *Nijenhuis torsion* of ϕ .

Definition 4. A complete Sasakian manifold M is said to be a *Sasakian space form* if it is of constant holomorphic sectional curvature.

A contact metric manifold M is said to be *H -contact* if its Reeb vector field ξ is an eigenvector field of the Ricci operator Q . One can see that every Sasakian manifold is H -contact.

When M is a weakly η -Einstein K -contact manifold of dimension greater than 3, then it is η -Einstein (see *e.g.*, [12, Proposition 11.8.1]). On the other hand, every Sasakian 3-manifold is weakly η -Einstein. 3-dimensional Sasakian space forms are η -Einstein. There exist weakly η -Einstein Sasakian 3-manifolds that are not η -Einstein. For instance Brieskorn 3-manifolds admit Sasakian structures of non constant holomorphic sectional curvature (see [1, 37, 38, 41] and [13, p. 193]). More examples of 3-dimensional Sasakian 3-manifolds of nonconstant holomorphic sectional curvature, we refer to [12, §10.1.11]. For higher dimensional examples of η -Einstein Sasakian manifolds, we refer to [13].

Example 1 ([23]). On the Cartesian 3-space $\mathbb{R}^3(x, y, z)$, we set

$$e_1 = 2\frac{\partial}{\partial y}, \quad e_2 = 2\left(\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right), \quad e_3 = 2\frac{\partial}{\partial z}$$

and introduce a Riemannian metric g so that $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. Then we obtain a contact metric structure (ϕ, ξ, η, g) by

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0, \quad \xi = e_3, \quad \eta = g(\xi, \cdot).$$

Then the resulting contact metric 3-manifold is Sasakian with Ricci operator

$$Q = -(2 + 24y^2)I + (4 + 24y^2)\eta \otimes \xi.$$

2.3 Critical associated metrics

For a contact $(2n-1)$ -manifold (M, η) , we denote by $\mathcal{M}(\eta)$ the space of all Riemannian metrics associated to η . All the metrics g in $\mathcal{M}(\eta)$ have *common* volume element (3).

Theorem 1 ([9]). *On a compact contact manifold (M, η) , an associated metric g is a critical point of the total scalar curvature functional*

$$I(g) = \int_M s \, dv_g$$

in $\mathcal{M}(\eta)$ if and only if its associated almost contact structure (ϕ, ξ, η) satisfies

$$Q\phi = \phi Q \quad \text{on} \quad \mathfrak{D}. \quad (6)$$

An associated metric $g \in \mathcal{M}(\eta)$ is said to be a *critical associated metric* if it satisfies (6).

2.4 S -curvature tensor fields

On a Sasakian manifold M , the Riemannian curvature R satisfies

$$R(X, Y)\phi Z - \phi R(X, Y)Z = g(Z, \phi X)Y - g(Y, Z)\phi X + g(Z, X)\phi Y - g(\phi Y, Z)X, \quad (7)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \quad (8)$$

and

$$S(\phi X, \phi Y) - S(X, Y) = -2(n-1)\eta(X)\eta(Y) \quad (9)$$

for all fields X and Y on M (see e.g. [45, p. 277]). Moreover

$$S(X, \xi) = 2(n-1)\eta(X) \quad (10)$$

for all vector field X on M . Thus both R and S are *never* ϕ -invariant. Obviously, every Sasakian manifold has weakly ϕ -invariant Ricci tensor field.

Here we recall the following notion introduced by Yoshimatsu [46].

Definition 5. A curvature-like tensor field L on a Sasakian manifold M is said to be

1. *S-curvature tensor field* if it satisfies (7)-(8).
2. *S-curvature-like tensor field* if it is ϕ -invariant, that is

$$L(X, Y) \circ \phi = \phi \circ L(X, Y)$$

and

$$L(\xi, X) = 0$$

for all vector fields X and Y on M .

For example,

$$R_1(X, Y)Z = (X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y$$

is an S -curvature tensor field.

Yoshimatsu pointed out the following interesting fact:

Proposition 1. *Let L be a curvature-like tensor field on a Sasakian manifold. Then L is an S -curvature tensor field if and only if $\tilde{L} := L - R_1$ is an S -curvature-like tensor field. The Ricci operators $Q_{\tilde{L}}$ and Q_L of $\tilde{L}$ and L are related by $Q_{\tilde{L}} = Q_L - 2(n-1)I$.*

S -curvature-like tensor fields on Sasakian manifolds are investigated by Yoshimatsu [46]. Matzeu studied curvature-like tensor fields on normal almost contact metric manifolds [26]. See also Janssens and Vanhecke [22].

Remark 1. On an almost contact metric manifold (M, ϕ, ξ, η, g) , the tensor field $\tilde{S} = S - 2(n-1)g$ is ϕ -invariant if and only if the Ricci tensor field S satisfies (10). In such a case S is weakly ϕ -invariant.

Question 1. Can we consider ϕ -invariance of $\tilde{S} = S - 2(n-1)g$ for any almost contact metric manifolds, especially real hypersurfaces in almost Hermitian manifolds ?

3 Contact metric 3-manifolds

Let us start our investigation of weak ϕ -invariance of Ricci tensor fields on contact metric 3-manifolds.

3.1 h -eigenframe field

Let M be a contact metric 3-manifold. Denote by $\mathcal{U}$ the open subset of M consisting of points p such that $h \neq 0$ around p . Next let $\mathcal{U}_0$ the open subset of M consisting of

points $p \in M$ such that $h = 0$ around p . Since h is smooth, $\mathcal{U} \cup \mathcal{U}_0$ is an open dense subset of M . So any property satisfied in $\mathcal{U} \cup \mathcal{U}_0$ is also satisfied in whole M . For any point $p \in \mathcal{U} \cup \mathcal{U}_0$, there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2 = \phi e_1, e_3 = \xi\}$ around p , where e_1 is an eigenvector field of h . More precisely this local orthonormal frame field satisfies the following properties:

Lemma 1. (cf. [14, 21]) *Let M be a contact metric 3-manifold. Then there exists a local orthonormal frame field $\mathcal{E} = \{e_1, e_2, e_3\}$ such that :*

$$he_1 = \lambda e_1, \quad e_2 = \phi e_1, \quad e_3 = \xi.$$

With respect to $\mathcal{E}$, the Levi-Civita connection ∇ is given by

$$\begin{aligned} \nabla_{e_1} e_1 &= be_2, & \nabla_{e_1} e_2 &= -be_1 + (1 + \lambda)\xi, & \nabla_{e_1} \xi &= -(1 + \lambda)e_2, \\ \nabla_{e_2} e_1 &= -ce_2 + (\lambda - 1)\xi, & \nabla_{e_2} e_2 &= ce_1, & \nabla_{e_2} \xi &= (1 - \lambda)e_1, \\ \nabla_{\xi} e_1 &= ae_2, & \nabla_{\xi} e_2 &= -ae_1, & \nabla_{\xi} \xi &= 0. \end{aligned}$$

The Ricci operator Q is given by

$$\begin{aligned} Qe_1 &= S_{11}e_1 + \xi(\lambda)e_2 + (2b\lambda - e_2(\lambda))\xi, \\ Qe_2 &= \xi(\lambda)e_1 + S_{22}e_2 + (2c\lambda - e_1(\lambda))\xi, \\ Q\xi &= (2b\lambda - e_2(\lambda))e_1 + (2c\lambda - e_1(\lambda))e_2 + 2(1 - \lambda^2)\xi, \end{aligned}$$

where

$$S_{11} = \frac{s}{2} + \lambda^2 - 2a\lambda - 1, \quad S_{22} = \frac{s}{2} + \lambda^2 + 2a\lambda - 1.$$

The covariant derivative $\nabla_{\xi} h$ of h by ξ is given by

$$\nabla_{\xi} h = -2ah\phi + \nu h.$$

The function ν is given by $\nu = 0$ when M is Sasakian and $\nu = \xi(\lambda)/\lambda$ when M is non-Sasakian.

Such a local orthonormal frame field $\{e_1, e_2, e_3\}$ is referred as to an h -eigenframe field. Note that, on a non-Sasakian contact metric 3-manifold,

$$a = g(\nabla_{\xi} X, \phi X)$$

is independent of the choice of unit eigenvector field X of h .

3.2 The torsion invariant

Let $M = (M, \phi, \xi, \eta, g)$ be a contact metric 3-manifold. The *torsion invariant* of M introduced by Chern and Hamilton [15] is the square norm $|\tau|^2$ of $\tau = \mathcal{L}_{\xi} g$. The torsion invariant is computed as

$$|\tau|^2 = -2S(\xi, \xi) + 4.$$

Moreover we have the following equations:

$$\tau(X, Y) = 2g(\phi X, hY), \quad (\nabla_\xi \tau)(X, Y) = 2g(\phi X, (\nabla_\xi h)Y).$$

From these equations, one can confirm that

$$\nabla_\xi \tau = 0 \iff \nabla_\xi h = 0.$$

Proposition 2 ([15]). *A contact metric 3-manifold is Sasakian if and only if $\tau = 0$.*

Assume that M is compact and take a local h -eigenframe field $\{e_1, e_2, e_3\}$, then one can confirm that the Euler-Lagrange equation (6) is equivalent to $\nabla_\xi h = 0$. Thus we retrieve the following result due to Perrone [34]

Corollary 1. *Let (M, ϕ, ξ, η, g) be a compact contact metric 3-manifold. Then the metric g is a critical associated metric to η if and only if $\nabla_\xi h = 0$.*

Contact metric 3-manifolds satisfying $\nabla_\xi h = 0$ are called *3- τ -manifolds* in [5].

4 Homogeneous contact metric 3-manifolds

A contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is said to be a *homogeneous contact metric manifold* if there exists a Lie group G of isometries which acts transitively on M such that every element of G is a *strictly* contact transformation. Perrone obtained the following classification.

Theorem 2 ([35]). *Let M be a simply connected homogeneous contact metric 3-manifold, then M is a Lie group equipped with left invariant contact metric structure.*

4.1 Unimodular Lie groups

Let G be a 3-dimensional unimodular Lie group with a left invariant metric $\langle \cdot, \cdot \rangle$. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathfrak{g}$ (called a *unimodular basis*) such that

$$[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad c_i \in \mathbb{R}. \quad (11)$$

Three-dimensional unimodular Lie groups are classified by Milnor [27] as follows:

Signature of (c_1, c_2, c_3)	Simply connected Lie group	Property
$(+, +, +)$	SU_2	compact and simple
$(-, -, +)$ or $(-, +, +)$	$\widetilde{SL}_2\mathbb{R}$	non-compact and simple
$(0, +, +)$	$\widetilde{E}_2$	solvable
$(0, -, +)$	$E_{1,1}$	solvable
$(0, 0, +)$	Heisenberg group Nil_3	nilpotent
$(0, 0, 0)$	$(\mathbb{R}^3, +)$	Abelian

Proposition 3. *The unimodular basis $\{e_1, e_2, e_3\}$ of a 3-dimensional unimodular Lie group diagonalizes the Ricci operator Q . The principal Ricci curvatures are given by*

$$Qe_i = q_i e_i, \quad q_1 = 2\mu_2\mu_3, \quad q_2 = 2\mu_1\mu_3, \quad q_3 = 2\mu_1\mu_2,$$

where

$$\mu_i = \frac{1}{2}(c_1 + c_2 + c_3) - c_i.$$

Theorem 3. *Let (G, ϕ, ξ, η, g) be a simply connected 3-dimensional Lie group together with a left invariant contact metric structure. If G is unimodular, then the Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1, e_2, e_3\}$ as in (11) with $c_3 = 2$. The left invariant contact metric structure is determined by*

$$\xi = e_3, \quad \phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi \xi = 0.$$

Hereafter we denote by $G(c_1, c_2)$ the unimodular Lie group (whose Lie algebra is determined by (11) with $c_3 = 2$) equipped with a left invariant contact metric structure.

The global orthonormal frame field $\{e_1, e_2, e_3\}$ on $G(c_1, c_2)$ is an example of the frame field given in Lemma 1 with $b = c = 0$ and $a = \mu_3$.

Proposition 4. *The endomorphism field h and the torsion invariant of a unimodular Lie group G equipped with a left invariant homogeneous contact metric structure are given by*

$$he_1 = -\frac{1}{2}(c_1 - c_2)e_1, \quad he_2 = \frac{1}{2}(c_1 - c_2)e_2, \quad |\tau|^2 = (c_1 - c_2)^2.$$

The holomorphic sectional curvature of G is

$$H = -3 + \frac{1}{4}(c_1 - c_2)^2 + c_1 + c_2.$$

Corollary 2. *If a unimodular Lie group $G(c_1, c_2)$ is non-Sasakian, i.e., $c_1 \neq c_2$, then G satisfies the so-called contact (κ, μ) -condition*

$$R(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y) \quad (12)$$

with

$$\kappa = 1 - \frac{1}{4}(c_1 - c_2)^2, \quad \mu = 2 - (c_1 + c_2).$$

Conversely, Blair, Koufogiorgos and Papantoniou proved the following theorem.

Theorem 4 ([11]). *Let M be a contact metric 3-manifold satisfying the (κ, μ) -condition (12) for some real constants κ and μ , then M is either*

- Sasakian ($\kappa = 1, h = 0$) or
- locally isomorphic to a non-Sasakian unimodular Lie group $G(c_1, c_2)$.

Remark 2. Let us denote by $\tilde{E}_2$ the universal covering of E_2 equipped with the flat associated metric. Then $\tilde{E}_2$ is realized as $\mathbb{R}^3(x, y, z)$ with contact form $\eta = (dz - ydx)/2$ with

$$g = \begin{pmatrix} 1 + y^2 + z^2 & z & -y \\ z & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

This example can be generalized to arbitrary odd-dimension as follows (cf. [6, p. 121]):

$$g = \begin{pmatrix} \delta_{ij} + y_i y_j + \delta_{ij} z & \delta_{ij} z & -y_i \\ \delta_{ij} z & \delta_{ij} & 0 \\ -y_j & 0 & 1 \end{pmatrix}, \quad \eta = \frac{1}{2} \left(dz - \sum_{i=1}^n y_i dx_i \right)$$

on $\mathbb{R}^{2n+1}$. This contact metric manifold is non-flat for $n > 1$.

Remark 3. The universal covering $\tilde{E}_2$ of E_2 equipped with the left invariant flat contact metric structure is realized as $\mathbb{R}^3(x, y, z)$ with metric $(dx^2 + dy^2 + dz^2)/4$ and contact form

$$\eta = \frac{1}{2}(\cos z \, dx + \sin z \, dy).$$

One can see that this contact structure is invariant under the action of discrete subgroup $\Gamma = 2\pi\mathbb{Z}^3$ of $\tilde{E}_2$. Hence we obtain the flat torus $\mathbb{T}^3 = \tilde{E}_2/\Gamma$ equipped with the left invariant contact metric structure.

4.2 Non-unimodular Lie groups

Now let us consider 3-dimensional non-unimodular Lie groups equipped with left invariant contact metric structure.

Theorem 5 ([35]). *Let G be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact metric structure. Then the Lie algebra $\mathfrak{g} = \mathfrak{g}(\alpha, \gamma)$ satisfies the commutation relations*

$$[e_1, e_2] = \alpha e_2 + 2e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = -\gamma e_2,$$

with $e_3 = \xi$, $e_1 = -\phi e_2$ and $\alpha \neq 0$.

Proposition 5. ([35, p. 251]) *The Ricci tensor field S of a non-unimodular Lie group G with Lie algebra $\mathfrak{g}(\alpha, \gamma)$ has components $S_{ij} = S(e_i, e_j)$:*

$$S_{11} = -\alpha^2 - 2 - 2\gamma - \frac{\gamma^2}{2}, \quad S_{22} = -\alpha^2 - 2 + \frac{\gamma^2}{2}, \quad S_{33} = 2 - \frac{\gamma^2}{2}, \quad S_{23} = -\alpha\gamma.$$

The Lie algebra $\mathfrak{g}(\alpha, \gamma)$ is classified by the Milnor's invariant $\mathcal{D} = -8\gamma/\alpha^2$.

Remark 4. The global orthonormal frame field $\{e_1, e_2, e_3\}$ is an example of orthonormal frame field given in Lemma 1 with $a = -(\gamma + 2)/2$, $b = 0$, $c = \alpha$.

4.3 The commutativity of S and ϕ

From Lemma 1 we deduce the following results.

Proposition 6. *On a contact metric 3-manifold with a local h -eigenframe field $\mathcal{E}$ as in Lemma 1, then $Q\phi = \phi Q$ holds if and only if*

$$2b\lambda - e_2(\lambda) = 0, \quad 2c\lambda - e_1(\lambda) = 0, \quad \xi(\lambda) = 0, \quad S_{11} = S_{22}.$$

Proposition 7. ([18, Proposition 5.2]) *Let M be a contact metric 3-manifold with local orthonormal frame field $\mathcal{E}$ as in Lemma (1). Then $S_{11} = S_{22}$ if and only if $a = 0$ or M is Sasakian.*

Hence we obtain

Corollary 3. *Let M be a contact metric 3-manifold with a local h -eigenframe field $\mathcal{E}$ as in Lemma 1. Then $Q\phi = \phi Q$ holds if and only if M is Sasakian or*

$$a = 0, \quad 2b\lambda - e_2(\lambda) = 0, \quad 2c\lambda - e_1(\lambda) = 0, \quad \xi(\lambda) = 0.$$

In particular when $b = c = 0$, $Q\phi = \phi Q$ holds if and only if $a = 0$ and λ is constant.

Here we recall the classification of contact metric 3-manifolds satisfying $Q\phi = \phi Q$ obtained in [10].

Theorem 6 ([10]). *Let M be a contact metric 3-manifold. Then the following two conditions are mutually equivalent:*

1. $Q\phi = \phi Q$;
2. M satisfies $(\kappa, 0)$ -condition

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}, \quad \kappa \leq 1$$

for all vector fields X and Y on M .

Alternatively, the constancy of κ is deduced from the formula $h \operatorname{grad} \mu = \operatorname{grad} \kappa$.

Next we recall the following result.

Theorem 7 ([11]). *Let M be a contact metric 3-manifold. Then M satisfies $Q\phi = \phi Q$ if and only if M is either*

1. *a Sasakian 3-manifold,*
2. *flat contact metric 3-manifold, or*
3. *non-Sasakian contact metric 3-manifold of constant holomorphic sectional curvature $-\kappa$ and constant ξ -sectional curvature κ .*

In the third case, $\kappa < 1$.

We describe the third case in detail (see [7]). One can see that in the third case, λ is constant and hence so is $\kappa = 1 - \lambda^2$. Moreover

$$[\xi, e_1] = \left(\frac{s}{4} + \frac{(1 + \lambda)^2}{2} \right) e_2, \quad [e_1, e_2] = 2\xi, \quad [e_2, \xi] = (1 - \lambda)e_1.$$

with

$$s = 2(1 - \lambda^2) = 2\kappa.$$

Inserting the scalar curvature formula to $[\xi, e_1]$, we get

$$[\xi, e_1] = (1 + \lambda)e_2.$$

Hence M admits a local structure of unimodular Lie group. Put

$$(c_1, c_2, c_3) := (1 - \lambda, 1 + \lambda, 2).$$

- $\kappa > 0 \iff \lambda^2 < 1$: In this case, the signature of (c_1, c_2, c_3) is $(+, +, +)$. Hence the possible Lie algebra is $\mathfrak{su}_2$.

- $\kappa = 0 \iff \lambda^2 = 1$: In this case the signature of (c_1, c_2, c_3) is $(0, +, +)$ or $(+, 0, +)$. The Lie algebra is $\mathfrak{e}_2$ and M is flat.
- $\kappa < 0 \iff \lambda^2 > 1$: In this case, the signature of (c_1, c_2, c_3) is $(-, +, +)$ or $(+, -, +)$. Hence the Lie algebra is $\mathfrak{sl}_2\mathbb{R}$.

5 Weak ϕ -invariance of S

5.1 ϕ -invariance condition

On an almost contact metric manifold M of arbitrary odd dimension, we have

$$S(\phi X, \phi Y) = g(Q(\phi X), \phi Y) = -g(\phi Q\phi X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Hence S is ϕ -invariant if and only if

$$\phi Q\phi + Q = 0.$$

Proposition 8. *If the Ricci operator Q of an almost contact metric manifold M commutes with ϕ , then*

- S is weakly ϕ -invariant.
- S is ϕ -invariant if and only if $S\xi = 0$ holds.

Proof. Assume that $Q\phi = \phi Q$, then we have

$$\phi Q\phi X + QX - \eta(QX)\xi = 0 \tag{13}$$

for all $X \in \mathfrak{X}(M)$. Thus the result follows. $\square$

5.2 Contact metric 3-manifolds with ϕ -invariant S

Let us concentrate our attention to *contact* metric 3-manifolds. Take a local h -eigenframe field as in Lemma 1, then we obtain the following result.

Proposition 9. *With respect to an h -eigenframe field, we have*

1. S is weakly ϕ -invariant if and only if $S_{11} = S_{22}$ and $S_{12} = 0$.
2. $Q\phi = \phi Q$ holds if and only if $S_{11} = S_{22}$, $S_{12} = 0$ and $S_{13} = S_{23} = 0$.
3. S is ϕ -invariant if and only if $S_{11} = S_{22}$, $S_{12} = 0$ and $S_{13} = S_{23} = S_{33} = 0$.

Hence we notice that

1. S is ϕ -invariant $\implies Q\phi = \phi Q \implies S$ is weakly ϕ -invariant.
2. S is weakly ϕ -invariant and H -contact $\iff Q\phi = \phi Q$.
3. S is ϕ -invariant $\iff Q\phi = \phi Q$ and $S\xi = 0$.

Moreover we notice the following fact.

Proposition 10. *On a contact metric 3-manifold M , the following properties are mutually equivalent.*

- S is weakly ϕ -invariant.

- g is a critical associated metric, i.e., $Q\phi = \phi Q$ holds on $\mathfrak{D}$.
- $\nabla_\xi h = 0$.
- $\nabla_\xi \tau = 0$.

Proof. From (1) of Proposition 9, we know that the weak ϕ -invariance of S is equivalent to $S_{11} = S_{22}$ and $S_{12} = 0$ with respect to any local h -eigenframe field. This property is equivalent to $\nabla_\xi h = 0$. Indeed, the weak ϕ -invariance of ρ is equivalent to (13).

With respect to a local h -eigenframe field $\{e_1, e_2, e_3\}$, the weak ϕ -invariance 13 is described as the system

$$\xi(\lambda) = 0 \quad \text{and} \quad a\lambda = 0. \quad (14)$$

This is equivalent to $\nabla_\xi h = 0$. $\square$

It should be remarked that Proposition 10 can be proved by avoiding the use of local h -eigenframe fields. Indeed, Proposition 10 can be deduced from the formula (see [34]):

$$\begin{aligned} (\nabla_\xi \tau)(X, Y) &= g(Q\phi X, \phi Y) - g(QX, Y) + \eta(X)g(Q\xi, Y) + \eta(Y)g(Q\xi, X) \\ &\quad - \eta(X)\eta(Y)g(Q\xi, \xi). \end{aligned}$$

Let us consider ϕ -invariant Ricci tensor fields.

Proposition 11. *Let M be a contact metric 3-manifold. Then S is ϕ -invariant if and only if M is locally isomorphic to E_2 equipped with flat contact metric structure.*

Proof. If $Q\phi = \phi Q$, then M is either, Sasakian or contact $(\kappa, 0)$ -space with $\kappa < 1$. In case M is Sasakian, ρ is not ϕ -invariant.

In case $(\kappa, 0)$ -space, from Theorem 4 together with Proposition 4, the Ricci operator Q satisfies $Q\xi = 2\kappa\xi$. Thus S is ϕ -invariant when and only when $\kappa = 0$. Hence M is locally isomorphic to E_2 equipped with flat contact metric structure.

Conversely, flat contact metric manifolds have ϕ -invariant Ricci tensor fields. $\square$

5.3 Contact metric 3-manifolds with weakly ϕ -invariant S

Proposition 11 shows that the ϕ -invariance of S is too restrictive for contact metric 3-manifolds. Now we weaken the ϕ -invariance assumption to weakly ϕ -invariance. Since the weak ϕ -invariance of S is equivalent to $\nabla_\xi h = 0$, we obtain the following examples:

Example 2. Let $G = G(c_1, c_2)$ be a unimodular Lie group. Then its Ricci tensor field is weakly ϕ -invariant if and only if $c_1 = c_2$ or $c_1 + c_2 = 2$. In the former case, G is a Sasakian space form. In case $c_1 + c_2 = 2$, G satisfy $Q\phi = \phi Q$. Hence G is a $(\kappa, 0)$ -space locally isomorphic to SU_2 , E_2 or $SL_2\mathbb{R}$. In particular the Ricci tensor fields of SU_2 and $SL_2\mathbb{R}$ are weakly ϕ -invariant but not ϕ -invariant. Note that $G(c_1, c_2)$ is weakly η -Einstein if and only if either $c_1 = c_2$ or $c_1 + c_2 = 2$. Thus for a unimodular Lie group $G(c_1, c_2)$ has weakly ϕ -invariant Ricci tensor field if and only if it is η -Einstein.

Example 3. Let $G = G(\alpha, \gamma)$ be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact metric structure. Then its Ricci tensor field S is weakly ϕ -invariant if and only if $\gamma = 0$ or $\gamma = -2$. In the former case, G is a Sasakian space form and satisfies $Q\phi = \phi Q$. In case $\gamma = -2$, G does not satisfy $Q\phi = \phi Q$. Moreover it is not η -Einstein.

It follows from Theorem 2, Examples 2 and 3 that homogeneous contact metric 3-manifolds admitting weakly ϕ -invariant Ricci tensor field are classified as follows:

Corollary 4. *Any homogeneous contact metric 3-manifold M whose Ricci tensor field S is weakly ϕ -invariant is locally isomorphic to one of the following homogeneous spaces:*

- 3-dimensional Sasakian space forms $SU_2 = (SU_2 \times U_1)/U_1$, $Nil_3 = (Nil_3 \ltimes U_1)/$, $\widetilde{SL}_2\mathbb{R} = (\widetilde{SL}_2\mathbb{R} \times SO_2)/SO_2$.
- non-Sasakian SU_2 , $\widetilde{SL}_2\mathbb{R}$ or $\widetilde{E}_2$ equipped with flat left invariant contact metric structure.
- A non-unimodular Lie group associated with the Lie algebra $\mathfrak{g}(\alpha, -2)$ for some α .

There exists a non-homogeneous contact metric 3-manifold for which the Ricci tensor field S is weakly φ -invariant. The following example is due to Blair [5] (see also [6]).

Example 4. Let $f(x, y)$ be a smooth non-constant function on $\mathbb{R}^3(x, y, z)$ bounded below by a positive constant c depending only on x and y . We take the standard Darboux contact form $\eta = \frac{1}{2}(dz - y dx)$. The Reeb vector field is given by $\xi = 2\frac{\partial}{\partial z}$. We consider a compatible metric

$$g = \frac{1}{4} \begin{pmatrix} (e^{zf} + (1 + f^2))e^{-zf} - 2)/f^2 + y^2 & (e^{zf} - 1)/f & -y \\ (e^{zf} - 1)/f & e^{zf} & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

The endomorphism fields ϕ is defined by

$$\phi = \begin{pmatrix} (e^{zf} - 1)/f & e^{zf} & 0 \\ -((e^{zf} + (1 + f^2))e^{-zf} - 2)/f^2 & -(e^{zf} - 1)/f & 0 \\ y(e^{zf} - 1)/f & ye^{zf} & 0 \end{pmatrix}.$$

Then h -operator is given by

$$h = \begin{pmatrix} e^{zf} & fe^{zf} & 0 \\ -((fe^{zf} + (1 + f^2))(-f)e^{-zf})/f^2 & -e^{zf} & 0 \\ ye^{zf} & yfe^{zf} & 0 \end{pmatrix}.$$

This example satisfies the condition $\nabla_\xi h = 0$. Thus, by the aid of Proposition 10, S is weakly ϕ -invariant. We remark that the eigenvalue of h is $\lambda = \sqrt{1 + f^2}$. Since f is non-constant, λ is also non-constant. Thus M is non-homogeneous.

5.4 Reeb flow invariance

The class of 3- τ -manifolds is pretty large. Here we shall attempt to classify them with the additional condition of Reeb flow invariance. Pan [32] classified almost cosymplectic 3-manifolds for which the Ricci tensor field S is weakly ϕ -invariant and the Ricci curvatures are invariant along the Reeb flow. We give here a result for contact metric 3-manifolds corresponding to that of [32].

Theorem 8. *Let M be a non-Sasakian contact metric 3-manifold. Then the Ricci tensor field S is weakly ϕ -invariant and the Ricci curvatures are invariant along Reeb flow if and only if M is locally isomorphic to either SU_2 , $\widetilde{SL}_2\mathbb{R}$ or $\widetilde{E}_2$ equipped with flat left invariant contact metric structure.*

Proof. It follows from Lemma 1, (14) and $\lambda \neq 0$ that we have

$$\begin{aligned} \nabla_{e_1} e_1 &= be_2, & \nabla_{e_1} e_2 &= -be_1 + (1 + \lambda)\xi, & \nabla_{e_1} \xi &= -(1 + \lambda)e_2, \\ \nabla_{e_2} e_1 &= -ce_2 + (\lambda - 1)\xi, & \nabla_{e_2} e_2 &= ce_1, & \nabla_{e_2} \xi &= (1 - \lambda)e_1, \\ \nabla_{\xi} e_1 &= 0, & \nabla_{\xi} e_2 &= 0, & \nabla_{\xi} \xi &= 0. \end{aligned}$$

These relations imply that

$$[\xi, e_1] = (1 + \lambda)e_2, [e_1, e_2] = -be_1 + ce_2 + 2\xi, [e_2, \xi] = (1 - \lambda)e_1. \quad (15)$$

By virtue of Jacobi identity

$$[\xi, [e_1, e_2]] + [e_1, [e_2, \xi]] + [e_2, [\xi, e_1]] = 0,$$

we have

$$c(1 - \lambda) + \xi(b) + e_1(\lambda) = 0, \quad (16)$$

$$-b(1 + \lambda) + \xi(c) + e_2(\lambda) = 0. \quad (17)$$

It is well-known that the functions b and c are expressed as (see [21]):

$$b = \frac{1}{2\lambda}(e_2(\lambda) + \sigma(e_1)), \quad c = \frac{1}{2\lambda}(e_1(\lambda) + \sigma(e_2)).$$

By using these relations, (15), (16) and (17), we have

$$\begin{aligned} \frac{1}{2\lambda}\sigma(e_2)(1 - \lambda) + e_1(\lambda) &= 0, \\ \frac{1}{2\lambda}\sigma(e_1)(1 + \lambda) + e_2(\lambda) &= 0, \end{aligned}$$

where we used $\xi(\sigma(e_1)) = \xi(\sigma(e_2)) = 0$. These relations mean that

$$e_1(\lambda) = -\frac{1 - \lambda}{2\lambda}\sigma(e_2), \quad e_2(\lambda) = \frac{1 + \lambda}{2\lambda}\sigma(e_1).$$

These, combined with (14) and (15), yield

$$(1 - \lambda)e_1(\lambda) = 0, \quad (1 + \lambda)e_2(\lambda) = 0.$$

Then we can see that λ is constant on $\mathcal{U}$. Thus we know that the functions b and c are vanish. That is, we obtain

$$[\xi, e_1] = (1 + \lambda)e_2, [e_1, e_2] = 2\xi, [e_2, \xi] = (1 - \lambda)e_1.$$

If we assume that $\lambda > 0$. By virtue of Milnor's list (see Subsection 4.1), we obtain the following cases.

- When $\lambda = 1$, M is locally isomorphic to E_2 equipped with flat left invariant contact metric structure.
- When $0 < \lambda < 1$, M is locally isomorphic to SU_2 .
- When $\lambda > 1$, M is locally isomorphic to $SL_2\mathbb{R}$.

Conversely, by Proposition 11 and Example 2, we can see that SU_2 , $SL_2\mathbb{R}$ or E_2 equipped with flat left invariant contact metric structure have weakly ϕ -invariant Ricci tensor field. $\square$

6 Contact metric real hypersurfaces

The weak ϕ -invariance of the Ricci tensor field S and the parallelism of h along ξ -direction are not equivalent on contact metric manifolds of dimension 5 or higher.

We give an example of the contact metric manifold with weakly ϕ -invariant Ricci tensor field but does *not* satisfy the condition $\nabla_\xi h = 0$. To this end, we recall here real hypersurface geometry in complex space forms.

6.1 Almost contact metric structures on real hypersurfaces

Let M^{2n-1} be a real hypersurface isometrically immersed in an almost Hermitian $(2n)$ -manifold $(\widetilde{M}^{2n}, J, \tilde{g})$.

Since we will work locally, we may assume that M is orientable. Take a unit normal vector field $\mathcal{N}$ of M^{2n-1} in $\widetilde{M}^{2n}$. Then the Levi-Civita connections $\tilde{\nabla}$ of $\widetilde{M}^{2n}$ and ∇ of M^{2n-1} are related by the *Gauss formula*

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \quad X, Y \in \mathfrak{X}(M),$$

where g is the induced metric on M and A is *shape operator* derived from $\mathcal{N}$ defined by the *Weingarten formula*

$$\tilde{\nabla}_X \mathcal{N} = -AX, \quad X, Y \in \mathfrak{X}(M).$$

The almost Hermitian structure $(J, \tilde{g})$ of $\widetilde{M}^{2n}$ induces an almost contact metric structure (ϕ, ξ, η, g) on M^{2n-1} by

$$\xi = -\varepsilon J\mathcal{N}, \quad \eta = g(\xi, \cdot), \quad \varepsilon = \pm 1 \tag{18}$$

and

$$\phi X = JX - g(JX, \mathcal{N})\mathcal{N}, \quad X \in \mathfrak{X}(M). \tag{19}$$

respectively. The vector field ξ is called the *structure vector field*.

An orientable real hypersurface M^{2n-1} in $\widetilde{M}^{2n}$ is said to be a *Hopf hypersurface* if its structure vector field is a principal curvature vector at each point of M^{2n-1} . On the other hand $M^{2n-1} \subset \widetilde{M}$ is said to be *contact metric* if the induced almost contact metric structure is contact metric, *i.e.*, it satisfies $d\eta = \Phi$.

6.2 Contact metric hypersurfaces in complex space forms

Hereafter we assume that the ambient space is a complex space form of constant holomorphic sectional curvature c and denote it by $\widetilde{M}_n(c)$. It should be remarked that real hypersurfaces in $\widetilde{M}_n(c)$ are *not* contact metric, in general. Indeed, one can see that the almost contact metric structure (ϕ, ξ, η, g) satisfies $\Phi = d\eta$ if and only if

$$A\phi + \phi A = -2\varepsilon\phi. \quad (20)$$

In such a case, the sign $\varepsilon = \pm 1$ is chosen so that the formula (3) holds.

Sharma investigated contact metric real hypersurfaces in $\widetilde{M}_n(c)$.

Theorem 9 ([39]). *Let M^{2n-1} ($n \geq 2$) be a contact metric real hypersurface of a complex space form $\widetilde{M}_n(c)$. Then M is either*

1. *Sasakian manifold of constant holomorphic sectional curvature and η -umbilical.*
2. *locally isomorphic to the unit tangent sphere bundle of some real space form of constant curvature $\bar{c} \neq 1$ up to $\mathfrak{D}$ -homothety.*
 - *If $c > -4$, then the principal curvatures are $-c/2, 1 \pm \sqrt{1 + \frac{c}{4}}$,*
 - *If $c = 0$, then the principal curvatures are 0 and 2. If in addition $n = 2$, M is flat. If $n > 2$, then M is locally isometric to the unit tangent sphere bundle $U\mathbb{E}^n$.*

Adachi, Kameda and Maeda [2] gave a more detailed classification for contact metric real hypersurfaces in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$.

Theorem 10 ([2]). *Let M^{2n-1} ($n \geq 2$) be an orientable real hypersurface of $\mathbb{C}P^n(c)$. If it is contact metric, then it is locally congruent to one of the following real hypersurface;*

- *a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{c}) \tan^{-1}(\sqrt{c}/2), 0 < r < \pi/\sqrt{c}$ and $\varepsilon = -1$, or*
- *a tube of radius $r = (2/\sqrt{c}) \tan^{-1}\{(\sqrt{c+4} - \sqrt{c})/2\} < \pi(2\sqrt{c})$ around complex hyperquadric $\mathbb{C}Q^{n-1}$ with $\varepsilon = +1$.*

In the former case, $G(r)$ is a Sasakian space form of constant holomorphic curvature $H = c + 1 > 1$. In the latter case, M^{2n-1} is a non-Sasakian contact (κ, μ) -space with $\kappa = -c/4, \mu = -c/2$ (see [16]).

On the other hand, contact metric real hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n(c)$ were classified by Vernon (see also [2]).

Theorem 11 ([2, 43]). *Let M^{2n-1} ($n \geq 2$) be an orientable real hypersurface of $\mathbb{C}H^n(c)$. If it is contact metric, then it is locally congruent to one of the following real hypersurface;*

- *a horosphere of $\mathbb{C}H^n(-4)$ and $\varepsilon = -1$. It is a Sasakian space form of constant holomorphic sectional curvature -3 ,*

- a geodesic sphere $G(r)$ of radius $r = (\log(2 + \sqrt{|c|}) - \log(2 - \sqrt{|c|}))/\sqrt{|c|}$ with $\varepsilon = -1$ and $-4 < c < 0$. It is a Sasakian space form constant holomorphic curvature $-3 < H = c + 1 < 1$.
- a tube of radius $r = (\log(2 + \sqrt{|c|}) - \log(2 - \sqrt{|c|}))/(\sqrt{|c|})$ the totally geodesic complex hyperbolic space $\mathbb{C}H^{n-1}(c)$ of constant holomorphic sectional curvature $c < -4$. The real hypersurface is a Sasakian space form constant holomorphic curvature $H = c + 1 < -3$.
- a tube of radius $r = (\log(2 + \sqrt{|c|}) - \log(2 - \sqrt{|c|}))/(\sqrt{|c|})$ around totally geodesic Lagrangian real hyperbolic space $\mathbb{R}H^n(c/4)$ with $\varepsilon = -1$ and $-4 < c < 0$. M^{2n-1} is a non-Sasakian contact (κ, μ) -space with $\kappa = 3c/4$, $\mu = -c/2$ (see [16]).

6.3 Hopf hypersurfaces with weakly ϕ -invariant Ricci tensor field

The second named author investigated Hopf hypersurfaces whose Ricci tensor field is weakly ϕ -invariant (for detail [30]). In particular, we obtain the following example:

Example 5. Let M^5 be a tube of radius $r = \tan^{-1}(\sqrt{2} - 1) = \pi/8 \approx 0.3926$ around complex hyperquadric $\mathbb{C}Q^2$ in $\mathbb{C}P^3(4)$. This real hypersurface is a contact metric one. Furthermore, the Ricci tensor field of M^5 is weakly ϕ -invariant, but the condition $\nabla_{\xi}h = 0$ does not hold (for details, see [31]).

Remark 5. Let M^{2n-1} be a contact metric real hypersurface of the complex projective space $\mathbb{C}P^n(c)$. Then M has weakly ϕ -invariant Ricci tensor field if and only if M^{2n-1} is (weakly) η -Einstein. On the other hand, non-Sasakian contact metric hypersurface (of type B) in the complex hyperbolic space $\mathbb{C}H^n(c)$ is not η -Einstein and its Ricci tensor field is not weakly ϕ -invariant. If we relax the contact metric assumption, then there exist non η -Einstein real hypersurfaces which has weakly ϕ -invariant Ricci tensor field.

Indeed, a tube of radius $r \neq (2/\sqrt{c})\cot^{-1}(\sqrt{\ell/(n-\ell-1)})$ around a totally geodesic $\mathbb{C}P^\ell(c)$ ($1 \leq \ell \leq n-2$) in $\mathbb{C}P^n(c)$ is not η -Einstein and its Ricci tensor field is weakly ϕ -invariant.

Kim and Ryan [24] proved the following fact.

Proposition 12. Every weakly η -Einstein real hypersurface M^3 in a non-flat complex space form $\widetilde{M}_2(c)$ is η -Einstein and Hopf. Moreover a weakly η -Einstein real hypersurface M^3 of $\mathbb{C}P^2(c)$ is locally congruent to either

- a geodesic sphere, or
- a tube of radius $\pi/(2\sqrt{c})$ around a holomorphic curve in $\mathbb{C}P^2(c)$.

The only one of these that is a contact metric hypersurface is a geodesic sphere $G(r)$ of radius $r = (2/\sqrt{c})\tan^{-1}(\sqrt{c}/2)$. We note that a tube of radius $\pi/(2\sqrt{c})$ around a holomorphic curve which lies on a complex projective line $\mathbb{C}P^1(c)$ is essentially a geodesic sphere $G(r)$ of radius $\pi/(2\sqrt{c})$. This is *not* contact metric. On the other hand, we take a non-totally geodesic holomorphic curve C without principal curvatures $\pm(\sqrt{c}/2)$. Then a tube of radius $\pi/(2\sqrt{c})$ around a non-totally geodesic holomorphic

curve C is *not* contact metric. Indeed, the shape operator A is given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & (\sqrt{c}/2) \cot((\pi/4) - \theta) & 0 \\ 0 & 0 & (\sqrt{c}/2) \cot((\pi/4) + \theta) \end{pmatrix}.$$

with respect to a ϕ -basis $\{\xi, e, \phi e\}$, where $(\pm\sqrt{c}/2) \cot \theta$ are principal curvatures of C (see [24, 25]). Since $(\sqrt{c}/2) \cot((\pi/4) - \theta) + (\sqrt{c}/2) \cot((\pi/4) + \theta)$ is not constant, this real hypersurface does not satisfy the condition (20).

Remark 6. There is an oversight in the list of Theorem 1 in [30]. More precisely, a non-homogeneous Hopf hypersurface with $A\xi = 0$ in $\mathbb{C}H^2(c)$ is omitted from the list. In [17], they studied three-dimensional real hypersurfaces satisfying the condition $Q\phi = \phi Q$. The above real hypersurface satisfies the condition $Q\phi = \phi Q$. On Hopf hypersurfaces, the condition $Q\phi = \phi Q$ and the weakly ϕ -invariance of the Ricci tensor field S are equivalent (see [30]). Thus the above real hypersurface also admits a weakly ϕ -invariant Ricci tensor field.

7 Tangent sphere bundles

Let $(\overline{M}, \bar{g})$ be a Riemannian n -manifold, then the metric $\bar{g}$ induces a Riemannian metric $\tilde{g}$ on the tangent bundle $T\overline{M}$ (called the *Sasaki lift metric*). Moreover the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$ induces the splitting

$$T(T\overline{M}) = \mathcal{H} \oplus \mathcal{V},$$

where $\mathcal{H}$ and $\mathcal{V}$ are *horizontal* and *vertical* subbundles of $T(T\overline{M})$. The Sasaki-lift metric $\tilde{g}$ is defined by

$$\tilde{g}_{(x;u)}(X^h, Y^h) = \tilde{g}_{(x;u)}(X^v, Y^v) = \bar{g}_x(X, Y)$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. Here h and v are horizontal and vertical lift operations. The canonical almost complex structure J on $T\overline{M}$ with respect to $\tilde{g}$ is defined by

$$JX^h = X^v, \quad JX^v = -X^h.$$

One can see that $(T\overline{M}, J, \tilde{g})$ is *almost Kähler*, that is, its fundamental 2-form is closed. The *tangent sphere bundle*

$$T^{(r)}\overline{M} = \{(x; u) \in T\overline{M} \mid \bar{g}_x(u, u) = r^2\}$$

of radius $r > 0$ is a hypersurface of $(T\overline{M}, J, \tilde{g})$ with globally defined unit normal vector field

$$\mathcal{N} = \frac{1}{r} \sum_{i=1}^n u^i \frac{\partial}{\partial u^i},$$

where $(u^1, u^2, \dots, u^n)$ is a fiber coordinate system derived from a local coordinate system $(x^1, x^2, \dots, x^n)$ of $\overline{M}$. Then the almost Kähler structure $(J, \tilde{g})$ induces an almost metric structure (ϕ, ξ', η', g') on $T^{(r)}\overline{M}$ by the formulas (18)–(19) with $\varepsilon = 1$:

$$\xi' = -J\mathcal{N}, \quad \eta' = g'(\cdot, \xi')$$

and

$$\phi W = JW - \tilde{g}(JW, \mathcal{N})\mathcal{N}, \quad W \in \mathfrak{X}(T\overline{M}).$$

Here g' is the Riemannian metric on $T^{(r)}\overline{M}$ induced from the Sasaki-lift metric $\tilde{g}$. It is well known that η' is the canonical contact form. However the structure is *not* contact metric. Let us perform the normalization

$$\xi = 2r \xi', \quad \eta = \frac{1}{2r} \eta', \quad g = \frac{1}{4r^2} g'.$$

Then (ϕ, ξ, η, g) is a contact metric structure. The endomorphism field ϕ is given explicitly by

$$\phi X_u^h = X_u^t, \quad \phi X_u^t = -X_u^h + \frac{1}{2r^2} g(X, u) \xi_u,$$

where X^t is the *tangential lift* of X defined by

$$X_u^t = X_u^v - \frac{1}{r} \mathcal{N}_u.$$

The resulting contact metric structure (ϕ, ξ, η, g) of $T^{(r)}\overline{M}$ is referred as to the *standard contact metric structure* of $T^{(r)}\overline{M}$. In particular $(T^{(1)}\overline{M}, \phi, \xi, \eta, g)$ is denoted by $U\overline{M}$ and called the *unit tangent sphere bundle equipped with the standard contact metric structure*.

The tangent space $T_u(T^{(r)}\overline{M})$ of $T^{(r)}\overline{M}$ at $u = (x; u)$ is given by

$$T_u(T^{(r)}\overline{M}) = \{X_u^h + Y_u^t \mid X, Y \in T_x\overline{M}, \bar{g}_x(Y, u) = 0\}.$$

For any $X \in T_x(\overline{M})$, we have $g_u(X^t, \xi) = 0$. On the other hand, $g_u(X^h, \xi) = 0$ if and only if $g(X, u) = 0$. Thus the contact distribution $\mathfrak{D}$ is given by

$$\mathfrak{D}_u = \{X_u^h + Y_u^t \mid X, Y \in T_x\overline{M}, \bar{g}_x(X, u) = \bar{g}_x(Y, u) = 0\}.$$

The Ricci tensor field S of $(T^{(r)}\overline{M}, g)$ is given by

$$\begin{aligned} S_u(X^h, Y^h) &= \bar{S}_x(X, Y) - \frac{1}{2} \sum_{i=1}^{n-1} \bar{g}(\bar{R}(u, e_i)X, \bar{R}(u, e_i)Y), \\ S_u(X^t, Y^h) &= \frac{1}{2} ((\bar{\nabla}_u \bar{S})(X, Y) - (\bar{\nabla}_X \bar{S})(u, Y)), \\ S_u(X^t, Y^t) &= \frac{1}{4} \sum_{i=1}^{n-1} \bar{g}(\bar{R}(u, X)e_i, \bar{R}(u, Y)e_i) + \frac{n-2}{r^2} \{g_x(X, Y) - \frac{1}{r^2} \bar{g}(X, u) \bar{g}(Y, u)\}, \end{aligned}$$

where $\{e_1, e_2, \dots, e_n = u/r\}$ is a orthonormal basis of $T_x \overline{M}$ and $X, Y \in \mathfrak{X}(M)$.

Let us assume that $\overline{M}$ is of constant curvature c , then the Riemannian curvature $\bar{R}$ of $\overline{M}$ has the form

$$\bar{R}(X, Y)Z = c(\bar{g}(Y, Z)X - \bar{g}(Z, X)Y).$$

For any $X, Y \in T_x \overline{M}$ orthogonal to u , we have

$$\begin{aligned} S_u(\phi X^h, \phi Y^h) &= S_u(X^t, Y^t) = \frac{c^2 r^2}{2} \bar{g}_x(X, Y) + \frac{n-2}{r^2} \bar{g}_x(X, Y), \\ S_u(\phi X^t, \phi Y^t) &= S_u(X^h, Y^h) = \bar{S}_x(X, Y) - \frac{c^2 r^2}{2} \bar{g}_x(X, Y). \end{aligned}$$

On the other hand

$$S_u(\phi X^h, \phi Y^t) = S_u(X^h, Y^t)$$

holds since $\nabla \bar{S} = 0$.

Thus S is weakly ϕ -invariant if and only if

$$\bar{S}(X, Y) = \left(c^2 r^2 + \frac{n-2}{r^2} \right) g(X, Y).$$

By the assumption $\bar{S}$ has the form

$$\bar{S} = c(n-1)\bar{g}.$$

Thus $M = T^{(r)}\overline{M}(c)$ has weakly ϕ -invariant Ricci tensor field if and only if the sectional curvature c satisfies

$$r^4 c^2 - (n-1)r^2 c + n-2 = 0.$$

This quadratic equation has solutions

$$c = \frac{1}{r^2}, \quad \frac{n-2}{r^2}.$$

Theorem 12. *The tangent sphere bundle $M = T^{(r)}\overline{M}(c)$ of a n -dimensional space form of curvature c has weakly ϕ -invariant Ricci tensor field if and only if $c = 1/r^2$ or $(n-2)/r^2$.*

Corollary 5. *The unit tangent sphere bundle $M = U\overline{M}(c)$ of an n -dimensional space form $\overline{M}(c)$ of curvature c has weakly ϕ -invariant Ricci tensor field if and only if $c = 1$ or $n-2$.*

In particular, the Euclidean motion group $E_2 = SO_2 \ltimes \mathbb{R}^2$ which is identified with the unit tangent sphere bundle $U\mathbb{E}^2$ of the Euclidean plane has weakly ϕ -invariant Ricci tensor field. Park and Sekigawa [33] proved the following fact.

Theorem 13. *The tangent sphere bundle $M = T^{(r)}\overline{M}$ of a Riemannian n -manifold is η -Einstein if and only if $\overline{M}$ is of constant curvature $c = 1/r^2$ or $(n-2)/r^2$.*

Note for tangent sphere bundles $M = T^{(r)}\overline{M}$ over a Riemannian n -manifold ($n \geq 2$), weakly η -Einstein property implies η -Einstein property. As a result, we deduce the following corollary.

Corollary 6. *For the tangent sphere bundle $M = T^{(r)}\overline{M}$ of a Riemannian n -manifold $\overline{M}$, the following properties are mutually equivalent:*

- M has weakly ϕ -invariant Ricci tensor field.
- M is weakly η -Einstein.
- M is η -Einstein.

On the other hand, Perrone proved the following fact [36].

Theorem 14. *The unit tangent sphere bundle $M = U\overline{M}$ of a Riemannian n -manifold satisfies $\nabla_\xi h = 0$ $\overline{M}$ is of constant curvature $c = 1$ or 0 .*

Comparing Theorem 12 with Theorem 14, we notice that the tangent sphere bundle

$$T^{(r)}\mathbb{E}^n = \mathbb{E}^n \times \mathbb{S}^{n-1}(1/r^2)$$

satisfies:

- $n = 2$: S is weakly ϕ -invariant and h satisfies $\nabla_\xi h = 0$.
- $n > 2$: S is not weakly ϕ -invariant, but h satisfies $\nabla_\xi h = 0$.

8 A relationship between the weakly ϕ -invariance of the Ricci tensor field and the J -invariance of the Ricci tensor field

8.1 J -invariance of the Ricci tensor field

Let $(\tilde{M}, J, \tilde{g})$ be an almost Hermitian manifold, then the 2-form defined by

$$\Omega(X, Y) = \tilde{g}(X, JY), \quad X, Y \in \mathfrak{X}(\tilde{M})$$

is called the *fundamental 2-form* of $(\tilde{M}, J, \tilde{g})$. An almost Hermitian manifold $(\tilde{M}, J, \tilde{g})$ is said to be *almost Kähler* if $d\Omega = 0$. It should be remarked that the fundamental 2-form Ω of an almost Kähler manifold $(\tilde{M}, J, \tilde{g})$ is *symplectic*.

Conversely, let $(\tilde{M}, \Omega)$ be a symplectic manifold. Then a Riemannian metric $\tilde{g}$ on $(\tilde{M}, \Omega)$ is said to be *compatible* if there exists an almost complex structure J such that the fundamental 2-form coincides with Ω . Note that an almost Kähler manifold $(\tilde{M}, J, \tilde{g})$ is Kähler if and only if J is integrable.

The J -invariance of the Ricci tensor field $\tilde{S}$ on compact almost Kähler manifolds has a variational characterization. Let (M, Ω) be a compact symplectic manifold. Denote by $\mathcal{M}(\Omega)$ the space of all compatible Riemannian metrics on (M, Ω) . Blair and Ianus [8] proved that a compatible Riemannian metric $\tilde{g} \in \mathcal{M}(\Omega)$ is a critical point of the total scalar curvature, if and only if its Ricci tensor field S is J -invariant, *i.e.*, $QJ = JQ$.

Blair and Ianus posed a question whether a compact almost-Kähler manifold whose J -invariant Ricci operator must be Kähler. This question is related to the so-called

Goldberg conjecture. Goldberg showed that if an almost Kähler manifold $(\tilde{M}, J, \tilde{g})$ has J -invariant Riemannian curvature, then it is Kähler. He conjectured that a compact almost Kähler-Einstein manifold is Kähler. Up to now, both the question posed by Blair and Ianus and Goldberg conjecture are still open.

Here we mention that 4-dimensional almost Hermitian, especially almost Kähler manifolds with J -invariant Ricci operator are investigated in [19, 20, 44].

On the other hand, on the Riemannian cone $C(M)$ over an almost contact metric manifold (M, ϕ, ξ, η, g) admits an orthogonal almost complex structure J . In this section, we study relations between J -invariance of the Ricci tensor field on the Riemannian cone $C(M)$ and weakly ϕ -invariance of the Ricci tensor field on M .

8.2 A relationship between J -invariance of the Ricci tensor field $\tilde{S}$ and weakly ϕ -invariance of the Ricci tensor field S

Let (M, g) be Riemannian manifold. Take a *Riemannian cone* $C(M) = \mathbb{R}_+ \times_t M$ over M . The Riemannian cone $C(M)$ is the product manifold of the positive half line $\mathbb{R}_+ = \{t \in \mathbb{R}; t > 0\}$ and M equipped with the warped product metric (*cone metric*) $\tilde{g} = dt^2 + t^2g$. There is a one-to-one correspondence between almost contact structure (ϕ, ξ, η) compatible to g and orthogonal complex structure J on $C(M)$ satisfying ([12, Proposition 6.5.2]):

$$JX = \phi X + t\eta(X)\partial_t, \quad J\partial_t = -\frac{1}{t}\xi$$

for all $X \in \mathfrak{X}(M)$ and $\partial_t = \partial/\partial t$. We have the following fundamental fact ([12, Proposition 6.5.5]):

Proposition 13. *An almost contact metric manifold (M, ϕ, ξ, η, g) is a contact metric manifold if and only if its almost Hermitian cone $\tilde{M} = (C(M), J, \tilde{g})$ is an almost Kähler manifold.*

On the other hand, (M, ϕ, ξ, η, g) is normal if and only if J is integrable ([12, Proposition 6.5.8]). Thus an almost contact metric manifold (M, ϕ, ξ, η, g) is Sasakian if and only if its almost Hermitian cone $\tilde{M}$ is Kähler.

The purpose of this subsection is to investigate the relationship between the weakly ϕ -invariance of the Ricci tensor field on an almost contact metric manifold M and the J -invariance of the Ricci tensor field of the almost Hermitian cone $\tilde{M}$.

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}$ is given by [28]:

$$\begin{aligned} \tilde{\nabla}_{\partial_t}\partial_t &= 0, \quad \tilde{\nabla}_X\partial_t = \frac{1}{t}X, \quad \tilde{\nabla}_{\partial_t}X = \frac{1}{t}X, \\ \tilde{\nabla}_XY &= \nabla_XY - tg(X, Y)\partial_t, \end{aligned} \tag{21}$$

for any $X, Y \in \mathfrak{X}(M)$ and $t \in \mathbb{R}_+$, where ∇ is the Levi-Civita connection of M . Let $\tilde{R}$ and R be the Riemannian curvatures of $\tilde{M}$ and M , respectively. According to (21), the curvature tensor field $\tilde{R}$ of $\tilde{M}$ is expressed as (see [3, 28]):

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z - g(Y, Z)X + g(X, Z)Y, \\ \tilde{R}(\partial_t, X)\partial_t &= 0, \quad \tilde{R}(X, Y)\partial_t = 0, \quad \tilde{R}(\partial_t, X)Y = 0 \end{aligned} \tag{22}$$

for any $X, Y, Z \in \mathfrak{X}(M)$. We denote by $\tilde{S}$ (resp. S) the Ricci tensor field of $\tilde{M}$ (resp. M). By using (22), the Ricci tensor field $\tilde{S}$ of $\tilde{M}$ is given as follows (cf. [3, 28]):

$$\begin{aligned}\tilde{S}(X, Y) &= S(X, Y) - 2(n-1)g(X, Y) \quad \text{for } X, Y \in \mathfrak{X}(M), \\ \tilde{S}(X, \partial_t) &= 0 \quad \text{for } X \in \mathfrak{X}(M), \\ \tilde{S}(\partial_t, \partial_t) &= 0.\end{aligned}\tag{23}$$

The following formulas for $\tilde{S}$ and S are given in [3]:

$$\begin{aligned}\tilde{S}(JX, JY) &= S(\phi X, \phi Y) - 2(n-1)g(X, Y) + 2(n-1)\eta(X)\eta(Y), \\ \tilde{S}(JX, J\partial_t) &= S(\phi X, \xi), \\ \tilde{S}(J\partial_t, J\partial_t) &= S(\xi, \xi) - 2(n-1).\end{aligned}\tag{24}$$

for any $X, Y \in \mathfrak{X}(M)$. Comparing (23) with (24), the Ricci tensor field $\tilde{S}$ is J -invariant if and only if

$$S(\phi X, \phi Y) = S(X, Y) - 2(n-1)\eta(X)\eta(Y) \quad \text{for } X, Y \in \mathfrak{X}(M),\tag{25}$$

$$S(\phi X, \xi) = 0 \quad \text{for } X \in \mathfrak{X}(M),\tag{26}$$

$$S(\xi, \xi) = 2(n-1).\tag{27}$$

The relation (25) is exactly weakly ϕ -invariance of the Ricci tensor field S of M . By auxiliary (26) and (27), the Ricci operator Q of M satisfies

$$Q\xi = 2(n-1)\xi.$$

Note that, from (25)–(27), we obtain the formula (9).

In summary, we have the following:

Theorem 15. *The almost Hermitian cone $\tilde{M} = (C(M), J, \tilde{g})$ over an almost contact metric $(2n-1)$ -manifold M has J -invariant Ricci tensor field if and only if M has weakly ϕ -invariant Ricci tensor field and the characteristic vector field ξ is an eigenvector field of the Ricci operator Q with eigenvalue $2(n-1)$.*

By the aid of Proposition 13 and [4, Theorem A], the relation (27) is equivalent to K -contact condition on contact metric manifolds. We obtain the following Corollary:
Corollary 7. *An almost Kähler cone $\tilde{M}$ over a contact metric manifold M has J -invariant Ricci tensor field if and only if M is K -contact, and the Ricci tensor field of M is weakly ϕ -invariant.*

In 3-dimensional case, a K -contact manifold is a Sasakian manifold (see e.g., [6, Corollary 6.5]). It follows from the definition of a Sasakian manifold [12, Definition 6.5.15], that we obtain the following result:

Corollary 8. *Let M be a contact metric 3-manifold. Then its almost Kähler cone $\tilde{M}$ has a J -invariant Ricci tensor field if and only if $\tilde{M}$ is a Kähler manifold.*

Remark 7. Let $M = G(\alpha, -2)$ be a 3-dimensional non-unimodular Lie group equipped with a left invariant contact metric structure. By Proposition 5, the almost Kähler cone

$\widetilde{M}$ does not have a J -invariant Ricci tensor field. However, M has weakly ϕ -invariant Ricci tensor field.

Remark 8. On a contact metric manifold (M, ϕ, ξ, η, g) , the metric g is critical associated metric if and only if (6) holds. Blair and Ledger [8] stated that:

In general a K -contact structure need not satisfy $Q\phi = \phi Q$, and it is an open question as to whether or not a K -contact structure satisfying $Q\phi = \phi Q$ is Sasakian.

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