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On the spectral synthesis for the unit circle in $\mathcal{F}L^q_s(\mathbf{R}^2)$

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Abstract

Let $\mathcal{F}L^q_s(\mathbf{R}^2)$ denote the set of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^2)$ such that the norm $\|f\|_{\mathcal{F}L^q_s} := (\int_{\mathbf{R}^2}(|\mathcal{F}[f](\xi)|(1+|\xi|)^s)^q d\xi)^{1/q}$ is finite, where $\mathcal{F}[f]$ denotes the Fourier transform of f. We investigate the spectral synthesis for the unit circle $S^1 \subset \mathbf{R}^2$ in $\mathcal{F}L^q_s(\mathbf{R}^2)$ with $1 < q < \infty$ and $s > 2/q'$, where q' denotes the conjugate exponent of q.

Keywords: Fourier-Wermer algebra, Fourier-Lebesgue spaces, Banach algebra, Set of spectral synthesis

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1 Introduction

In commutative harmonic analysis, there is an important problem of determining whether the spectral synthesis holds for a given Banach space B and a given subset E of $\mathbf{R}^n$. To be more precise, let $(B, \|\cdot\|_B)$ be a Banach space of bounded continuous, complex-valued functions, i.e., $B \subset C_b(\mathbf{R}^n)$, and assume that $C_c^\infty(\mathbf{R}^n)$ is dense in B and convergence in B implies pointwise convergence. For every closed subset E of $\mathbf{R}^n$,

we set $I(E) := \{f \in B \mid f|_E = 0\}$ and define $J(E)$ as the closure in B of the set

$$\{f \in C_c^\infty(\mathbf{R}^n) \mid f = 0 \text{ in a neighborhood of } E\}.$$

Here, $f|_E = 0$ means that $f(x) = 0$ for all $x \in E$. Since $I(E) \supset J(E)$, it is natural to ask whether $I(E) \subset J(E)$ holds. If $I(E) = J(E)$, then E is called a *set of spectral synthesis* in B. Let $A(\mathbf{R}^n)$ be the Fourier algebra of all functions on $\mathbf{R}^n$ which are the Fourier transform of functions in $L^1(\mathbf{R}^n)$, also denoted by $\mathcal{F}L_0^1(\mathbf{R}^n)$ elsewhere. It is well known that $A(\mathbf{R}^n) \subset C_0(\mathbf{R}^n)$ by the Riemann-Lebesgue lemma and $A(\mathbf{R}^n)$ is a Banach algebra with respect to pointwise multiplication and the norm $\|f\|_A := \|\mathcal{F}[f]\|_{L^1}$, where $\mathcal{F}[f](\xi) := \int_{\mathbf{R}^n} f(x)e^{-ix\xi}dx$. The first counterexample of the spectral synthesis was given by Schwartz [25] and he showed that the unit sphere $S^{n-1} := \{x \in \mathbf{R}^n \mid |x| = 1\}$ is not a set of spectral synthesis in $A(\mathbf{R}^n)$, if $n \geq 3$. Surprisingly, Herz [17] showed that the spectral synthesis holds for S^1 in $A(\mathbf{R}^2)$. Motivated by these results, Varopoulos [29] extended Herz's method to higher dimensions to study this gap and showed that S^{n-1} $(n \geq 3)$ is of weak synthesis, i.e., the closure in $A(\mathbf{R}^n)$ of the set $\{f \in C_c^\infty(\mathbf{R}^n) \mid f|_E = 0\}$ is equal to $I(S^{n-1})$. On the other hand, Reiter [22] showed similar results in the Fourier-Beurling algebras $\mathcal{F}L_s^1(\mathbf{R}^n)$ and proved that S^1 is a set of spectral synthesis in $\mathcal{F}L_s^1(\mathbf{R}^2)$ if $0 < s < 1/2$, and S^1 is not a set of spectral synthesis in $\mathcal{F}L_s^1(\mathbf{R}^2)$ if $s \geq 1/2$. The study of spectral synthesis is also related to a uniqueness property for PDEs (see for example [16] and Section 6 below). The reader is referred to [5], [6], [7], [23], [24], [27] for more details on the spectral synthesis.

In this paper, we investigate the spectral synthesis for S^1 in the Fourier-Wermer algebra $\mathcal{F}L_s^q(\mathbf{R}^2)$ with $1 < q < \infty$ and $s > 2/q'$, which gives an answer to the problem proposed by Domar [6]. The precise definition of $\mathcal{F}L_s^q(\mathbf{R}^n)$ will be given in Section 2.2, but we remark that $\mathcal{F}L_s^q(\mathbf{R}^n) \subset C_0(\mathbf{R}^n)$ and $\mathcal{F}L_s^q(\mathbf{R}^n)$ is a Banach algebra with respect to pointwise multiplication if $1 < q < \infty$ and $s > n/q'$. "Spectral synthesis in $\mathcal{F}L_s^q$" is a classical topic, but recently it has been found to be useful in determining "Spectral synthesis in modulation spaces $M_s^{p,q}$" (see [11], [12]). Our results are as follows:

Theorem 1 *Let $1 < q < \infty$ and $s > 2/q' + 1/2$. Then S^1 is not a set of spectral synthesis in $\mathcal{F}L_s^q(\mathbf{R}^2)$.*

Theorem 2 *Let $1 < q < 2$ and $2/q' < s < 2/q' + 1/2$. Then S^1 is a set of spectral synthesis in $\mathcal{F}L_s^q(\mathbf{R}^2)$.*

Moreover, we have the following result for $n \geq 3$.

Theorem 3 *Let $1 < q < \infty$ and $s > n/q'$. Then S^{n-1} $(n \geq 3)$ is not a set of spectral synthesis in $\mathcal{F}L_s^q(\mathbf{R}^n)$.*

We explain the organization of this paper. After the next preliminary section devoted to the definitions and basic properties of $\mathcal{F}L^q_s(\mathbf{R}^n)$ and a set of spectral synthesis, we prove Theorem 1 in Section 3. Theorems 2 and 3 are treated in Sections 4 and 5, respectively. Finally, we give an application of spectral synthesis to PDEs and some remarks on closed ideals in $\mathcal{F}L^q_s(\mathbf{R}^2)$ in Sections 6 and 7.

2 Preliminaries

The following notation will be used throughout this paper. The notation $I \lesssim J$ means that I is bounded by a constant times J and we write $I \approx J$ if $I \lesssim J$ and $J \lesssim I$. We use C to denote various positive constants which may change from line to line. The closed ball with center $x_0 \in \mathbf{R}^n$ and radius $R > 0$ is defined by $B_n(x_0, R) := \{x \in \mathbf{R}^n \mid |x - x_0| \le R\}$ and its Lebesgue measure is denoted by $|B_n(x_0, R)|$. For $x \in \mathbf{R}^n$ and $E \subset \mathbf{R}^n$, we set $d(x, E) := \inf\{|x - y| \mid y \in E\}$. We define for $1 \le q < \infty$ and $s \in \mathbf{R}$

$$\|f\|_{L^q_s} := \left(\int_{\mathbf{R}^n} (|f(x)|\langle x\rangle^s)^q \, dx \right)^{1/q}$$

and $\|f\|_{L^\infty_s} := \operatorname{ess.sup}_{x\in\mathbf{R}^n} \langle x\rangle^s |f(x)|$ with $\langle x\rangle := 1 + |x|$. We simply write $L^q(\mathbf{R}^n)$ instead of $L^q_0(\mathbf{R}^n)$. For $1 \le q < \infty$, q' denotes the conjugate exponent of q, i.e., $1/q + 1/q' = 1$. Let $C_c(\mathbf{R}^n)$ be the set of all $f \in C(\mathbf{R}^n)$ such that supp f is compact. Also let $C^\infty_c(\mathbf{R}^n)$ be the set of all $f \in C^\infty(\mathbf{R}^n)$ such that supp f is compact. If $1 \le q < \infty$ and $s \in \mathbf{R}$, then $C^\infty_c(\mathbf{R}^n)$ is dense in $L^q_s(\mathbf{R}^n)$. As in the proof of [23, Proposition 1.6.11], we can easily see

$$\lim_{|t|\to 0} \|f(\cdot - t) - f(\cdot)\|_{L^q_s} = 0 \tag{1}$$

for $1 \le q < \infty$ and $f \in L^q_s(\mathbf{R}^n)$ (see [8], [15], [26] for more details on the weighted function spaces). Let $\mathcal{S}(\mathbf{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$ and $\mathcal{S}'(\mathbf{R}^n)$ be the set of all tempered distributions, i.e., the topological dual of $\mathcal{S}(\mathbf{R}^n)$. We use $\langle f, g\rangle$ to denote the extension of the inner product $\langle f, g\rangle_{L^2(\mathbf{R}^n)} = \int_{\mathbf{R}^n} f(t)\overline{g(t)}dt$ on $L^2(\mathbf{R}^n)$ to $\mathcal{F}L^q_s(\mathbf{R}^n) \times \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ (see Section 2.2 below). The Fourier transform of $f \in L^1(\mathbf{R}^n)$ is $\mathcal{F}[f](\xi) := \int_{\mathbf{R}^n} f(x)e^{-ix\xi}dx$, and the inverse Fourier transform of $h \in L^1(\mathbf{R}^n)$ is $\mathcal{F}^{-1}[h](x) := (2\pi)^{-n}\mathcal{F}[h](-x)$. We note that $\langle f, g\rangle_{L^2(\mathbf{R}^n)} = (2\pi)^{-n}\langle \mathcal{F}[f], \mathcal{F}[g]\rangle_{L^2(\mathbf{R}^n)}$, $\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$ and $\mathcal{F}[fg] = (2\pi)^{-n}(\mathcal{F}[f] * \mathcal{F}[g])$, where the convolution $f * g$ is defined by $(f * g)(x) = \int_{\mathbf{R}^n} f(x - y)g(y)dy$. Moreover, χ_E denotes the characteristic function of a set $E \subset \mathbf{R}^n$, i.e., $\chi_E(x) = 1$ (if $x \in E$), $= 0$ (if $x \notin E$).

2.1 The Bessel functions

Let $k \in \mathbf{Z}$. Then the *Bessel function* of the first kind of order k is defined by

$$J_k(t) := \frac{1}{2\pi} \int_0^{2\pi} e^{it\sin\theta} e^{-ik\theta} d\theta \quad (t > 0). \tag{2}$$

We also note

$$J_0(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{it\cos\theta}d\theta = \frac{1}{\pi}\int_0^{\pi} e^{it\cos\theta}d\theta.$$

The Bessel function has the following interesting properties (see for instance [13, Appendix B], [28, p.153, p.158], [30, p.18]).

Lemma 4 We have the following.

(i) $J_k(t) = O(t^{-1/2})$ as $t \to \infty \quad (k = 0, 1, 2\cdots)$.
(ii) $J_k(t) = (-1)^k J_{-k}(t) \quad (k \in \mathbf{Z})$.
(iii) $\frac{d}{dt}\{t^{-k}J_k(t)\} = -t^{-k}J_{k+1}(t) \quad (k = 0, 1, 2\cdots)$.
(iv) $\frac{d}{dt}\{t^{k}J_k(t)\} = t^{k}J_{k-1}(t) \quad (k = 1, 2, \cdots)$.

2.2 The Fourier-Wermer algebra

Let $1 \le q < \infty$ and $s \in \mathbf{R}$. Then $\mathcal{F}L^q_s(\mathbf{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{\mathcal{F}L^q_s} := \left(\int_{\mathbf{R}^n} (|\mathcal{F}[f](\xi)|\langle\xi\rangle^s)^q \, d\xi\right)^{1/q}$$

is finite. It is well known that $(\mathcal{F}L^q_s(\mathbf{R}^n), \|\cdot\|_{\mathcal{F}L^q_s})$ is a Banach space and its dual space is $\mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$, i.e., $\ell \in (\mathcal{F}L^q_s(\mathbf{R}^n))'$ if and only if

$$\ell(f) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \mathcal{F}[f](\xi)\overline{\mathcal{F}[\phi](\xi)}d\xi =: \langle f, \phi\rangle \quad (f \in \mathcal{F}L^q_s(\mathbf{R}^n))$$

for some $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ with $\|\ell\|_{op} = (2\pi)^{-n}\|\phi\|_{\mathcal{F}L^{q'}_{-s}}$. We note that if $1 < q < \infty$, $s > n/q'$ and $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$, then $\mathcal{F}[f] \in L^1(\mathbf{R}^n)$ by the Hölder inequality, and thus f is continuous and vanishes at infinity, i.e., $f \in C_0(\mathbb{R}^n)$, by the Riemann-Lebesgue lemma. Thus the inversion formula $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$ holds for all $x \in \mathbf{R}^n$. Moreover since $\mathcal{F}[\overline{f}](\xi) = \overline{\mathcal{F}[f](-\xi)}$, we have $\overline{f} \in \mathcal{F}L^q_s(\mathbf{R}^n)$ whenever $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$. Therefore for $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$, we can define $\overline{\phi} \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ by

$$\langle f, \overline{\phi}\rangle := \overline{\langle \overline{f}, \phi\rangle} \quad (f \in \mathcal{F}L^q_s(\mathbf{R}^n))$$

and obtain

$$\langle f, \overline{\phi}\rangle = (2\pi)^{-n}\int_{\mathbf{R}^n} \mathcal{F}[f](-\xi)\mathcal{F}[\phi](\xi)d\xi = (2\pi)^{-n}(\mathcal{F}[f] * \mathcal{F}[\phi])(0).$$

We also note that if $1 < q < \infty$, then $\mathcal{F}L^q_s(\mathbf{R}^n)$ is a multiplication algebra if and only if $s > n/q'$ (see for example [3], [19, Proposition B.2]). In some literature, $\mathcal{F}L^q_s(\mathbf{R}^n)$ are called the *Fourier-Lebesgue spaces*, but following a suggestion of Feichtinger [10], we call $\mathcal{F}L^q_s(\mathbf{R}^n)$ the *Fourier-Wermer algebra* because they are the Fourier images of convolution algebras $L^q_s(\mathbf{R}^n)$ that were studied in the early paper of Wermer [31] (see also [5]).

In the following, we often use the following lemma.

Lemma 5 Let $1 < q < \infty$ and $s > n/q'$. Then for $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$ and $\Phi \in L^{q'}_{-s}(\mathbf{R}^n)$, we have the following:

(*i*) $\mathcal{F}[f] * \Phi$ is a continuous function on $\mathbf{R}^n$.
(*ii*) $\mathcal{F}[f] * \Phi \in L^{q'}_{-s}(\mathbf{R}^n)$.

Proof (*i*) Let $\xi_0 \in \mathbf{R}^n$ and $\varepsilon > 0$. Since $\mathcal{F}[f] \in L^q_s(\mathbf{R}^n)$ and $C^\infty_c(\mathbf{R}^n)$ is dense in $L^q_s(\mathbf{R}^n)$, we can find $\Psi \in C^\infty_c(\mathbf{R}^n)$ with $\|\mathcal{F}[f] - \Psi\|_{L^q_s} < \varepsilon$ and supp $\Psi \subset B_n(0, R)$ for some $R > 0$. Moreover, since Ψ is uniformly continuous on $\mathbf{R}^n$, there exists $0 < \delta < 1$ such that

$$\sup_{|\eta_1 - \eta_2| < \delta} |\Psi(\eta_1) - \Psi(\eta_2)| < \varepsilon (1 + |\xi_0| + R)^{-(s+n/q)}. \tag{3}$$

Now we show $\mathcal{F}[f] * \Phi$ is continuous at ξ_0. Suppose $|\xi_0 - \xi| < \delta$. Then the Hölder inequality implies

$$\begin{aligned} |(\mathcal{F}[f] * \Phi)(\xi_0) - (\mathcal{F}[f] * \Phi)(\xi)| &= \left| \int_{\mathbf{R}^n} \left(\mathcal{F}[f](\xi_0 - \eta) - \mathcal{F}[f](\xi - \eta) \right) \Phi(\eta) d\eta \right| \\ &\le \|\Phi\|_{L^{q'}_{-s}} \, \|\mathcal{F}[f](\xi_0 - \cdot) - \mathcal{F}[f](\xi - \cdot)\|_{L^q_s}. \end{aligned}$$

On the other hand, we note that

$$\begin{aligned} \|\mathcal{F}[f](\xi_0 - \cdot) - \Psi(\xi_0 - \cdot)\|_{L^q_s} &= \left(\int_{\mathbf{R}^n} \left(|\mathcal{F}[f](\eta) - \Psi(\eta)| \langle \xi_0 - \eta \rangle^s \right)^q d\xi \right)^{1/q} \\ &\le \langle \xi_0 \rangle^s \|\mathcal{F}[f] - \Psi\|_{L^q_s} < \varepsilon \langle \xi_0 \rangle^s \end{aligned}$$

and similarly

$$\|\Psi(\xi - \cdot) - \mathcal{F}[f](\xi - \cdot)\|_{L^q_s} \le \langle \xi \rangle^s \|\mathcal{F}[f] - \Psi\|_{L^q_s} < 2^s \varepsilon \langle \xi_0 \rangle^s.$$

Therefore, we have

$$\begin{aligned} &\|\mathcal{F}[f](\xi_0 - \cdot) - \mathcal{F}[f](\xi - \cdot)\|_{L^q_s} \\ &\le \|\mathcal{F}[f](\xi_0 - \cdot) - \Psi(\xi_0 - \cdot)\|_{L^q_s} + \|\Psi(\xi_0 - \cdot) - \Psi(\xi - \cdot)\|_{L^q_s} + \|\Psi(\xi - \cdot) - \mathcal{F}[f](\xi - \cdot)\|_{L^q_s} \\ &\le (1 + 2^s) \varepsilon \langle \xi_0 \rangle^s + \|\Psi(\xi_0 - \cdot) - \Psi(\xi - \cdot)\|_{L^q_s}. \end{aligned}$$

Moreover, since supp $\Psi(\xi_0 - \cdot) \subset B_n(0, |\xi_0| + R)$ and supp $\Psi(\xi - \cdot) \subset B_n(0, 1 + |\xi_0| + R)$, we have

$$\text{supp } |\Psi(\xi_0 - \cdot) - \Psi(\xi - \cdot)| \subset B_n(0, R_0)$$

with $R_0 := 1 + |\xi_0| + R$. Combining these estimates with (3), we have

$$\begin{aligned} \|\Psi(\xi_0 - \cdot) - \Psi(\xi - \cdot)\|_{L^q_s} &= \left(\int_{B_n(0,R_0)} \left(|\Psi(\xi_0 - \eta) - \Psi(\xi - \eta)| \langle \eta \rangle^s \right)^q d\eta \right)^{1/q} \\ &\le \varepsilon R_0^{-(s+n/q)} \left(\int_{B_n(0,R_0)} \langle \eta \rangle^{sq} d\eta \right)^{1/q} \le 2^s \varepsilon |B_n(0,1)|^{1/q}, \end{aligned}$$

which implies the desired result.
(*ii*) Let $g \in L^q_s(\mathbf{R}^n)$. Then the Hölder and Young inequalities imply

$$\left| \int_{\mathbf{R}^n} (\mathcal{F}[f] * \Phi)(\xi) g(\xi) d\xi \right| \le \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} |\mathcal{F}[f](\xi - \eta)| |g(\xi)| d\xi \right) |\Phi(\eta)| d\eta$$

$$\le \|\Phi\|_{L^{q'}_{-s}} \left\| \int_{\mathbf{R}^n} |\mathcal{F}[f](\xi - \cdot)||g(\xi)|d\xi \right\|_{L^q_s}$$
$$\le \|\Phi\|_{L^{q'}_{-s}} (\|\mathcal{F}[f]\|_{L^q_s}\|g\|_{L^1} + \|\mathcal{F}[f]\|_{L^1}\|g\|_{L^q_s})$$
$$\lesssim \|\Phi\|_{L^q_{-s}}\|f\|_{\mathcal{F}L^q_s}\|g\|_{L^q_s}$$

(cf. [19, Proposition 3.4]). Hence $\mathcal{F}[f] * \Phi \in L^{q'}_{-s}(\mathbf{R}^n)$.

□

Remark 1 We can also prove Lemma 5 as follows: Since $L^q_s(\mathbf{R}^n)$ is a convolution algebra and $\langle \mathcal{F}[f] * \Phi, g\rangle = \langle \Phi, \mathcal{F}[\overline{f}] * g\rangle$ (at least for test functions), we have

$$|\langle \mathcal{F}[f] * \Phi, g\rangle| \le \|\Phi\|_{L^{q'}_{-s}} \|\mathcal{F}[\overline{f}] * g\|_{L^q_s} \lesssim \|\Phi\|_{L^{q'}_{-s}} \|f\|_{\mathcal{F}L^q_s}\|g\|_{L^q_s}.$$

Hence $\mathcal{F}[f]*\Phi \in L^{q'}_{-s}(\mathbf{R}^n)$. The continuous translation property (1) then also gives continuity.

We also consider the differentiability of f in $\mathcal{F}L^q_s(\mathbf{R}^2)$.

Lemma 6 Let $1 < q < \infty$ and $s > 2/q' + 1/2$. Suppose that $f \in \mathcal{F}L^q_s(\mathbf{R}^2)$ be radial, i.e., $f(x) = f^+(|x|)$ for some $f^+(\rho)$ $(\rho \ge 0)$. Then $f^+(\rho)$ is differentiable and satisfies

$$|(f^+)'(\rho)| \le C_\rho \|f\|_{\mathcal{F}L^q_s} \quad (\rho > 0). \tag{4}$$

Proof Let $\rho > 0$. To see this, we use the polar coordinates $\xi = (r\cos\theta, r\sin\theta)$. Since $\mathcal{F}[f]$ is also radial, $\mathcal{F}[f](\xi) = \mathcal{F}[f](|\xi|, 0)$ and thus

$$\begin{aligned}(f^+)(\rho) = f(\rho, 0) &= \frac{1}{(2\pi)^2}\int_{\mathbf{R}^2} \mathcal{F}[f](\xi) e^{i\xi\cdot(\rho,0)} d\xi \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_0^\infty \mathcal{F}[f](r,0)e^{i\rho r\cos\theta} r dr\right) d\theta \\ &= \frac{1}{2\pi}\int_0^\infty \mathcal{F}[f](r,0) J_0(\rho r) r dr,\end{aligned}$$

where J_0 denotes the Bessel function defined in (2). Then by the mean value theorem and Lemma 4 (iii), we have for $h \in \mathbf{R}$ with $|h| < \rho/2$

$$\begin{aligned}\frac{f^+(\rho+h) - f^+(\rho)}{h} &= \frac{1}{2\pi}\int_0^\infty \mathcal{F}[f](r,0)\frac{J_0((\rho+h)r) - J_0(\rho r)}{rh} r^2 dr \\ &= \frac{1}{2\pi}\int_0^\infty \mathcal{F}[f](r,0)(\mathrm{Re}J_0'((\rho+\theta_1 h)r) + i\mathrm{Im}J_0'((\rho+\theta_2 h)r))r^2 dr \\ &= \frac{1}{2\pi}\int_0^\infty \mathcal{F}[f](r,0)(\mathrm{Re}J_1((\rho+\theta_1 h)r) + i\mathrm{Im}J_1((\rho+\theta_2 h)r))r^2 dr\end{aligned}$$

for some $\theta_1, \theta_2 \in (0,1)$. We note $|J_1(t)| \le 1$ $(t \in \mathbf{R})$. Moreover, since $J_1(r) = O(r^{-1/2})$ $(r \to \infty)$ and $(\rho + \theta_i h)r > \rho r/2$ $(i = 1, 2)$, we have

$$\left|\int_1^\infty \mathcal{F}[f](r,0) r^2 r^{-1/2} dr\right| \le C_\rho \int_0^{2\pi}\int_0^\infty |\mathcal{F}[f](r,0)|\langle r\rangle^{1/2} r dr d\theta$$

$$= \int_{\mathbf{R}^2} |\mathcal{F}[f](\xi)| \langle \xi \rangle^{1/2} d\xi \lesssim \|f\|_{\mathcal{F}L^q_s}.$$

Therefore, the Lebesgue dominated convergence theorem implies that $f^+(\rho)$ is differentiable and (4) holds.

□

The following lemma will be used in Section 7.

Lemma 7 Let $1 \le q \le 2$ and $2/q' < s < 2/q' + 1/2$. For $h \in C_c(\mathbf{R}^2) \cap \mathcal{F}L^1_s(\mathbf{R}^2)$ with $h(0) = 1$ and $\mathcal{F}[h](\xi) \ge 0$, we define

$$S_\rho(\mathcal{F}[h])(\xi) := \rho^2 \mathcal{F}[h](\rho\xi) \quad (\xi \in \mathbf{R}^2).$$

Then we have

$$\lim_{\rho \to \infty} \|S_\rho(\mathcal{F}[h]) * \mathcal{F}[f] - \mathcal{F}[f]\|_{L^q_s} = 0 \tag{5}$$

for all $f \in \mathcal{F}L^q_s(\mathbf{R}^2)$. Equivalently, we have

$$\lim_{\rho \to \infty} \left\| h\Big(\frac{\cdot}{\rho}\Big) f(\cdot) - f(\cdot) \right\|_{\mathcal{F}L^q_s} = 0 \tag{6}$$

for all $f \in \mathcal{F}L^q_s(\mathbf{R}^2)$.

Remark 2 Let $g = \frac{1}{2}\chi_{[-1,1]}(t)$ $(t \in \mathbf{R})$ and define a B-spline type function h by

$$h(x) := c\,((g * g * g * g)(x_1) \cdot (g * g * g * g)(x_2))$$

with $x = (x_1, x_2) \in \mathbf{R}^2$ or equivalently $\mathcal{F}[h](\xi) = c(\frac{\sin \xi_1}{\xi_1})^4 (\frac{\sin \xi_2}{\xi_2})^4$ with $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$. Here c is so chosen that $h(0) = \int_{\mathbf{R}^2} \mathcal{F}[h](\xi) d\xi = 1$. Since $\mathcal{F}[\frac{1}{2}\chi_{[-1,1]}](t) = \frac{\sin t}{t}$ and $s < 2$, we can easily see $h \in C_c(\mathbf{R}^2) \cap \mathcal{F}L^1_s(\mathbf{R}^2)$.

The proof of Lemma 7 We note $h(0) = \int_{\mathbf{R}^2} \mathcal{F}[h](\eta) d\eta = 1$ and

$$(S_\rho(\mathcal{F}[h]) * \mathcal{F}[f])(\xi) = \int_{\mathbf{R}^2} \mathcal{F}[h](\eta) \mathcal{F}[f]\left(\xi - \frac{\eta}{\rho}\right) d\eta.$$

By the Minkowski inequality for integral we have

$$\|S_\rho(\mathcal{F}[h]) * \mathcal{F}[f] - \mathcal{F}[f]\|_{L^q_s} \le \int_{\mathbf{R}^2} |\mathcal{F}[h](\eta)| \left\| \mathcal{F}[f]\left(\cdot - \frac{\eta}{\rho}\right) - \mathcal{F}[f](\cdot) \right\|_{L^q_s} d\eta.$$

On the other hand, for all $\eta \in \mathbf{R}^n$, we have

$$\lim_{\rho \to \infty} \left\| \mathcal{F}[f]\Big(\cdot - \frac{\eta}{\rho}\Big) - \mathcal{F}[f](\cdot) \right\|_{L^q_s} = 0$$

and

$$\left\| \mathcal{F}[f]\Big(\cdot - \frac{\eta}{\rho}\Big) - \mathcal{F}[f](\cdot) \right\|_{L^q_s} \le \left\langle \frac{\eta}{\rho} \right\rangle^s \|f\|_{\mathcal{F}L^q_s} + \|f\|_{\mathcal{F}L^q_s} \le C_f \langle \eta \rangle^s$$

for $\rho \ge 1$. Since $h \in \mathcal{F}L^1_s(\mathbf{R}^2)$, the Lebesgue convergence theorem shows (5), and thus (6).

□

2.3 Closed Ideals in $\mathcal{F}L^q_s(\mathbf{R}^n)$

Throughout Section 2.3, we assume $1 < q < \infty$ and $s > n/q'$. Let I be a linear subspace of $\mathcal{F}L^q_s(\mathbf{R}^n)$. Then I is called an *ideal* in $\mathcal{F}L^q_s(\mathbf{R}^n)$ if $fg \in I$ whenever $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$ and $g \in I$. Moreover, if an ideal I in $\mathcal{F}L^q_s(\mathbf{R}^n)$ is a closed subset of $\mathcal{F}L^q_s(\mathbf{R}^n)$, then I is called a *closed ideal* in $\mathcal{F}L^q_s(\mathbf{R}^n)$. For a subset S of $\mathcal{F}L^q_s(\mathbf{R}^n)$, let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of all ideals containing S. Then the set $\bigcap_{\lambda\in\Lambda} I_\lambda$ is called the *ideal generated by* S. We note that the closed ideal generated by S coincides with the closure of

$$\left\{ \sum_{j=1}^{n} f_j g_j \;\middle|\; f_j \in \mathcal{F}L^q_s(\mathbf{R}^n), g_j \in S, n \in \mathbf{N} \right\}.$$

Definition 1 Let I be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. Then the *zero-set* of I is defined by $Z(I) := \bigcap_{f\in I} f^{-1}(\{0\})$ with $f^{-1}(\{0\}) := \{x \in \mathbf{R}^2 \mid f(x) = 0\}$.

We note that $x \in Z(I)$ if and only if $f(x) = 0$ for all $f \in I$. Moreover, $Z(I)$ is a closed subset of $\mathbf{R}^n$ whenever I is a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. In fact, if $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$, then f is continuous on $\mathbf{R}^n$ and thus $f^{-1}(\{0\})$ is a closed subset of $\mathbf{R}^n$.

The following lemma seems to be known to many people, but for the reader's convenience, we give the proof.

Lemma 8 Let E be a closed subset of $\mathbf{R}^n$ and set

$$I(E) := \{f \in \mathcal{F}L^q_s(\mathbf{R}^n) \mid f|_E = 0\}.$$

Then $I(E)$ is a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$ with $E = Z(I(E))$.

Proof It is clear that $I(E)$ is an ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. To see $I(E)$ is closed, let $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$, $\{f_n\}_{n=1}^\infty \subset I(E)$ and $\|f_n - f\|_{\mathcal{F}L^q_s} \to 0$ $(n \to \infty)$. Since

$$\|f_n - f\|_{L^\infty} \lesssim \|\mathcal{F}[f_n] - \mathcal{F}[f]\|_{L^1} \lesssim \|f_n - f\|_{\mathcal{F}L^q_s},$$

we see that $\{f_n\}_{n=1}^\infty$ converges pointwise to f on $\mathbf{R}^n$. Since $f_n|_E = 0$, we have $f|_E = 0$, and thus $f \in I(E)$. Hence $I(E)$ is closed. Next, we prove $E = Z(I(E))$. Since $E \subset Z(I(E))$ is clear, we show $Z(I(E)) \subset E$. Suppose $x_0 \notin E$. Since E is closed and $C^\infty_c(\mathbf{R}^n) \subset \mathcal{F}L^q_s(\mathbf{R}^n)$, there exists $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$ such that $f(x_0) = 1$ and $f|_E = 0$. Then $f \in I(E)$ and $f(x_0) \neq 0$. Thus $x_0 \notin Z(I(E))$, which implies the desired result.

□

Definition 2 Let E be a closed set $\mathbf{R}^n$. Then E is called a *set of spectral synthesis* for $\mathcal{F}L^q_s(\mathbf{R}^n)$ if we have $I_1 = I_2$ when $Z(I_1) = Z(I_2) = E$ for two closed ideals I_1 and I_2 (i.e., a closed ideal I with $Z(I) = E$ is a unique closed ideal).

We also prepare the following lemma, which also characterizes the closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$ (cf. [24, Theorem 7.1.2]). In the following, we call a closed ideal I in $\mathcal{F}L^q_s(\mathbf{R}^n)$ *modulation invariant* if $[M_\eta f](x) := e^{ix\eta} f(x) \in I$ whenever $f \in I$ and $\eta \in \mathbf{R}^n$.

Lemma 9 Let I be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$ and $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$. Then

$$\langle M_{-\eta} f, \phi \rangle = 0 \tag{7}$$

for all $f \in I$ and $\eta \in \mathbf{R}^n$ if and only if

$$\langle f, \phi \rangle = 0 \tag{8}$$

for all $f \in I$. Moreover, every closed modulation invariant subspace of $\mathcal{F}L^q_s(\mathbf{R}^n)$ is an ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. Conversely, every closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$ is modulation invariant.

Proof We give the proof only for (8) implies (7); the other case is clear. Assume that (8) holds for all $f \in I$. By Lemma 5, $\mathcal{F}[f] * \mathcal{F}[\overline{\phi}]$ is continuous and $f\overline{\phi} \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$. Moreover, since

$$\langle M_{-\eta} f, \phi \rangle = (2\pi)^{-n} (\mathcal{F}[f] * \mathcal{F}[\overline{\phi}])(\eta) = \mathcal{F}[f\overline{\phi}](\eta),$$

we need to prove $f\overline{\phi} = 0$. By duality, it suffices to prove $\langle g, f\overline{\phi} \rangle = 0$ for all $f \in I$ and $g \in \mathcal{F}L^q_s(\mathbf{R}^n)$. Let $f \in I$ and $g \in \mathcal{F}L^q_s(\mathbf{R}^n)$. As in the proof of Lemma 5 (ii), we have

$$\iint_{\mathbf{R}^n} |\mathcal{F}[f](\xi - \eta)||\mathcal{F}[\overline{\phi}](\eta)||\overline{\mathcal{F}[g](\xi)}| d\xi d\eta \lesssim \|f\|_{\mathcal{F}L^q_s} \|g\|_{\mathcal{F}L^q_s} \|\overline{\phi}\|_{\mathcal{F}L^{q'}_{-s}}.$$

Thus it follows from the Fubini Theorem that

$$\begin{aligned}
\overline{\langle g, f\overline{\phi} \rangle} &= (2\pi)^{-2n} \int_{\mathbf{R}^n} (\mathcal{F}[f] * \mathcal{F}[\overline{\phi}])(\xi) \overline{\mathcal{F}[g](\xi)} d\xi \\
&= (2\pi)^{-2n} \int_{\mathbf{R}^n} (\mathcal{F}[f] * \mathcal{F}[\overline{g}])(\xi) \overline{\mathcal{F}[\phi](\xi)} d\xi = \langle f\overline{g}, \phi \rangle.
\end{aligned}$$

Since $\overline{g} \in \mathcal{F}L^q_s(\mathbf{R}^n)$ and I is an ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$, we have $f\overline{g} \in I$ and thus $\langle f\overline{g}, \phi \rangle = 0$. Hence we have $\langle g, f\overline{\phi} \rangle = 0$.

Finally, we prove the characterization of a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. Let I be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$. Suppose that $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ and (8) holds for all $f \in I$. Then by (7)

$$\langle M_{-\eta} f, \phi \rangle = 0 \tag{9}$$

for all $f \in I$ and $\eta \in \mathbf{R}^n$. If I is not modulation invariant, then there exist $f_0 \in I$ and $\eta_0 \in \mathbf{R}^n$ such that $M_{-\eta_0} f_0 \notin I$. Then the Hahn-Banach theorem and $(\mathcal{F}L^q_s(\mathbf{R}^n))' = \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ imply that there exists $\phi_0 \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ such that $\langle f, \phi_0 \rangle = 0$ for all $f \in I$ and $\langle M_{-\eta_0} f_0, \phi_0 \rangle = 1$. This contradicts (9). Conversely, we assume that I is a closed modulation invariant subspace of $\mathcal{F}L^q_s(\mathbf{R}^n)$. If I is not an ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$, then there exist $f_0 \in I$ and $g_0 \in \mathcal{F}L^q_s(\mathbf{R}^n)$ such that $f_0\overline{g_0} \notin I$. Then the Hahn-Banach theorem implies that there exists $\phi_0 \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^n)$ such that

$$\langle f, \phi_0 \rangle = 0 \tag{10}$$

for all $f \in I$, and

$$\langle f_0\overline{g_0}, \phi_0 \rangle = 1. \tag{11}$$

On the other hand, since I is modulation invariant, we have $M_{-\xi} f_0 \in I$ for all $\xi \in \mathbf{R}^n$. Thus we have by (10)

$$\mathcal{F}[f_0\overline{\phi_0}](\xi) = (2\pi)^{-n} (\mathcal{F}[f_0] * \mathcal{F}[\overline{\phi_0}])(\xi) = \langle M_{-\xi} f_0, \phi_0 \rangle = 0. \tag{12}$$

Hence $f_0\overline{\phi_0} = 0$ and thus $\langle f_0\overline{g_0}, \phi_0 \rangle = \overline{\langle g_0, f_0\overline{\phi_0} \rangle}) = 0$. However, this contradicts (11).

□

3 The proof of Theorem 1

Throughout Section 3, we simply write $I(S^1)$ in Lemma 8 as I. Let I_0 be the set of all functions g on $\mathbf{R}^2$ with $g = f^2$, where $f \in \mathcal{F}L^q_s(\mathbf{R}^2)$, $f|_{S^1} = 0$ and f is the inverse Fourier transform for some radial function on $\mathbf{R}^2$, and define I_1 as the closed ideal generated by I_0 in $\mathcal{F}L^q_s(\mathbf{R}^2)$. We note $Z(I) = S^1$ by Lemma 8. Moreover, by Lemma 9, I_1 is the smallest closed modulation invariant subspace of $\mathcal{F}L^q_s(\mathbf{R}^2)$ with $I_0 \subset I_1$. Thus I_1 coincides with the closure of the set

$$H := \left\{ \sum_{j=1}^{m} e^{ix\eta_j} g_j(x) \;\middle|\; \eta_j \in \mathbf{R}^2,\ g_j \in I_0,\ m \in \mathbf{N} \right\}. \tag{13}$$

In the following, we show $Z(I_1) = S^1$ and $I \neq I_1$, which implies S^1 is not a set of spectral synthesis in $\mathcal{F}L^q_s(\mathbf{R}^2)$. To prove $I \neq I_1$, we construct $\ell_0 \in (\mathcal{F}L^q_s(\mathbf{R}^2))'$ and $f_0 \in I \setminus I_1$ such that $\ell_0 = 0$ on I_1 and $\ell_0(f_0) \neq 0$.

3.1 Technical Lemma

In the following lemma, μ denotes the Radon measure in $\mathbf{R}^2$ corresponding to a mass distributed homogeneously over the unit circle S^1, i.e.,

$$\int_{\mathbf{R}^2} f(x) d\mu(x) := \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\infty} f(r\cos\theta, r\sin\theta) r d\delta_1(r) \right) d\theta,$$

where δ_1 is defined by $\int_0^{\infty} h(r) d\delta_1(r) := h(1)$ for $h \in C([0,\infty))$ (see for instance [21], [25]).

Lemma 10 Let $1 < q < \infty$ and $s > 2/q' + 1/2$. Define the function k_0 defined by

$$k_0(\xi) := \int_{\mathbf{R}^2} e^{ix\xi} d\mu(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i|\xi|\cos\theta} d\theta \quad (\xi \in \mathbf{R}^2)$$

and set $K_0(\xi) := i\xi_2 k_0(\xi)$ with $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$. Then

$$\ell_0(f) := \int_{\mathbf{R}^2} \mathcal{F}[f](\xi) K_0(\xi) d\xi \qquad (f \in \mathcal{F}L^q_s(\mathbf{R}^2)) \tag{14}$$

defines a bounded linear functional on $\mathcal{F}L^q_s(\mathbf{R}^2)$.

Proof We first note that $|k_0(\xi)| \leq 1$ $(\xi \in \mathbf{R}^2)$ and $k_0(\xi) = O(|\xi|^{-1/2})$ as $|\xi| \to \infty$ (see Lemma 4 (i)). Moreover, since $|\xi_2| \leq |\xi| \leq |\xi|^{1/2}$ whenever $|\xi| \leq 1$, the Hölder inequality show that

$$\begin{aligned}
|\ell_0(f)| &\leq \int_{\mathbf{R}^2} |\mathcal{F}[f](\xi)||\xi_2||k_0(\xi)| d\xi \\
&\lesssim \int_{|\xi|\leq 1} |\mathcal{F}[f](\xi)| \langle\xi\rangle^{1/2} d\xi + \int_{|\xi|\geq 1} |\mathcal{F}[f](\xi)| \langle\xi\rangle \langle\xi\rangle^{-1/2} d\xi \\
&\leq 2 \| \langle\cdot\rangle^{-s+1/2} \|_{L^{q'}} \|f\|_{\mathcal{F}L^q_s}.
\end{aligned}$$

We note that $s > 1/2 + 2/q'$ implies $(-s + 1/2)q' < -2$. Thus ℓ_0 defines a bounded linear functional on $\mathcal{F}L^q_s(\mathbf{R}^2)$.

□

Remark 3 Let $1 < q < \infty$, $s = 1/2 + 2/q'$ and define f_0 by

$$\mathcal{F}[f_0](\xi) = \chi_{|\xi|\le 2}(\xi)|\xi|^{-5/2}(\log|\xi|)^{-1}\mathrm{sgn}(K_0(\xi)).$$

Then $f_0 \in \mathcal{F}L^q_s(\mathbf{R}^2)$ and $|\ell_0(f_0)| = \infty$. Thus ℓ_0 is not bounded on $\mathcal{F}L^q_s(\mathbf{R}^2)$.

3.2 The proof of Theorem 1

We divide the proof into three parts.

Step 1. We first prove $Z(I_1) = S^1$. Let $x_0 \in S^1$ and $f \in I_1$. Since I_1 is the closure of H defined in (13), there exists $\{f_n\}_{n=1}^\infty \subset H$ such that $\|f - f_n\|_{\mathcal{F}L^q_s} \to 0$ $(n \to \infty)$. We note that $f_n(x_0) = 0$, and thus

$$|f(x_0)| = |f(x_0) - f_n(x_0)| \lesssim \|\mathcal{F}[f] - \mathcal{F}[f_n]\|_{L^1} \lesssim \|f - f_n\|_{\mathcal{F}L^q_s},$$

which implies $f(x_0) = 0$. Therefore $S^1 \subset Z(I_1)$. On the other hand, if $x_0 \notin S^1$, then there exists a radial function $f \in C_c^\infty(\mathbf{R}^2)$ such that $f|_{S^1} = 0$ and $f(x_0) \neq 0$. Thus $g := f^2 \in I_0$ and $x_0 \notin g^{-1}(\{0\})$, which implies $x_0 \notin Z(I_1)$.

Step 2. To prove $I \neq I_1$, we first show $\ell_0 = 0$ on I_1, where ℓ_0 denotes the bounded linear functional on $\mathcal{F}L^q_s(\mathbf{R}^2)$ defined in Lemma 10. We note that since I_1 is the closure of H, it suffices to show $\ell_0(M_\eta g) = 0$ for all $g \in I_0$ and $\eta \in \mathbf{R}$. Let $g \in I_0$ and $\eta \in \mathbf{R}$. Then

$$\begin{aligned}
\ell_0(M_\eta g) &= \int_{\mathbf{R}^2} \mathcal{F}_{x\to\xi}[e^{ix\eta}g(x)](\xi)K_0(\xi)d\xi \\
&= \int_{\mathbf{R}^2} \mathcal{F}[g](\xi-\eta)\left(i\xi_2\int_{\mathbf{R}^2} e^{iy\xi}d\mu(y)\right)d\xi \\
&= \int_{\mathbf{R}^2}\left(\int_{\mathbf{R}^2}\mathcal{F}[g](\xi-\eta)\frac{\partial}{\partial y_2}[e^{iy\xi}]d\xi\right)d\mu(y) \\
&= \int_{\mathbf{R}^2}\left(\frac{\partial}{\partial y_2}\int_{\mathbf{R}^2}\mathcal{F}[g](\xi-\eta)e^{iy\xi}d\xi\right)d\mu(y) \\
&= (2\pi)^2\int_{\mathbf{R}^2}\left(\frac{\partial}{\partial y_2}\left[e^{iy\eta}g(y)\right]\right)d\mu(y).
\end{aligned}$$

On the other hand, since $g = f^2$ for some radial $f \in \mathcal{F}L^q_s(\mathbf{R}^2)$ with $f|_{S^1} = 0$, we have

$$\frac{\partial}{\partial y_2}\left[e^{iy\eta}g(y)\right] = \left(\frac{\partial}{\partial y_2}[e^{iy\eta}]\right)g(y) + e^{iy\eta}\left(\frac{\partial g}{\partial y_2}\right)(y)$$

$$= \left(\frac{\partial}{\partial y_2}[e^{iy\eta}]\right)(f(y))^2 + 2e^{iy\eta}f(y)\frac{\partial f}{\partial y_2}(y) = 0$$

for all $y \in S^1$. Therefore, we obtain $\ell_0(M_\eta g) = 0$.

Step 3. Finally, we show $I \neq I_1$. The proof is completed by showing that there exists $f_0 \in I$ such that $\ell_0(f_0) \neq 0$. Let τ be a nonnegative function in $C_c^\infty(\mathbf{R}^2)$ such that $\tau(x) = 1$ on $B_2(x^{(0)}, 10^{-5})$ and supp $\tau \subset B_2(x^{(0)}, 10^{-4})$ with $x^{(0)} := (1/2, \sqrt{3}/2)$, and define $f_0 \in C_c^\infty(\mathbf{R}^2)$ by

$$f_0(x) = \tau(x)\Big(x_2 - \sqrt{1 - x_1^2}\Big), \quad x = (x_1, x_2) \in \mathbf{R}^2.$$

We note that $f_0|_{S^1} = 0$ and $\frac{\partial f_0}{\partial x_2}(x) = \frac{\partial \tau}{\partial x_2}(x)(x_2 - \sqrt{1 - x_1^2}) + \tau(x)$, and thus $\frac{\partial f_0}{\partial x_2} \geq 0$ on S^1. Therefore in the same way as Step 2, we obtain

$$\begin{aligned}
\ell_0(f_0) &= \int_{\mathbf{R}^2} \mathcal{F}[f_0](\xi)K_0(\xi)d\xi \\
&= \int_{\mathbf{R}^2} \left(\frac{\partial}{\partial x_2}\int_{\mathbf{R}^2} \mathcal{F}[f_0](\xi)e^{ix\xi}d\xi\right) d\mu(x) \\
&= (2\pi)^2 \int_{\mathbf{R}^2} \frac{\partial f_0}{\partial x_2}(x)d\mu(x) \\
&= (2\pi)^2 \int_{\mathbf{R}^2} \left(\frac{\partial \tau}{\partial x_2}(x)\left(x_2 - \sqrt{1 - x_1^2}\right) + \tau(x)\right) d\mu(x) \\
&= (2\pi)^2 \int_{\mathbf{R}^2} \tau(x)d\mu(x) = 2\pi \int_0^{2\pi} \tau(\cos\theta, \sin\theta)d\theta > 0,
\end{aligned}$$

which implies the desired result.

4 The proof of Theorem 2

Throughout Section 4, we assume $1 < q < 2$ and $2/q' < s < 2/q' + 1/2$. Moreover, we simply write I instead of $I(S^1)$, and J denotes the closed ideal in $\mathcal{F}L_s^q(\mathbf{R}^2)$ generated by

$$J^0 := \{f \in \mathcal{F}L_s^q(\mathbf{R}^2) \mid f = 0 \text{ in a neighborhood of } S^1\}. \tag{15}$$

We note that I is the biggest closed ideal I' in $\mathcal{F}L_s^q(\mathbf{R}^2)$ with $Z(I') = S^1$, and J is the smallest closed ideal I'' in $\mathcal{F}L_s^q(\mathbf{R}^2)$ with $Z(I'') = S^1$ (see Theorem 17 below). In the following, we show $I = J$, which implies S^1 is a set of spectral synthesis in $\mathcal{F}L_s^q(\mathbf{R}^2)$. Since $J \subset I$ is clear, we need to prove $I \subset J$. The proof is completed by showing the following:

Proposition 11 *Let I' be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$ and $f_0 \in \mathcal{F}L^q_s(\mathbf{R}^2)$. If every $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ such that $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $\langle f, \phi\rangle = 0$ for all $f \in I'$ satisfies $\langle f_0, \phi\rangle = 0$, then f_0 is in I'.*

We use Proposition 11 with $I' = J$ and $f_0 \in I$. Moreover, to prove "every $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ such that $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $\langle f, \phi\rangle = 0$ for all $f \in J$ satisfies $\langle f_0, \phi\rangle = 0$", we use an idea derived from Herz [17]. In the following, $\mathcal{F}M(\mathbf{R}^2)$ denotes the set of all Fourier-Stieltjes transform of the bounded regular Borel measures $M(\mathbf{R}^2)$, i.e.,

$$\mathcal{F}M(\mathbf{R}^2) = \left\{ \mathcal{F}[\mu] \;\middle|\; \mathcal{F}[\mu](x) = \int_{\mathbf{R}^2} e^{-ix\xi} d\mu(\xi),\ \mu \in M(\mathbf{R}^2) \right\}.$$

Proposition 12 *Let $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$, $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $\langle f, \phi\rangle = 0$ for all $f \in J$. Then there exists $\{\sigma_n[\phi]\}_n \subset \mathcal{F}M(\mathbf{R}^2)$ such that*

(i) $\sigma_n[\phi] = \mathcal{F}[\mu_n]$ *for some* $\mu_n \in M(\mathbf{R}^2)$ *with* $\operatorname{supp} \mu_n \subset S^1$.
(ii) $\|\sigma_n[\phi]\|_{L^{q'}_{-s}} \le C\|\phi\|_{\mathcal{F}L^{q'}_{-s}}$.
(iii) $\{\sigma_n[\phi]\}_n$ *converges to* $\mathcal{F}[\phi]$ *uniformly on any compact set of* $\mathbf{R}^2$.

In fact, if Propositions 11 and 12 are true, then we can prove Theorem 2 as follows: Let $f_0 \in I$ and $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ be such that $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $\langle f, \phi\rangle = 0$ for all $f \in J$. Then there exists $\{\sigma_n[\phi]\}_n \subset \mathcal{F}M(\mathbf{R}^2)$ satisfying (i)-(iii) in Proposition 12. Since $\mathcal{F}[f_0]\cdot\mathcal{F}[\phi] \in L^1(\mathbf{R}^2)$ and $\mathcal{F}[f_0] \in L^q_s(\mathbf{R}^2)$, for any $\varepsilon > 0$, there exists a compact set K such that

$$\int_{\mathbf{R}^2\setminus K} |\mathcal{F}[f_0](\xi)\mathcal{F}[\phi](\xi)| d\xi < \frac{\varepsilon}{4}$$

and

$$\left(\int_{\mathbf{R}^2\setminus K} (|\mathcal{F}[f_0](\xi)|\langle\xi\rangle^s)^q d\xi \right)^{1/q} < \frac{\varepsilon}{4C\|\phi\|_{\mathcal{F}L^{q'}_{-s}}},$$

where C denotes the constant in Proposition 12 (ii). Then we have

$$\int_{\mathbf{R}^2\setminus K} |\mathcal{F}[f_0](\xi)\sigma_n[\phi](\xi)| d\xi \le \|\mathcal{F}[f_0]\|_{L^q_s(\mathbf{R}^2\setminus K)} \|\sigma_n[\phi]\|_{L^{q'}_{-s}(\mathbf{R}^2\setminus K)} < \frac{\varepsilon}{4}.$$

Therefore,

$$\begin{aligned} |\langle f_0, \phi\rangle - \langle f_0, \mu_n\rangle| &\lesssim \left| \int_K \mathcal{F}[f_0](\xi) \left(\overline{\mathcal{F}[\phi](\xi)} - \overline{\sigma_n[\phi](\xi)} \right) d\xi \right| \\ &\quad + \int_{\mathbf{R}^2\setminus K} |\mathcal{F}[f_0](\xi)| \left(|\mathcal{F}[\phi](\xi)| + |\sigma_n[\phi](\xi)| \right) d\xi \\ &\le \sup_{\xi\in K} |\mathcal{F}[\phi](\xi) - \sigma_n[\phi](\xi)| \int_K |\mathcal{F}[f_0](\xi)| d\xi + \frac{\varepsilon}{2}. \end{aligned}$$

On the other hand, since $f_0|_{S^1} = 0$ and supp $\mu_n \subset S^1$, we have

$$\langle f_0, \mu_n \rangle = (2\pi)^{-2} \int_{\mathbf{R}^2} \mathcal{F}[f_0](\xi) \overline{\left(\int_{\mathbf{R}^2} e^{-ix\xi} d\mu_n(x) \right)} d\xi$$
$$= (2\pi)^{-2} \overline{\int_{\mathbf{R}^2} \overline{\left(\int_{\mathbf{R}^2} \mathcal{F}[f_0](\xi) e^{ix\xi} d\xi \right)} d\mu_n(x)} = \overline{\int_{\mathbf{R}^2} \overline{f_0(x)} d\mu_n(x)} = 0.$$

For this reason $\langle f_0, \phi \rangle = 0$, and we conclude $f_0 \in J$ by Proposition 11.

4.1 The proof of Proposition 11

Let $\varphi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ be such that $\langle f, \varphi \rangle = 0$ for all $f \in I'$. To show $f_0 \in I'$, it suffices to prove $\langle f_0, \varphi \rangle = 0$. In fact, if $f_0 \notin I'$, then the Hahn-Banach theorem yields a contradiction. Moreover, since $\langle f_0, \varphi \rangle = \mathcal{F}[f_0 \overline{\varphi}](0)$, it suffices to prove $f_0 \overline{\varphi} = 0$. We prove this by the duality argument. Let $g \in \mathcal{F}L^q_s(\mathbf{R}^2)$. Since $\overline{g} \in \mathcal{F}L^q_s(\mathbf{R}^2)$ and I' is an ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$, we see $f\overline{g} \in I'$ whenever $f \in I'$. Thus $\langle f, g\varphi \rangle = \langle f\overline{g}, \varphi \rangle = 0$ for all $f \in I'$. On the other hand, by Lemma 5 we have $g\varphi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ and $\mathcal{F}[g\varphi] \in C(\mathbf{R}^2)$. Thus by our assumption, $\langle f_0, g\varphi \rangle = 0$. Therefore $\overline{\langle g, f_0 \overline{\varphi} \rangle} = \langle f_0, g\varphi \rangle = 0$, and thus $f_0 \overline{\varphi} = 0$.

4.2 Technical lemmas

Throughout Section 4.2, we assume that $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ satisfies $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$ and $\langle f, \phi \rangle = 0$ for all $f \in J$. In the following, we show that ϕ satisfies $\mathcal{F}[\phi] + \Delta[\mathcal{F}[\phi]] = 0$, where Δ stands for the Laplace operator. For this, we introduce auxiliary functions G_0 and Φ_h in Lemmas 13 and 14.

Lemma 13 Define $F_0(\xi) := \mathcal{F}^{-1}[\chi_{B_2(0,1/2)}](\xi)$ and $G_0(\xi) := (F_0(\xi))^2$ for $\xi \in \mathbf{R}^2$. Then we have

(i) $G_0(\xi) = O(|\xi|^{-3})$ $(|\xi| \to \infty)$.
(ii) $\mathcal{F}[G_0](x) = 0$ $(|x| \geq 1)$.
(iii) $G_0 \in L^q_s(\mathbf{R}^2)$.

Proof (i) We use the polar coordinates $\xi = (R\cos\eta, R\sin\eta)$ and $x = (r\cos\theta, r\sin\theta)$. Since $x\xi = Rr\cos(\theta - \eta)$, we have

$$F_0(\xi) = \frac{1}{(2\pi)^2} \int_0^{1/2} r \left(\int_0^{2\pi} e^{iRr\cos(\theta-\eta)} d\theta \right) dr = \frac{1}{2\pi} \int_0^{1/2} r J_0(Rr) dr$$

by the periodicity of the cosine function and the definition of the Bessel function J_0. Moreover, it follows from Lemma 4 (i) and (iv) that

$$F_0(\xi) = \frac{1}{2\pi R^2} \int_0^{R/2} r J_0(r) dr = \frac{1}{2\pi R^2} \left[r J_1(r) \right]_{r=0}^{r=R/2} = O(R^{-3/2})$$

as $R = |\xi| \to \infty$. Hence $G_0(\xi) = (F_0(\xi))^2 = O(|\xi|^{-3})$ as $|\xi| \to \infty$.

(ii) Since $G_0 \in C(\mathbf{R}^2)$, (i) implies $G_0 \in L^1(\mathbf{R}^2)$. Therefore

$$\mathcal{F}[G_0](x) = (2\pi)^{-2} \left(\chi_{B_2(0,1/2)} * \chi_{B_2(0,1/2)} \right)(x).$$

Hence we obtain $\operatorname{supp} \mathcal{F}[G_0] \subset B_2(0,1/2) + B_2(0,1/2) = B_2(0,1)$.
(iii) By (i) and the continuity of G_0, we see that $G_0 \in L^q_s(\mathbf{R}^2)$ if and only if $\int_{|\xi|\geq 1} (|G_0(\xi)|\langle\xi\rangle^s)^q d\xi < \infty$, which is equivalent to $(3-s)q > 2$. Hence, we get $G_0 \in L^q_s(\mathbf{R}^2)$ by our assumption on s.

□

Lemma 14 Let $h > 0$ and G_0 be the function defined in Lemma 13. Define

$$\Phi_h(x) := \int_{\mathbf{R}^2} e^{-ix\xi} G_0(h\xi) \overline{\mathcal{F}[\phi](\xi)} d\xi \quad (x \in \mathbf{R}^2).$$

Then $\operatorname{supp} \Phi_h \subset K_h := \{x \in \mathbf{R}^2 \mid d(x, S^1) \leq 2h\}$.

Proof We first note

$$\mathcal{F}^{-1}_{\xi \to u}[e^{-ix\xi} G_0(h\xi)](u) = \frac{1}{(2\pi h)^2} \mathcal{F}[G_0]\left(\frac{x-u}{h}\right)$$

and thus

$$\Phi_h(x) = \frac{1}{(2\pi h)^2} \int_{\mathbf{R}^2} \mathcal{F}\left[\mathcal{F}[G_0]\left(\frac{x-\cdot}{h}\right)\right](\xi) \overline{\mathcal{F}[\phi](\xi)} d\xi.$$

By our assumption on ϕ, it suffices to show $\mathcal{F}[G_0]\left(\frac{x-\cdot}{h}\right) \in J^0 \subset J$ if $d(x, S^1) > 2h$. Suppose $d(x, S^1) > 2h$ and let $y \in \mathbf{R}^2$ be such that $d(y, S^1) < h$. Since $|x-y| \geq d(x, S^1) - d(y, S^1) > h$, we have $\mathcal{F}[G_0]\left(\frac{x-y}{h}\right) = 0$ by Lemma 13 (ii). Hence, $\mathcal{F}[G_0]\left(\frac{x-\cdot}{h}\right) \in J^0$.

□

Lemma 15 ϕ satisfies $\mathcal{F}[\phi] + \Delta[\mathcal{F}[\phi]] = 0$.

Proof We will divide the proof into four parts.

Step 1. Let $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$ and Φ_h be the function in Lemma 14. Firstly, we show

$$|\langle M_{-\eta} f, \phi \rangle| \lesssim \liminf_{h \downarrow 0} \left(\int_{K_h} |f(x)|^2 dx \right)^{1/2} \left(\int_{K_h} |\Phi_h(x)|^2 dx \right)^{1/2}.$$

To see this, we first note that $\Phi_h \in L^1(\mathbf{R}^2)$ and $G_0(h\xi)\overline{\mathcal{F}[\phi](\xi)} \in L^1(\mathbf{R}^2_\xi)$, since

$$|\Phi_h(x)| \leq \int_{\mathbf{R}^2} |G_0(h\xi)||\mathcal{F}[\phi](\xi)| d\xi \leq C_h \|G_0\|_{L^q_s} \|\phi\|_{\mathcal{F}L^{q'}_{-s}}$$

and $\operatorname{supp} \Phi_h \subset K_h$. Then the Fourier inversion formula yields

$$G_0(h\xi)\overline{\mathcal{F}[\phi](\xi)} = (2\pi)^{-2} \mathcal{F}^{-1}[\Phi_h](\xi) = (2\pi)^{-2} \mathcal{F}[\Phi_h](-\xi)$$

for all $\xi \in \mathbf{R}^2$. From the above equality with $G_0(0) = 1$, we obtain

$$(2\pi)^2 \langle M_{-\eta} f, \phi \rangle = \langle \mathcal{F}[f], \mathcal{F}[M_\eta \phi] \rangle_{L^2(\mathbf{R}^2)}$$

$$
\begin{aligned}
&= \langle \mathcal{F}[f](\xi), \overline{G_0(0)\overline{\mathcal{F}[\phi](\xi-\eta)}} \rangle_{L^2(\mathbf{R}^2)} \\
&= \liminf_{h\downarrow 0} \langle \mathcal{F}[f](\xi), \overline{G_0(h(\xi-\eta))\overline{\mathcal{F}[\phi](\xi-\eta)}} \rangle_{L^2(\mathbf{R}^2)} \\
&= (2\pi)^{-2} \liminf_{h\downarrow 0} \langle \mathcal{F}[f](\xi), \overline{\mathcal{F}[\Phi_h](\eta-\xi)} \rangle_{L^2(\mathbf{R}^2)} \\
&= (2\pi)^{-2} \liminf_{h\downarrow 0} \langle \mathcal{F}[f](\xi), \mathcal{F}[\overline{M_\eta \Phi_h}] \rangle_{L^2(\mathbf{R}^2)} \\
&= \liminf_{h\downarrow 0} \langle f, \overline{M_\eta \Phi_h} \rangle.
\end{aligned}
$$

Hence, the Cauchy-Schwarz inequality and the Plancherel theorem yield the desired result.

Step 2. Secondly, we prove

$$
\left(\int_{K_h} |\Phi_h(x)|^2 dx \right)^{1/2} \le C h^{-1-s+2/q'} \|\phi\|_{\mathcal{F}L^{q'}_{-s}} \tag{16}
$$

for $0 < h < 1/2$. Since $\Phi_h(x) = \mathcal{F}_{\xi\to x}[G_0(h\xi)\overline{\mathcal{F}[\phi](\xi)}](x)$ and $\operatorname{supp} \Phi_h \subset K_h$, we obtain by the Plancherel theorem and the Hölder inequality with $\frac{1}{\frac{q'}{q'-2}} + \frac{1}{\frac{q'}{2}} = 1$,

$$
\begin{aligned}
\int_{K_h} |\Phi_h(x)|^2 dx &\approx \int_{\mathbf{R}^2} \left(|G_0(h\xi)| \langle \xi \rangle^s \right)^2 \left(|\mathcal{F}[\phi](\xi)| \langle \xi \rangle^{-s} \right)^2 d\xi \\
&\le \left(\int_{\mathbf{R}^2} \left(|G_0(h\xi)| \langle \xi \rangle^s \right)^{2q'/(q'-2)} d\xi \right)^{(q'-2)/q'} \|\phi\|^2_{\mathcal{F}L^{q'}_{-s}}.
\end{aligned}
$$

On the other hand, we note $\langle \xi/h \rangle \le \langle \xi \rangle / h$ for $0 < h < 1/2$, and $2q'(-3+s)/(q'-2) < -2$ if and only if $s < 2 + 2/q'$. Thus we obtain

$$
\int_{\mathbf{R}^2} \left(|G_0(h\xi)| \langle \xi \rangle^s \right)^{2q'/(q'-2)} d\xi = \frac{1}{h^2} \int_{\mathbf{R}^2} \left(|G_0(\xi)| \left\langle \frac{\xi}{h} \right\rangle^s \right)^{2q'/(q'-2)} d\xi \lesssim h^{-2-\frac{2sq'}{q'-2}},
$$

since G_0 is continuous and satisfies $G_0(\xi) = O(|\xi|^{-3})$ as $|\xi| \to \infty$. Combining these inequalities, we obtain (16).

Step 3. Thirdly, we show that there exists $f_0 \in \mathcal{F}L^q_s(\mathbf{R}^n)$ such that

$$
\int_{K_h} |f_0(x)|^2 dx = O(h^3) \quad (\text{as } h \to 0).
$$

Let $k \in C_c^\infty(\mathbf{R})$ be such that $\operatorname{supp} k \subset B_1(0, 3/2)$, $k(t) = 1$ on $B_1(0, 5/4)$ and $0 \le k(t) \le 1$ $(t \in \mathbf{R})$, and define real-valued $\tau \in C_c^\infty(\mathbf{R}^2)$ by $\tau(x) := k(|x|)$. We first show $\phi = \tau\phi$ by the duality argument. Let $g \in C_c^\infty(\mathbf{R}^2) \subset \mathcal{F}L^q_s(\mathbf{R}^2)$ and $\psi \in C_c^\infty(\mathbf{R}^2)$ be such that $\psi(x) = 1$ on $\operatorname{supp} g$. Moreover, define real-valued $\tau_0 \in C_c^\infty(\mathbf{R}^2)$ by

$$
\tau_0(x) := (1 - \tau(x))\psi(x).
$$

Then $\operatorname{supp} \tau_0 \subset \mathbf{R}^2 \backslash S^1$, $\tau(x) + \tau_0(x) = 1$ on $\operatorname{supp} g$ and $S^1 \subset \operatorname{int}(\operatorname{supp} \tau)$. Since $g = (\tau + \tau_0)g$, we obtain

$$
\langle g, \phi \rangle = \langle g, \tau\phi \rangle + \langle g, \tau_0\phi \rangle.
$$

We note that $\tau_0 = 0$ in a neighborhood of S^1, and thus $\tau_0 \in J$. Moreover, since J is closed modulation invariant subspace of $\mathcal{F}L^q_s(\mathbf{R}^n)$, we have $M_\eta \tau_0 \in J$ for all $\eta \in \mathbf{R}^2$. Thus, we obtain

$$
\mathcal{F}[\tau_0\phi](\eta) = \overline{\langle M_\eta \tau_0, \phi \rangle} = 0,
$$

which implies $\langle g, \tau_0\phi\rangle = 0$, and thus $\langle g, \phi\rangle = \langle g, \tau\phi\rangle$. Hence $\phi = \tau\phi$. From this we get

$$\mathcal{F}[\phi](\eta) = \mathcal{F}[\tau\phi](\eta) = \overline{\langle M_\eta \tau, \phi\rangle} \quad (\eta \in \mathbf{R}^2).$$

Moreover, since $\mathcal{F}[\tau] \in \mathcal{S}(\mathbf{R}^2)$, we have

$$\mathcal{F}[\phi](\eta) + \Delta[\mathcal{F}[\phi]](\eta) = \overline{\langle (1+\Delta_\eta)[M_\eta\tau], \phi\rangle} = \overline{\langle M_\eta f_0, \phi\rangle},$$

where we denote $f_0(x) := (1-|x|^2)\tau(x)$. Therefore,

$$\int_{K_h} |f_0(x)|^2 dx = \int_{K_h} (1-|x|^2)^2 |\tau(x)|^2 dx \le C\|\tau\|_{L^\infty}(1-|1+2h|^2)^2 \int_{1-2h}^{1+2h} r dr \le C_\tau h^3$$

for $0 < h < 1/2$. Hence, we obtain the desired result.

Step 4. Finally, we prove $\mathcal{F}[\phi] + \Delta[\mathcal{F}[\phi]] = 0$. Let f_0 be the function defined in Step 3. Then by Step 1. we have

$$|\langle M_\eta f_0, \phi\rangle| \lesssim \liminf_{h\downarrow 0} \left(\int_{K_h} |f_0(\xi)|^2 d\xi\right)\left(\int_{K_h} |\Phi_h(\xi)|^2 d\xi\right) \lesssim \liminf_{h\downarrow 0} h^{3-2-2s+\frac{4}{q'}} = 0$$

by our assumption $s < 1/2 + 2/q'$, and thus

$$\mathcal{F}[\phi](\eta) + \Delta[\mathcal{F}[\phi]](\eta) = \overline{\langle M_\eta f_0, \phi\rangle} = 0.$$

□

4.3 The proof of Proposition 12

Let $k \in \mathbf{Z}$. We consider the polar coordinates $\xi_1 = r\cos\theta$ and $\xi_2 = r\sin\theta$ in $\mathbf{R}^2$, and define a function ϕ_k on $\mathbf{R}^2$ by

$$\mathcal{F}[\phi_k](\xi) := C_k(\phi; r)e^{ik\theta} \quad (\xi = (\xi_1, \xi_2) \in \mathbf{R}^2)$$

with

$$C_k(\phi; r) := \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}[\phi](r\cos\omega, r\sin\omega)e^{-ik\omega} d\omega.$$

Step 1. We first show that ϕ_k satisfies $\mathcal{F}[\phi_k] \in C(\mathbf{R}^2)$, $\phi_k \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$ and $\langle f, \phi_k\rangle = 0$ for all $f \in J$. The continuity of $\mathcal{F}[\phi_k]$ follows from

$$\begin{aligned}\mathcal{F}[\phi_k](\xi) &= \frac{1}{2\pi}\int_{-\theta}^{2\pi-\theta} \mathcal{F}[\phi](r\cos(\omega+\theta), r\sin(\omega+\theta))e^{-ik\omega} d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}[\phi](r\cos(\omega+\theta), r\sin(\omega+\theta))e^{-ik\omega} d\omega\end{aligned}$$

and the continuity of $\mathcal{F}[\phi]$ with the Lebesgue convergence theorem. To prove $\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$, we note that by the Hölder inequality

$$\begin{aligned}|\mathcal{F}[\phi_k](\xi)| &\le \frac{1}{2\pi}\int_0^{2\pi} |\mathcal{F}[\phi](r\cos\omega, r\sin\omega)| d\omega \\ &\le (2\pi)^{1/q-1}\left(\int_0^{2\pi} |\mathcal{F}[\phi](r\cos\omega, r\sin\omega)|^{q'} d\omega\right)^{1/q'}\end{aligned}$$

and also we have

$$\|\phi\|^{q'}_{\mathcal{F}L^{q'}_{-s}} = \int_0^\infty \left(\int_0^{2\pi} |\mathcal{F}[\phi](r\cos\omega, r\sin\omega)|^{q'} d\omega \right) \langle r\rangle^{-sq'} r dr.$$

Therefore,

$$\begin{aligned}\|\phi_k\|^{q'}_{\mathcal{F}L^{q'}_{-s}} &= \int_0^{2\pi} \left(\int_0^\infty |\mathcal{F}[\phi_k](r\cos\theta, r\sin\theta)|^{q'} \langle r\rangle^{-sq'} r dr \right) d\theta \\ &\lesssim \int_0^{2\pi} \left(\int_0^\infty \left(\int_0^{2\pi} |\mathcal{F}[\phi](r\cos\omega, r\sin\omega)|^{q'} d\omega \right) \langle r\rangle^{-sq'} r dr \right) d\theta \lesssim \|\phi\|^{q'}_{\mathcal{F}L^{q'}_{-s}},\end{aligned}$$

which gives $\phi_k \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2)$. Finally, we prove $\langle f, \phi_k\rangle = 0$ for all $f \in J$. Since J is the closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$ generated by J^0, it is enough to prove $\langle M_{-\eta} f_0, \phi_k\rangle = 0$ for all $f_0 \in J^0$ and $\eta = (\eta_1, \eta_2) \in \mathbf{R}^2$ (cf. (13)). By the polar coordinates

$$\begin{aligned}&(2\pi)^2 \langle M_{-\eta} f_0, \phi_k\rangle = \langle \mathcal{F}[M_{-\eta} f_0], \mathcal{F}[\phi_k]\rangle_{L^2(\mathbf{R}^2)} \\ &= \int_0^\infty \left(\int_0^{2\pi} \mathcal{F}[f_0](r\cos\theta + \eta_1, r\sin\theta + \eta_2) \overline{\mathcal{F}[\phi_k](r\cos\theta, r\sin\theta)} d\theta \right) r dr\end{aligned}$$

and

$$\overline{\mathcal{F}[\phi_k](r\cos\theta, r\sin\theta)} = \frac{1}{2\pi} \int_0^{2\pi} \overline{\mathcal{F}[\phi](r\cos(\omega+\theta), r\sin(\omega+\theta))} e^{ik\omega} d\omega.$$

Combining these estimates with the Fubini theorem and the change of variables, we obtain

$$\begin{aligned}\langle M_{-\eta} f_0, \phi_k\rangle = \frac{1}{(2\pi)^3} \int_0^{2\pi} \Big(\int_0^\infty \Big(\int_0^{2\pi} &\mathcal{F}[f_0](r\cos(\theta-\omega) + \eta_1, r\sin(\theta-\omega) + \eta_2) \\ &\times \overline{\mathcal{F}[\phi](r\cos\theta, r\sin\theta)} d\theta \Big) r dr \Big) e^{ik\omega} d\omega.\end{aligned}$$

We note $r\cos(\theta-\omega) = \xi_1\cos\omega + \xi_2\sin\omega$, $r\sin(\theta-\omega) = \xi_2\cos\omega - \xi_1\sin\omega$. Then we can easily see

$$\begin{aligned}&\mathcal{F}[f_0](\xi_1\cos\omega + \xi_2\sin\omega + \eta_1, -\xi_1\sin\omega + \xi_2\cos\omega + \eta_2) \\ &= \mathcal{F}[f_0]((\xi_1, \xi_2)^t A + (\eta_1, \eta_2)) \\ &= \iint_{\mathbf{R}^2} f_0(x_1, x_2) e^{-i((\xi_1,\xi_2)^t A + (\eta_1,\eta_2))\cdot(x_1,x_2)} dx_1 dx_2 \\ &= \iint_{\mathbf{R}^2} e^{-i(\eta_1,\eta_2)\cdot(y_1,y_2)^t A} f_0((y_1, y_2)^t A) e^{-i(\xi_1,\xi_2)\cdot(y_1,y_2)} dy_1 dy_2 = \mathcal{F}[F_0](\xi)\end{aligned}$$

with $A = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}$ and $F_0(y) = e^{-i\eta\cdot y\,{}^tA} f_0(y\,{}^tA)$, where ${}^tA$ denotes the transpose of A. Therefore we have

$$\langle M_{-\eta} f_0, \phi_k \rangle = \frac{1}{(2\pi)^3} \int_0^{2\pi} \left(\int_{\mathbf{R}^2} \mathcal{F}[F_0](\xi) \overline{\mathcal{F}[\phi](\xi)} d\xi \right) e^{ik\omega} d\omega.$$

Moreover, since $f_0 \in J^0$ and A is a rotation matrix in $\mathbf{R}^2$, and also J is a closed modulation invariant subspace, we have $F_0 \in J$ and thus $\langle F_0, \phi \rangle = 0$. Hence $\langle M_{-\eta} f_0, \phi_k \rangle = 0$ for all $f_0 \in J^0$ and $\eta \in \mathbf{R}^2$.

Step 2. We note that ϕ_k satisfies $\mathcal{F}[\phi_k] + \Delta[\mathcal{F}[\phi_k]] = 0$. In fact, as we have already seen in Step 1, ϕ_k satisfies the same condition to ϕ. Thus we get the desired result.

Step 3. Now we give the proof of Proposition 12. By Step 2 and the polar coordinates, we have

$$\begin{aligned} 0 &= \mathcal{F}[\phi_k] + \Delta[\mathcal{F}[\phi_k]] = \mathcal{F}[\phi_k] + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} \right) \mathcal{F}[\phi_k] \\ &= C_k(\phi; r) e^{ik\theta} + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\partial r} \right) (C_k(\phi; r) e^{ik\theta}) \\ &= \left(C_k(\phi; r) + \frac{\partial^2}{\partial r^2} C_k(\phi; r) + \frac{1}{r^2} C_k(\phi; r)(-k^2) + \frac{1}{r}\frac{\partial}{\partial r} C_k(\phi; r) \right) e^{ik\theta}. \end{aligned}$$

Then, we get

$$\left(\frac{\partial^2}{\partial r^2} + \left(1 - \frac{k^2}{r^2} \right) + \frac{1}{r}\frac{\partial}{\partial r} \right) C_k(\phi; r) = 0.$$

We note that the above equality is just the Bessel differential equation, and we can state $C_k(\phi; r) = C_k(\phi) J_k(r)$ for some $C_k(\phi) \in \mathbf{R}$, where J_k is the Bessel function defined in (2).

The proof of Proposition 12 (*i*)

We consider the polar coordinates in $\mathbf{R}^2$, i.e., $\xi_1 = r\cos\theta$ and $\xi_2 = r\sin\theta$, and define a modified version of the Fejér kernel by

$$\sigma_N[\phi](\xi) := \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1} \right) C_k(\phi; r) e^{ik\theta}.$$

Moreover, let ν_k be the measure on $\mathbf{R}^2$ defined by

$$\int_{\mathbf{R}^2} f(x) d\nu_k(x) := \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\infty} f(r\cos\omega, r\sin\omega) r d\delta_1(r) \right) e^{ik\omega} d\omega$$

(cf. Section 3.1). Then we have

$$\begin{aligned}\mathcal{F}[\nu_k](\xi) = \mathcal{F}[\nu_k](r\cos\theta, r\sin\theta) &= \int_{\mathbf{R}^2} e^{ix\cdot(r\cos\theta, r\sin\theta)} d\nu_k(x) \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{ir\cos(\omega-\theta)} e^{ik\omega} d\omega = \frac{1}{2\pi}\int_{-\theta-\frac{\pi}{2}}^{2\pi-\theta-\frac{\pi}{2}} e^{ir\cos(\omega+\frac{\pi}{2})} e^{ik(\omega+\theta+\frac{\pi}{2})} d\omega \\ &= \frac{e^{ik(\theta+\frac{\pi}{2})}}{2\pi}\int_{-\theta-\frac{\pi}{2}}^{2\pi-\theta-\frac{\pi}{2}} e^{-ir\sin\omega} e^{ik\omega} d\omega = i^k J_k(r) e^{ik\theta}\end{aligned}$$

and thus $C_k(\phi; r)e^{ik\theta} = C_k(\phi)J_k(r)e^{ik\theta} = i^{-k}\mathcal{F}[\nu_k](\xi)$. Hence

$$\mu_k(x) := \sum_{k=-N}^{N} i^{-k}\left(1 - \frac{|k|}{N+1}\right) C_k(\phi)\nu_k(x)$$

satisfies $\sigma_N[\phi] = \mathcal{F}[\mu_k]$ and supp $\mu_N \subset S^1$.

The proof of Proposition 12 (*ii*)

We first note that

$$\begin{aligned}\sigma_N[\phi](\xi) &= \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N+1}\right)\left(\frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}[\phi](r\cos\omega, r\sin\omega) e^{-ik\omega} d\omega\right) e^{ik\theta} \\ &= \int_0^{2\pi} \mathcal{F}[\phi](r\cos\omega, r\sin\omega)\left(\frac{1}{2\pi}\sum_{k=-N}^{N}\left(1 - \frac{|k|}{N+1}\right) e^{ik(\theta-\omega)}\right) d\omega \\ &= (\mathcal{F}[\phi]_r \star K_N)(\theta),\end{aligned}$$

where $\mathcal{F}[\phi]_r(\theta) := \mathcal{F}[\phi](r\cos\theta, r\sin\theta)$, $K_N(\theta) := \frac{1}{2\pi}\sum_{k=-N}^{N}(1-\frac{|k|}{N+1})e^{ik\theta}$ (i.e., K_N is the Fejér kernel) and $\star$ is the operator defined by $(G\star H)(\theta) := \int_0^{2\pi} G(\omega)H(\theta-\omega)d\omega$. Then it is well-known that $K_N(\theta) \geq 0$ and $\int_0^{2\pi} K_N(\theta)d\theta = 1$. Therefore we have by the Hölder inequality

$$\begin{aligned}&\|\sigma_N[\phi]\|^{q'}_{L^{q'}_{-s}} = \int_0^{2\pi}\left(\int_0^{\infty}(|(\mathcal{F}[\phi]_r \star K_N)(\theta)|\langle r\rangle^{-s})^{q'} r dr\right) d\theta \\ &\lesssim \int_0^{2\pi}\left(\int_0^{\infty}\left(\int_0^{2\pi}|\mathcal{F}[\phi]_r(\theta-\omega)|K_N(\omega)d\omega\right)^{q'}\langle r\rangle^{-sq'} r dr\right) d\theta \\ &\leq \int_0^{2\pi}\Big(\int_0^{\infty}\Big(\int_0^{2\pi}|\mathcal{F}[\phi]_r(\theta-\omega)|^{q'}K_N(\omega)d\omega\Big)\Big(\int_0^{2\pi}K_N(\omega)d\omega\Big)^{\frac{q'}{q}}\langle r\rangle^{-sq'} r dr\Big)d\theta \\ &= \int_0^{2\pi}\Big(\int_0^{\infty}\Big(\int_0^{2\pi}|\mathcal{F}[\phi]_r(\theta-\omega)|^{q'}K_N(\omega)d\omega\Big)\langle r\rangle^{-sq'} r dr\Big)d\theta\end{aligned}$$

$$= \int_0^{2\pi} K_N(\omega)\Big(\int_0^{\infty} \Big(\int_0^{2\pi} |\mathcal{F}[\phi]_r(\theta-\omega)|^{q'} d\theta \Big) \langle r\rangle^{-sq'} r dr \Big) d\omega = \|\mathcal{F}[\phi]\|^{q'}_{L^{q'}_{-s}},$$

which implies the desired result.

The proof of Proposition 12 (*iii*)

For the sake of simplicity, we set

$$\mathcal{F}[\phi]_r(\theta) := \mathcal{F}[\phi](r\cos\theta, r\sin\theta), \quad \sigma_N[\phi]_r(\theta) := \sigma_N[\phi](r\cos\theta, r\sin\theta).$$

Let $R > 0$ and $\varepsilon > 0$. Since $\mathcal{F}[\phi] \in C(\mathbf{R}^2)$, $\mathcal{F}[\phi]$ is uniformly continuous on any compact set on $\mathbf{R}^2$. Thus there exists $\delta > 0$ such that for any $0 \le r \le R$, $0 \le \theta \le 2\pi$ and $|\eta| < \delta$, we have

$$|\mathcal{F}[\phi]_r(\theta-\eta) - \mathcal{F}[\phi]_r(\theta)| < \varepsilon. \tag{17}$$

Now suppose that $0 \le r \le R$ and $0 \le \theta \le 2\pi$. Since $\int_0^{2\pi} K_N(\theta) d\theta = 1$, we have

$$\begin{aligned}
|\sigma_N[\phi]_r(\theta) - \mathcal{F}[\phi]_r(\theta)| &= |(\mathcal{F}[\phi]_r \star K_N)(\theta) - \mathcal{F}[\phi]_r(\theta)| \\
&= \left| \int_0^{2\pi} (\mathcal{F}[\phi]_r(\theta-\eta) - \mathcal{F}[\phi]_r(\theta)) K_N(\eta) d\eta \right| \\
&\le \int_{|\eta|<\delta} |\mathcal{F}[\phi]_r(\theta-\eta) - \mathcal{F}[\phi]_r(\theta)| K_N(\eta) d\eta \\
&\quad + \int_{|\eta|\ge\delta} |\mathcal{F}[\phi]_r(\theta-\eta) - \mathcal{F}[\phi]_r(\theta)| K_N(\eta) d\eta \\
&\le \varepsilon + \int_{|\eta|\ge\delta} |\mathcal{F}[\phi]_r(\theta-\eta) - \mathcal{F}[\phi]_r(\theta)| K_N(\eta) d\eta.
\end{aligned}$$

Moreover, since $K_N(\eta) = \frac{1}{2\pi(N+1)} \frac{\sin^2 \frac{(N+1)\eta}{2}}{\sin^2 \frac{\eta}{2}}$, there exists $C_\delta > 0$ such that $K_N(\eta) \le \frac{C_\delta}{(N+1)}$ holds for all $\delta \le |\eta| \le 2\pi$. Therefore, we obtain

$$|\sigma_N[\phi]_r(\theta) - \mathcal{F}[\phi]_r(\theta)| < \varepsilon + \frac{C_\delta}{(N+1)} \sup_{|\xi|\le R} |\mathcal{F}[\phi](\xi)|.$$

If N is sufficiently large, we obtain

$$|\sigma_N[\phi]_r(\theta) - \mathcal{F}[\phi]_r(\theta)| < 2\varepsilon$$

for all $0 \le r \le R$ and $0 \le \theta \le 2\pi$, which implies that $\{\sigma_N[\phi]\}_{N=1}^{\infty}$ converges uniformly to $\mathcal{F}[\phi]$ on any compact set.

5 The proof of Theorem 3

Let $\sigma(t)$ be the Radon measure in $\mathbf{R}^n$ corresponding to a mass distributed homogeneously over S^{n-1} (cf. Section 3.1), and $h_0(\xi) := \int_{\mathbf{R}^n} e^{i\xi\cdot t} d\sigma(t)$. Then, it is known that $h_0(\xi) = O(|\xi|^{-(n-1)/2})$ $(|\xi| \to \infty)$ (cf. [28, p.154]). By this estimate, we can show the statement in the same way to the proof of Theorem 1. Then, we omit the details.

6 Appendix I : Application to PDEs

Let $1 < p < \infty$, $s \in \mathbf{R}$ and $\psi \in L^{p'}_{-s}(\mathbf{R}^2)$, and define the bounded linear functional ℓ_ψ on $\mathcal{F}L^p_s(\mathbf{R}^2)$ by

$$\ell_\psi(f) := \int_{\mathbf{R}^2} \mathcal{F}[f](\xi)\psi(\xi)d\xi \quad (f \in \mathcal{F}L^p_s(\mathbf{R}^2)),$$

and the support of ℓ_ψ by

$$\bigcap\{F \mid F \textit{ is a closed subset of } \mathbf{R}^2 \textit{ and } \ell_\psi|_{\mathbf{R}^2\setminus F} = 0\},$$

where $\ell_\psi|_{\mathbf{R}^2\setminus F} = 0$ means that $\ell_\psi(f) = 0$ for all $f \in \mathcal{F}L^{p'}_s(\mathbf{R}^2)$ with supp $f \subset \mathbf{R}^2 \setminus F$. Guo [16, Lemma 1 and Theorem 2′] showed the following : Let $2 \le p < 4$ and $u \in L^p(\mathbf{R}^2)$. If supp $\ell_u \subset S^1$ (i.e., supp $\mathcal{F}[u] \subset S^1$) and u satisfies the Helmholtz equation $\Delta u + u = 0$, then $u = 0$. This result is related to Hörmander [18]. On the other hand, as a by-product of the proof of Theorem 2, we obtain the following:

Corollary 16 Let $2 < p < \infty$ and $2/p < s < 2/p + 1/2$. Then there exists non-zero $u \in L^p_{-s}(\mathbf{R}^2) \cap C(\mathbf{R}^2)$ such that supp $\ell_u \subset S^1$ and $u + \Delta u = 0$.

Proof Define $q \in (1, 2)$ by $q = p'$, and let J be the closed ideal defined in Section 4. Set

$$A := \{\phi \in \mathcal{F}L^{q'}_{-s}(\mathbf{R}^2) \mid \mathcal{F}[\phi] \in C(\mathbf{R}^2) \text{ and } \langle f, \phi\rangle = 0 \text{ for all } f \in J\}.$$

If we can prove $A \neq \{0\}$, then putting $u := \mathcal{F}[\phi]$ for some non-zero $\phi \in A$ will give us the desired result (cf. Lemma 15). Now, contrary to our claim, we assume $A = \{0\}$. Then we have $\mathcal{F}L^q_s(\mathbf{R}^2) \subset J$. In fact, let $f_0 \in \mathcal{F}L^q_s(\mathbf{R}^2)$. Then, obviously, $\langle f_0, \phi\rangle = 0$ for all $\phi \in A$. Therefore, by Proposition 11, we have $f_0 \in J$. But this is impossible since $J \neq \mathcal{F}L^q_s(\mathbf{R}^2)$. □

7 Appendix II : Closed ideals in $\mathcal{F}L^q_s(\mathbf{R}^2)$

In this section, we shall prove the following theorem.

Theorem 17 *Let $1 \le q \le 2$, $2/q' < s < 2/q' + 1/2$ and J be the closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$ defined in Section 4. Then, J is the smallest closed ideal I' in $\mathcal{F}L^q_s(\mathbf{R}^2)$ with $Z(I') = S^1$.*

7.1 Technical Lemmas

To prove Theorem 17, we recall the following lemmas.

Lemma 18 ([19, Theorem 1.2]) Let $4/3 \leq q \leq \infty$. If $F \in C^\infty(\mathbf{R})$ and $F(0) = 0$, then $F(f) \in \mathcal{F}L^q_s(\mathbf{R}^n)$ for all real-valued $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$.

Lemma 19 ([19, Lemma A.1]) Let $1 \leq p, q \leq \infty$ and $s \in \mathbf{R}$. Suppose that $\varphi \in C^\infty_c(\mathbf{R}^n)\backslash\{0\}$ and supp $\varphi \subset B_n(x_0, R)$. Then we have

$$\|\varphi \cdot f\|_{M^{p,q}_s} \approx \|\varphi \cdot f\|_{\mathcal{F}L^q_s}.$$

Here, $M^{p,q}_s(\mathbf{R}^n) = M^{p,q}_s$ denotes the modulation space defined by the norm

$$\|f\|_{M^{p,q}_s} = \left\| \langle \xi \rangle^s \left\| \mathcal{F}_{y \to \xi}[f(y)e^{-|x-y|^2}](\xi) \right\|_{L^p(\mathbf{R}^n_x)} \right\|_{L^q(\mathbf{R}^n_\xi)}.$$

We refer to [1], [4], [9], [14] for more details on the modulation spaces.

Lemma 20 ([20, Theorem 4.13]) Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $s > n/q'$. Let μ be a complex measure on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} (1 + |\xi|)^{1+(s+\frac{n}{q})(1+\frac{1}{s-\frac{n}{q'}})} d|\mu|(\xi) < \infty$$

and $\mu(\mathbf{R}) = 0$. Furthermore, assume that the function F is the Fourier transform of μ. Then $F(f) \in M^{p,q}_s(\mathbf{R}^n)$ for all real-valued $f \in M^{p,q}_s(\mathbf{R}^n)$.

Making use of the fact that $\mathcal{F}L^q_s(\mathbf{R}^n)$ is a Wiener algebra in the sense of Reiter ([22], [23]), we give a lemma which corresponds to an ideal version of Bhimani-Ratnakumar's [2, Lemma 3.13].

Lemma 21 Let I be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^n)$, and $f \in \mathcal{F}L^q_s(\mathbf{R}^n)$. Suppose that

(i) For every $x \in \mathbf{R}^n$, there exists $g_x \in I$ such that $f = g$ in a neighborhood of x (i.e., $f \in I$ locally at every $x \in \mathbf{R}^n$).
(ii) There exists $\psi \in C^\infty_c(\mathbf{R}^n)$ such that $(1 - \psi)f \in I$ (i.e., $f \in I$ locally at ∞).

Then $f \in I$.

Proof Let $\psi \in C^\infty_c(\mathbf{R}^n)$ be a function that satisfies (ii) and divide f as $f = (1 - \psi)f + \psi f$. Since $(1 - \psi)f \in I$ and I is a linear subspace of $\mathcal{F}L^q_s(\mathbf{R}^n)$, to prove $f \in I$, it suffices to show $\psi f \in I$. By virtue of (i), for any $x \in$ supp ψ, there exists $g_x \in \mathcal{F}L^q_s(\mathbf{R}^n)$ such that $f = g_x$ in an open ball $U(x, r_x) := \{y \in \mathbf{R}^n \mid |x - y| < r_x\}$. Since supp ψ is compact, we can find $\{x_j\}_{j=1}^N \subset$ supp ψ such that supp $\psi \subset \cup_{j=1}^N U(x_j, r_{x_j})$. Also, let $\{\phi_{x_j}\}_{j=1}^N \subset C^\infty_c(\mathbf{R}^n)$ be a

partition of unity subordinate to this cover, and let $\{g_{x_j}\}_{j=1}^N \subset I$ be such that $f = g_{x_j}$ on $U(x_j, r_{x_j})$. Since supp $\phi_{x_j} \subset U(x_j, r_{x_j})$, we have $\phi_{x_j} f = \phi_{x_j} g_{x_j}$. Hence we get

$$\psi f = \sum_{j=1}^N \psi \phi_{x_j} f = \sum_{j=1}^N \psi \phi_{x_j} g_{x_j} \in I.$$

□

Lemma 22 Let $x_0 \in \mathbf{R}^2$ and f be a real-valued function in $\mathcal{F}L^q_s(\mathbf{R}^2)$ such that $f(x_0) = 1$ and supp f is in a compact neighborhood of x_0. Then $1/f$ belongs to $\mathcal{F}L^q_s(\mathbf{R}^2)$ locally at x_0.

Proof Choose a non-negative $v \in C^\infty(\mathbf{R})$ such that $v(x) = 1$ on $B_1(1, 1/10)$ and supp $v \subset B_1(1, 1/3)$. Define F by $F(x) = v(x)/x$ ($x \in$ supp v), and $= 0$ ($x \notin$ supp v). We first consider the case $q \geq 4/3$. Since $F \in \mathcal{S}(\mathbf{R})$ and $F(0) = 0$, Lemma 18 shows that $F(f) \in \mathcal{F}L^q_s(\mathbf{R}^2)$. Moreover, since $f \in \mathcal{F}L^q_s(\mathbf{R}^2) \subset C(\mathbf{R}^2)$ and $v(x) = 1$ on $B_1(1, 1/10)$, we see $F(f) = 1/f$ in some neighborhood of x_0, and thus $1/f$ belongs to $\mathcal{F}L^q_s(\mathbf{R}^2)$ locally at x_0. Next, we consider the case $1 < q < 4/3$. Since $F \in \mathcal{F}L^1_\alpha(\mathbf{R})$ for all $\alpha \in \mathbf{R}$, Lemma 20 shows that $F(f) \in M^{p,q}_s(\mathbf{R}^2)$. On the other hand, by the definitions of v and f, we can easily see supp $F(f)$ is compact. Thus Lemma 19 yields $F(f) \in \mathcal{F}L^q_s(\mathbf{R}^2)$. Moreover, since $F(f) = 1/f$ in some neighborhood of x_0, $1/f$ belongs to $\mathcal{F}L^q_s(\mathbf{R}^2)$ locally at x_0.

□

7.2 The proof of Theorem 17

Let I' be a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$ with $Z(I') = S^1$. Then, it suffices to prove $J \subset I'$. Before we prove this, we remark that there exist $x_0 \in \mathbf{R}^2 \setminus S^1$ and $h \in I'$ such that $h(x) = 1$ on a neighborhood of x_0. In fact, we first note that there exist $x_0 \in \mathbf{R}^2 \setminus S^1$ and $f \in I'$ such that $f(x_0) = 1$ by $Z(I') = S^1$. Additionally, since I' is a closed ideal in $\mathcal{F}L^q_s(\mathbf{R}^2)$, by taking $\phi \in C^\infty_c(\mathbf{R}^2)$ with $\phi(x_0) = 1$ and considering $f\phi$ instead of f, we may assume supp f is compact. Moreover, since $\overline{f} \in \mathcal{F}L^q_s(\mathbf{R}^2)$, we see $|f|^2 = f\overline{f} \in I'$ and $|f(x_0)|^2 = 1$. Therefore, Lemma 22 shows that there exists $k \in \mathcal{F}L^q_s(\mathbf{R}^2)$ such that $k(x) = 1/|f(x)|^2$ in some neighborhood of x_0. Hence, $h := k|f|^2 \in I'$ and $h(x) = 1$ in some neighborhood of x_0. Now we prove $J \subset I'$ by Lemma 21. By Lemma 7 and the definition of J (see also (15)), it suffices to show $f \in I'$ for all $f \in J^0$ with compact support. Let $f \in J^0$ and supp f is compact. Since $0 \in I'$ and $f = 0$ in a neighborhood of S^1, f belongs to I' locally near S^1. Moreover, if $x_0 \in \mathbf{R}^2 \setminus S^1$, then there exists $k \in I'$ such that $k(x) = 1$ near x_0. Hence, we have $f = k \cdot f$ near x_0 and $k \cdot f \in I'$, which shows f belongs to I' locally near $\mathbf{R}^2 \setminus S^1$. We also note $f \in I'$ locally at ∞, since supp f is compact. Hence, f is in I' at all point which contains within $\mathbf{R}^2$ and ∞, which implies $f \in I'$.

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References

[1] Bényi, Á.; Okoudjou, K.A.: Modulation Spaces. With Applications to Pseudodifferential Operators and Nonlinear Schrödinger Equations. Applied and Numerical Harmonic Analysis. New York, NY: Birkhäuser. xvi, 169 p. (2020).

[2] Bhimani, D.G.: Ratnakumar, P.K.: Functions operating on modulation spaces and nonlinear dispersive equations. J. Funct. Anal. 270, No. 2, 621-648 (2016).

[3] Brandenburg, L. H.: On identifying the maximal ideals in Banach algebras. J. Math. Anal. Appl. 50, 489-510 (1975).

[4] Cordero, E.; Rodino, L.: Time-Frequency Analysis of Operators. De Gruyter Studies in Mathematics 75. Berlin: De Gruyter. xiv, 442 p. (2020).

[5] Domar, Y.: Harmonic analysis based on certain commutative Banach algebras. Acta Math. 96, 1-66 (1956).

[6] Domar, Y.: On the spectral synthesis problem for $(n-1)$-dimensional subsets of $\mathbf{R}^n$, $n \geq 2$. Ark. Mat. 9, 23-37 (1971).

[7] Domar, Y.: On spectral synthesis in $\mathbb{R}^n$, $n \geq 2$. Euclidean harmonic analysis, Proc. Semin., Univ. Maryland 1979, Lect. Notes Math. 779, 46-72 (1980).

[8] Feichtinger, H. G.: Gewichtsfunktionen auf lokalkompakten Gruppen. Sitzungsber., Abt. II, Österr. Akad. Wiss., Math.-Naturwiss. Kl. 188, 451-471 (1979).

[9] Feichtinger, H.G.: Modulation spaces on locally compact abelian groups, in: M. Krishna, R. Radha and S. Thangavelu (Eds.), Wavelets and their Applications, Chennai, India, Allied Publishers, New Delhi, 2003, 99-140, Updated version of a technical report, University of Vienna, 1983.

[10] Feichtinger, H.G.: Personal communication (2024)

[11] Feichtinger, H.G., Kobayashi, M., Sato, E.: On some properties of modulation spaces as Banach algebras. Stud. Math. 280, No. 1, 55-86 (2025).

[12] Feichtinger, H.G., Kobayashi, M., Sato, E.: Further study of modulation spaces as Banach algebras. Ann. Univ. Sci. Budap. Rolando Eötvös, Sect. Comput. 56, 151-166 (2024).

[13] Grafakos, L.: Classical Fourier Analysis. 3rd ed. Graduate Texts in Mathematics 249. New York, Springer. xvii, 638 p. (2014).

[14] Gröchenig, K.: Foundations of Time-Frequency Analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, (2001)

[15] Gröchenig, K.: Weight functions in time-frequency analysis. Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis, 343–366. Fields Inst. Commun., 52 American Mathematical Society, Providence, RI, (2007).

[16] Guo, K.: On the spectral synthesis property and its application to partial differential equations. Ark. Mat. 30, No. 1, 93-103 (1992).

[17] Herz, C. S.: Spectral synthesis for the circle. Ann. Math. (2) 68, 709-712 (1958).

[18] Hörmander, L.: Lower bounds at infinity for solutions of differentiable equations with constant coefficients, Israel J. Math. 16, 103-116 (1973).

[19] Kato, T., Sugimoto, M., Tomita, N.: Nonlinear operations on a class of modulation spaces. J. Funct. Anal. 278, No. 9, Article ID 108447, 26 p. (2020).

[20] Reich, M., Sickel, W.: Multiplication and composition in weighted modulation spaces. Qian, Tao (ed.) et al., Mathematical Analysis, Probability and Applications – Plenary Lectures. Springer Proceedings in Mathematics & Statistics 177, 103-149 (2016).

[21] Reiter, H.: Contributions to harmonic analysis. IV. Math. Ann. 135, 467-476 (1958).

[22] Reiter, H.: Classical Harmonic Analysis and Locally Compact Groups. Oxford: At the Clarendon Press. XI, 200 p. (1968).

[23] Reiter, H., Stegeman, J D.: Classical Harmonic Analysis and Locally Compact Groups. 2nd ed. London Mathematical Society Monographs. New Series. 22. Oxford: Clarendon Press. xiii, 327 p. (2000).

[24] Rudin, W.: Fourier Analysis on Groups. Interscience Tracts in Pure and Applied Mathematics. 12. New York and London: Interscience Publishers, division of John Wiley and Sons. IX, 285 p. (1962).

[25] Schwartz, L.: Sur une propriété de synthese spectrale dans les groupes non compacts. C. R. Acad. Sci., Paris 227, 424-426 (1948).

[26] Stegeman, J. D.: Wiener - Ditkin sets for certain Beurling algebras. Monatsh. Math. 82, 337-340 (1976).

[27] Stegeman, J. D.: Analysis of P. Malliavin's proof of non spectral synthesis. New Integrals, Proc. Henstock Conf., Coleraine/Ireland 1988, Lect. Notes Math. 1419, 180-200 (1990).

[28] Stein, E. M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Apaces. Princeton Mathematical Series 32. Princeton, N. J.: Princeton University Press. X, 297 p. (1971).

[29] Varopoluos, N. T.: Spectral synthesis on spheres. Proc. Camb. Philos. Soc. 62, 379-387 (1966).

[30] Watson, G. N.: A Treatise on the Theory of Bessel Functions. 2nd ed. London: Cambridge University Press. vii, 804 p. (1966).

[31] Wermer, J.: On a class of normed rings. Ark. Mat. 2, 537-551 (1954).