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Immersions associated with links of complex surface singularities and their topology

（複素曲面特異点リンクにまつわるはめ込みとそのトポロジー）

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Abstract

In this thesis, we study links of isolated surface singularities in $\mathbb{C}^3$ from the viewpoint of immersions through two problems. First, for each singularity, we determine the regular homotopy class of the inclusion map of its link into the 5-sphere. Second, for each type of Arnol'd's simple singularities, we show that the inclusion map of its link is regularly homotopic to the immersion associated with the corresponding Dynkin diagram which was constructed by Kinjo. We prove these by computing the complete invariants of the immersions given by Wu and Saeki–Szűcs–Takase. As an application, we also determine the Smale invariants of Kinjo's immersions.

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Chapter 1

Introduction

Given a complex analytic surface-germ in $\mathbb{C}^3$ with an isolated singularity, an oriented closed 3-manifold, the *link* of the singularity, appears within a small 5-sphere centered at the singularity. The link is isotopic in the 5-sphere to the boundary of a complex analytic surface, called the *Milnor fiber* of the singularity. These subjects have been deeply studied in various contexts, such as singularity theory, low-dimensional topology, and contact/symplectic/complex geometry [Mil68, CNP06, Sea06, Sea19].

In this thesis, we study links of isolated surface singularities in $\mathbb{C}^3$ from the viewpoint of immersions through two problems. First, we pose the following problem.

Problem I. *Let K be the link of an isolated surface singularity in $\mathbb{C}^3$. Then which is the regular homotopy class of the inclusion map $f\colon K \hookrightarrow S^5_\varepsilon$ as an immersion?*

Let us review the classification of immersions up to regular homotopy. In the late fifties, S. Smale defined a complete invariant, now called the *Smale invariant*, for the classification of immersions of spheres into Euclidean spaces [Sma59b]. His result includes specific examples such as the classification of plane curves by Whitney–Graustein and the eversion of the 2-sphere in 3-space. Shortly thereafter, M. W. Hirsch generalized Smale's result to immersions between arbitrary manifolds [Hir59]. Thanks to their works, the classification problem of immersions up to regular homotopy was reduced to homotopy theory. These results are known as the Smale–Hirsch theory today.

Based on the theory, W. T. Wu studied immersions of oriented 3-manifolds into 5-space. He gave a complete invariant for them up to regular homotopy. This invariant consists of two components. The first one is an invariant on the 2-skeleton of the source 3-manifold — this is named the *Wu invariant* in [SST02]. The second one is an invariant on the 3-cells, which is an analogue of the Smale invariant — let us call this the *Smale-type invariant*. If the normal bundle of a given immersion is trivial, then the Wu invariant is a second cohomology class of the 3-manifold which is a 2-torsion, and the Smale-type invariant is an integer. In order to determine the regular homotopy class of a given immersion, it is necessary to compute these two invariants.

However, it has been difficult to compute these invariants, since their geometric meanings were unclear from Wu's original work. The first attempt to resolve the difficulty was made by Saeki–Szűcs–Takase [SST02]. They gave more geometric expressions of the two invariants in the case where the normal bundle is trivial. In particular, their work made the Smale-type invariant computable via certain (co)homological data. Their results have been applied to the study of immersions [ST02], generalized to the case where the normal bundle is non-trivial [Juh05], and also applied to the study of surface singularities in $\mathbb{C}^3$ [GP24, PS24, PT23]. In contrast, a framework to compute the Wu invariant has not yet been established. It seems to be due to the following reason: the Wu invariant is defined for a fixed parallelization (trivialization of the tangent bundle) of the 3-manifold. Furthermore, if the parallelization changes, the Wu invariant can attain every possible value (Appendix A). Hence, it is crucial to choose an appropriate parallelization depending on situation.

Let us return to our first problem. We give an answer to Problem I by computing the Wu and Smale-type invariants.

Main Theorem I. *Under the setup of Problem I, the Wu invariant $c_\tau(f)$ with respect to any almost contact parallelization τ and the Smale-type invariant $i(f)$ are given by*

$$c_\tau(f) = 0 \in H^2(K;\mathbb{Z}), \quad i(f) = \frac{3}{2}(\sigma(F) - \alpha(K)) \in \mathbb{Z},$$

where F is the Milnor fiber of the singularity, $\sigma(F)$ is the signature of F, and $\alpha(K)$ is an integer determined by the torsion part of $H_1(K;\mathbb{Z})$ (see Definition 3.5.5).

Here is an outline of the proof. The Smale-type invariant is computed by Saeki–Szűcs–Takase's formula. To compute the Wu invariant, we focus on the fact that the inclusion map $f\colon K \hookrightarrow S^5_\varepsilon$ forms a contact embedding into the standard contact 5-sphere. In fact, this contact structure on K can be trivialized as a 2-plane field by the result of Kasuya [Kas16]. Hence we can parallelize K along this trivialization. We call this kind of parallelization an *almost contact parallelization* in this thesis (Definition 5.2.5). Then we can prove that the Wu invariant with respect to any almost contact parallelization vanishes for contact embeddings (Theorem 6.2.1).

As an application of Main Theorem I, for several pairs of distinct singularity links that are diffeomorphic as abstract 3-manifolds, we show that their inclusion maps into the 5-sphere are not regularly homotopic (Chapter 8). We note that Katanaga–Némethi–Szűcs studied a problem corresponding to Problem I for hypersurface singularities in $\mathbb{C}^4$ of type A [KNS14], and Saeki studied the isotopy classification for a class of fibered 3-knots in the 5-sphere [Sae99].

Our second problem focuses on *simple singularities*, the most fundamental class in the classification of holomorphic function-germs. V. I. Arnol'd classified them into types of A, D, and E [AGV85]. It is known that their links admit several special structures, such as quotient spaces under actions of finite subgroups of SU(2) on S^3 [Kle56] and plumbed 3-manifolds along Dynkin diagrams. These manifolds and structures also have been deeply studied. Now, we recall two works which motivated our problem.

For any holomorphic map-germ $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ which is singular only at the origin, A. Némethi and G. Pintér [NP15] obtained an immersion $\Phi|_{\mathfrak{S}}\colon S^3 \looparrowright S^5_\varepsilon$ by taking the preimage $\mathfrak{S} = \Phi^{-1}(S^5_\varepsilon) \cong S^3$. They discovered a formula of the Smale invariant of $\Phi|_{\mathfrak{S}}$ by counting complex singularities of a holomorphic perturbation of Φ. As examples of map-germs Φ, they considered parametrizations of simple singularities, and determined the regular homotopy classes of $\Phi|_{\mathfrak{S}}$ for all A-D-E cases. Note that in these cases, each immersion is factorized into $\Phi|_{\mathfrak{S}} = f \circ p$, where $f\colon K \hookrightarrow S^5_\varepsilon$ is the inclusion map of the link and $p\colon S^3 \to K$ is the universal covering map.

S. Kinjo [Kin15] associated an immersion to each of the Dynkin diagrams of type A-D-E. The construction is as follows: let $M(G)$ denote the plumbed 3-manifold along the diagram G, which is orientation-preservingly diffeomorphic to the link of the corresponding singularity. Construct an immersion $\bar{g}_G = g_G \circ r\colon M(G) \looparrowright \mathbb{R}^4$, where $r\colon M(G) \to M(G)$ is an orientation-reversing diffeomorphism. Pull-back $\bar{g}_G$ by the universal covering map $p\colon S^3 \to M(G)$. Then she determined the regular homotopy classes of $\bar{g}_G \circ p$ for the A-D cases by computing their Smale invariants. She employed Ekholm–Takase's formula [ET11], constructing certain C^∞ maps so-called singular Seifert surfaces of the immersions and counting their real singularities (see also the end of this section). However, the method did not apply to the E cases, leaving their Smale invariants unknown.

The works of Némethi–Pintér and Kinjo had a connection despite the difference of their natures. They pointed out it as follows: let $j\colon \mathbb{R}^4 \hookrightarrow \mathbb{R}^5$ denote the standard inclusion. Then, for each of the A-D cases, the Smale invariants of $f \circ p$ and $j \circ \bar{g}_G \circ p$ coincide with opposite sign. This implies that $f \circ p$ and $j \circ g_G \circ p$ are regularly homotopic (under removing one point from S^5_ε). Then the following problem naturally arises.

Problem II. *Consider any of the singularities of type E.*

(a) Which is the regular homotopy class of Kinjo's immersion $\bar{g}_G \circ p\colon S^3 \looparrowright \mathbb{R}^4$?

(b) Are the two immersions $f \circ p$ and $j \circ g_G \circ p\colon S^3 \looparrowright \mathbb{R}^5$ regularly homotopic?

We solve these by showing that *f and $j \circ g_G$ are actually regularly homotopic*, i.e., they have the same invariants. More precisely, we show the following assertion.

Main Theorem II. *For each type of simple singularities and the corresponding Dynkin diagram G, the Wu invariant $c_\tau(j \circ g_G)$ with respect to any almost contact parallelization τ and the Smale-type invariant $i(j \circ g_G)$ are given by*

$$c_\tau(j \circ g_G) = 0 \in H^2(M(G); \mathbb{Z}), \quad i(j \circ g_G) = \frac{3}{2}(-\#V(G) - \alpha(M(G))) \in \mathbb{Z},$$

where $\#V(G)$ is the number of vertices of G.

As in the proof of Main Theorem I, we compute the Smale-type invariant by Saeki–Szűcs–Takase's formula. To compute the Wu invariant, we consider an almost contact structure on $M(G)$ and an almost contact parallelization along this (§5.3). However, the reason for the vanishing of the Wu invariant differs from the previous case. We regularly homotope the immersion g_G so that the image of the almost contact structure forms complex tangency for the standard complex structure on $\mathbb{R}^4 = \mathbb{C}^2$. Then we prove that if we push forward this immersion into $\mathbb{R}^5$, then the Wu invariant with respect to the almost contact parallelization vanishes (Theorem 6.3.1).

In fact, we can identify two manifolds K and $M(G)$ as almost contact manifolds (§5.3.3). Then Main Theorems I and II implies the following, by the coincidence of their complete invariants: for each of A-D-E cases, the inclusion map $f\colon K \hookrightarrow S^5_\varepsilon$ and the immersion $j \circ g_G\colon M(G) \looparrowright \mathbb{R}^5$ are regularly homotopic (Corollary 7.2.1). It is immediate that two immersions $f \circ p$ and $j \circ g_G \circ p$ are regularly homotopic. In particular, we obtain the affirmative answer to Problem II(b).

As an application of this regular homotopicity, we also determine the regular homotopy classes of Kinjo's immersions $\bar{g}_G \circ p$ for all A-D-E cases, by computing the Smale invariants. This result recovers that of Kinjo for the A-D cases, and also answers to Problem II(a).

Corollary (= Theorem 7.2.4)**.** *For each type of simple singularities and the corresponding Dynkin diagram G, the Smale invariant $\Omega(\bar{g}_G \circ p) \in \pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see §3.2 for the choice of generators) is given by*

$$\Omega(\bar{g}_G \circ p) = \begin{cases} (n^2-1, 0) & \text{for } A_{n-1}\ (n \geqslant 2); \\ (4n^2+12n-1, 0) & \text{for } D_{n+2}\ (n \geqslant 2); \\ (167, 0) & \text{for } E_6; \\ (383, 0) & \text{for } E_7; \\ (1079, 0) & \text{for } E_8. \end{cases}$$

To prove this, we additionally employs Némethi–Pintér's result and the computation of the normal mapping degree of $\bar{g}_G \circ p$. This argument is quite different from that of Kinjo (Remark 7.2.5). Note that it was expected by Pintér to find any direct relationship between Némethi–Pintér's and Kinjo's immersions [Pin18]. We realized the expectation and provided new aspects for the immersions, in the above sense. We summarize our study in Figure 1.1.

Finally, we mention Thom polynomials, although our present result does not use them directly. Several works on immersions, such as [SST02, ET11] mentioned above and other results [ES03, ES06, Tak07, Tak12], heavily relies on counting singularities of maps, which is nothing but Thom polynomial theory. A Thom polynomial is a universal polynomial counting the loci of a prescribed type of singularity appearing in maps. Its variables are universal characteristic classes — Chern, Stiefel–Whitney, Pontrjagin classes etc. — depending on the context. Given a map, by substituting the classes of the source manifold and the pull-back of the target manifold by the map, we can obtain the Poincaré dual of the singular locus of the map. Thus, one can say that *Thom polynomials relate the local classification of singularities to the global topology of*

$$
\begin{tikzcd}
 & K \arrow[drr, "f"] & & \\
S^3 \arrow[ur, "p"] \arrow[r, "p"] & M(G) \arrow[u, "\S 5.3.3"'] \arrow[r, "g_G"] \arrow[d, "r"'] & \mathbb{R}^4 \arrow[r, hook, "j"] & \mathbb{R}^5 = S^5_\varepsilon - \{1\text{pt.}\} \\
 & M(G) \arrow[ur, "\bar{g}_G"'] & &
\end{tikzcd}
$$

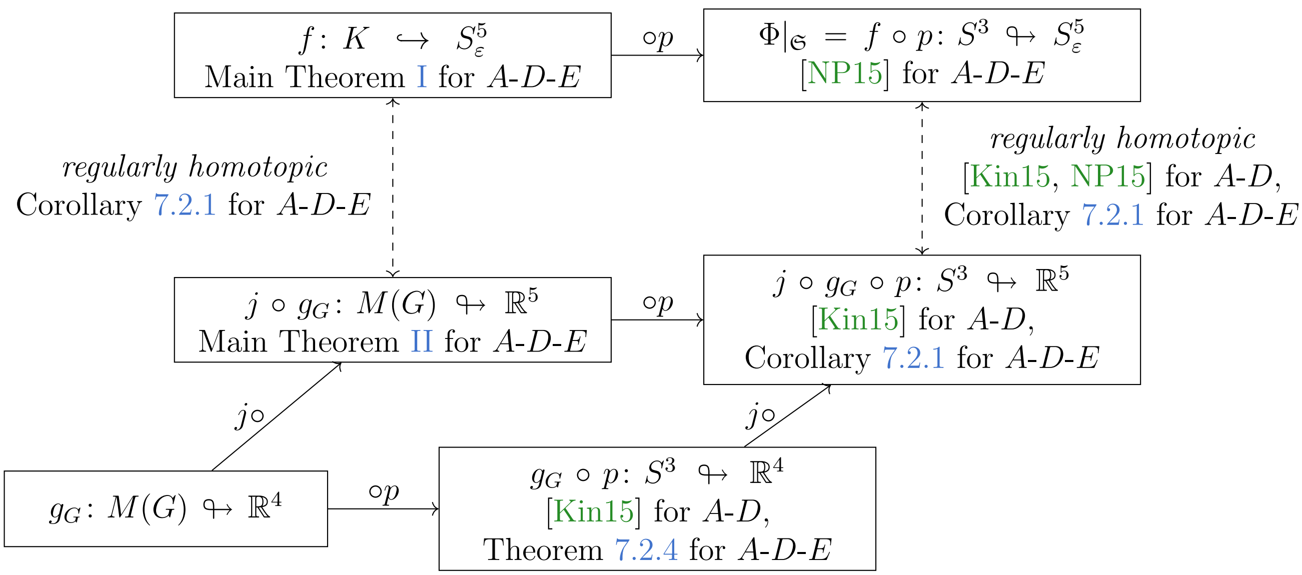

Figure 1.1: Overview of our study

manifolds and maps. The theory was established by R. Thom [Tho56] and developed into a huge unification of enumerative geometry, e.g., Schubert calculus [Por71] and Multiple point formulae [Her81], and has been studied up to now [Kaz06, Rim01, Rim24, Ohm09, Ohm16, Ohm24, Ohm25]. We will briefly review Thom polynomial theory in Chapter 2 and its applications to immersions in Chapter 3.

Organization

We organize this thesis as follows. In Chapter 2, we give a brief overview of singularity theory and Thom polynomial theory. In Chapter 3, we review the classification of immersions based on the Smale–Hirsch theory. We partially explain how Thom polynomial theory applies to the classification. In Chapter 4, we review isolated surface singularities in $\mathbb{C}^3$, in particular, simple singularities. We also list immersions studied in this thesis and summarize the work of Némethi–Pintér and Kinjo. In Chapter 5, we review contact and almost contact structures and reconstruct Kinjo's immersions. In Chapter 6, we study geometric properties of the Wu invariant. In Chapter 7, we determine the regular homotopy classes of immersions listed in Chapter 4. In Chapter 8, we give examples of pairs of distinct singularity links that are diffeomorphic as abstract 3-manifolds, and show that their inclusion maps into the 5-sphere are not regularly homotopic. In Chapter 9, we present our further problems. In Appendix A, we study additional properties of the Wu invariant.

In this thesis, the terms "manifold" and "map" always refer to those of class C^∞. We orient the boundary of an oriented manifold by the 'outward normal first' convention.

The contents of this thesis are based on the paper [Tan25a] of the author.

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Chapter 2

Singularities and Thom polynomials

In this chapter, we review the Thom polynomial theory by briefly reviewing several notions of singularity theory. For details, see [AGV85, Tho56, MatI, MatII, MatIII, MatIV, MatV, MatVI, Ohm94, Kaz06, Ohm16, MN20, Rua22, Rim24, Ohm25].

Let $\mathbb{K}$ denote the field $\mathbb{R}$ or $\mathbb{C}$. By manifolds, maps, and map-germs, we mean real C^∞ ones if $\mathbb{K} = \mathbb{R}$ and complex analytic (i.e., holomorphic) ones if $\mathbb{K} = \mathbb{C}$.

2.1 Jets

Let M, N be manifolds of dimensions m, n, respectively. For points $p \in M$ and $q \in N$, let $\mathcal{M}(M,N)_{(p,q)}$ denote the space of all map-germs $f\colon (M,p) \to (N,q)$. Let $\mathcal{A}$ denote the group consisting of all pairs (φ, ψ) of diffeomorphism-germs $\varphi\colon (M,p) \to (M,p)$ and $\psi\colon (N,q) \to (N,q)$. We define the action of $\mathcal{A}^k$ on $\mathcal{M}(m,n)_{(p,q)}$ by

$$(\varphi, \psi) \cdot f \coloneqq \psi \circ f \circ \varphi^{-1}.$$

Definition 2.1.1 ($\mathcal{A}$-equivalence)**.** Two map-germs are $\mathcal{A}$-*equivalent* if they belong to the same $\mathcal{A}^k$-orbit.

The above equivalence relation naturally arises in the study of singularities of map-germs. One of the difficulties in this study is the fact that the space of all map-germs is infinite-dimensional. The following notion enables us to apply techniques of finite-dimensional geometry.

Definition 2.1.2 (jet)**.** Let $k \geqslant 0$ be an integer, $p \in M$, and $q \in N$. Two map-germs $f, g\colon (M,p) \to (N,q)$ are said to *have the same k-jets* if they have the same l-th derivatives at p for every $1 \leqslant l \leqslant k$. The equivalence class which is represented by a map-germ f is called the *k-jet* of f at p, and denoted by $j^k f(p)$. The quotient space of $\mathcal{M}(m,n)_{(p,q)}$ by the equivalence relationship is denoted by $J^k(M,N)_{(p,q)}$.

We denote $J^k(m,n) \coloneqq J^k(\mathbb{K}^m, \mathbb{K}^n)_{(0,0)}$ and call it the *k-jet space*. This space is often identified with the Euclidean space of dimension $n \cdot \binom{m+k}{k} - 1$, by arranging differential coefficients. Note that $J^1(m,n)$ is the space of all linear maps from $\mathbb{K}^m$ to $\mathbb{K}^n$.

The notion of $\mathcal{A}$-equivalence descends to jet spaces in the following natural way. Let $\mathcal{A}^k$ denote the group consisting of all pairs $(j^k\varphi(0), j^k\psi(0))$ of k-jets of diffeomorphism-germs $\varphi\colon (\mathbb{K}^m, 0) \to (\mathbb{K}^m, 0)$ and $\psi\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$. We define an action of $\mathcal{A}^k$ on $J^k(m, n)$ by

$$(j^k\varphi(0), j^k\psi(0)) \cdot j^k f(0) \coloneqq j^k(\psi \circ f \circ \varphi^{-1})(0).$$

Definition 2.1.3 ($\mathcal{A}^k$-equivalence)**.** Two k-jets are $\mathcal{A}^k$-*equivalent* if they belong to the same $\mathcal{A}^k$-orbit.

Moreover, the notion of jet is globalized as follows.

Definition 2.1.4 (jet bundle)**.** The k-th jet bundle of M and N is the fiber bundle

$$\pi\colon J^k(M, N) \coloneqq \bigcup_{(p,q)\in M\times N} J^k(M, N)_{(p,q)} \to M \times N$$

with fiber $J^k(m, n)$, whose local trivializations are defined in the following way. Let $(U, \varphi), (V, \psi)$ be charts of M, N centered at p, q, respectively. Then we trivialize $\pi^{-1}(U \times V)$ by

$$\pi^{-1}(U \times V) \to U \times V \times J^k(m, n); \quad j^k(f)(p) \mapsto (p, f(p), j^k(\psi \circ f \circ \varphi^{-1})(0)).$$

Remark 2.1.5. The structure group of the bundle $J^k(M, N)$ is $\mathcal{A}^k$. Since $\mathcal{A}^1 = \mathrm{GL}(m) \times \mathrm{GL}(n)$, the group $\mathcal{A}^k$ retracts onto $\mathrm{O}(m) \times \mathrm{O}(n)$ if $\mathbb{K} = \mathbb{R}$ and $\mathrm{U}(m) \times \mathrm{U}(n)$ if $\mathbb{K} = \mathbb{C}$, and then the induced transition functions are the same as those of the product bundle $T^*M \times TN$.

Definition 2.1.6 (jet extension)**.** Let $f\colon M \to N$ be a map between manifolds and $k \geqslant 0$. The *k-jet extension of f* is the map

$$j^k f\colon M \to J^k(M, N); \quad x \mapsto j^k f(x).$$

2.2 Stable singularities

Again, let M, N be manifolds of dimensions m, n, respectively.

Definition 2.2.1 (deformation)**.** Let $f\colon (M, p) \to (N, q)$ be a map-germ. A *deformation* of f is a map-germ $F\colon (M \times \mathbb{K}^s, (p, 0)) \to (N, q)$ such that $F|_{M\times\{0\}} = f$. A deformation F is said to be *trivial* if there are two diffeomorphism-germs

$$\Phi\colon (M \times \mathbb{K}^s, (p, 0)) \to (M \times \mathbb{K}^s, (p, 0)), \quad \Psi\colon (N \times \mathbb{K}^s, (q, 0)) \to (N \times \mathbb{K}^s, (q, 0))$$

which are deformations of diffeomorphism-germs on $(M, p), (N, q)$, respectively, such that

$$\Psi \circ (F, \mathrm{pr}_2) \circ \Phi^{-1} = f \times \mathrm{id}_{(\mathbb{K}^s,0)},$$

where $\mathrm{pr}_2\colon (M \times \mathbb{K}^s, (p, 0)) \to (\mathbb{K}^s, 0)$ is the projection.

Definition 2.2.2 (local stability)**.** A map-germ $f\colon (M,p) \to (N,q)$ is *stable* if every deformation of the germ $f\colon (M,p) \to (N, f(p))$ is trivial. Moreover, a map $f\colon M \to N$ is *locally stable* if at each point $p \in M$, the germ $f\colon (M,p) \to (N, f(p))$ is stable.

We call stable map-germs which are singular at base points *stable singularities*. Stable singularities have been classified up to $\mathcal{A}$-equivalence for many dimension pairs. The following results were already known by Morse and Whitney.

Example 2.2.3 (classical $\mathcal{A}$-classification of stable singularities)**.** Stable singularities are classified into the following types.

- for $n = 1$, only Morse type $(x_1, \dots, x_m) \mapsto \pm x_1^2 \cdots \pm x_m^2$;
- for $n \geqslant 2m$, there are no stable singularities;
- for $n = 2m-1$, only Whitney umbrella: $(x, y_1, \dots, y_{m-1}) \mapsto (x^2, xy_1, \dots xy_{m-1}, y_1, \dots, y_{m-1})$. In particular, for $(m,n) = (2,3)$, only $(x,y) \mapsto (x^2, xy, y)$;
- for $m \geqslant n = 2$, Fold: $(x_1, \dots, x_m) \mapsto (x_1, \pm x_2^2 \cdots \pm x_m^2)$ and Cusp: $(x_1, \dots, x_m) \mapsto (x_1, x_1x_2 \pm x_2^3 \pm x_3^3 \cdots \pm x_m^2)$.

Furthermore, stable singularities in other dimensions were classified using the *Thom–Mather theory*, see [MatI, MatII, MatIII, MatIV, MatV, MatVI, AGV85, MN20] for the details.

Example 2.2.4. For the case where $\mathbb{K} = \mathbb{R}$ and $m = n = 4$, all the stable singularities and their representatives are listed as follows.

- Fold A_1: $(x,y,z,w) \mapsto (x^2, y, z, w)$;
- Cusp A_2: $(x,y,z,w) \mapsto (x^3 + xy, y, z, w)$;
- Swallowtail A_3: $(x,y,z,w) \mapsto (x^4 + x^2y + xz, y, z, w)$;
- Butterfly A_4: $(x,y,z,w) \mapsto (x^5 + x^3y + x^2z + xw, y, z, w)$;
- Parabolic umbilic $\mathrm{I}_{2,2}$: $(x,y,z,w) \mapsto (x^2 + y^2 + xz + yw, xy, z, w)$;
- Hyperbolic umbilic $\mathrm{II}_{2,2}$: $(x,y,z,w) \mapsto (x^2 - y^2 + xz + yw, xy, z, w)$.

Remark 2.2.5. As a consequence of the Thom–Mather theory, we have the following result. Let M^4 be a compact 4-manifold and N^4 a 4-manifold. Consider the space $C^\infty(M,N)$ of all maps from M^4 to N^4 endowed with the Whitney topology. Then the subset of all locally stable maps (i.e., maps which admit only the singularities listed in Example 2.2.4) is dense in $C^\infty(M,N)$. Such maps are also said to be *generic*.

2.3 Thom polynomials

For a space X, let $\bar{H}_*(X)$ denote the Borel–Moore homology group (the homology group made from cycles with closed support) of X.

Definition 2.3.1 (singularity type)**.** A *singularity type* is a $\mathcal{A}^k$-invariant submanifold $\eta \subset J^k(m,n)$ such that its closure $\overline{\eta}$ in $J^k(m,n)$ forms a Whitney stratified cycle of $J^k(m,n)$ with closed support, i.e, there exists a class of $\bar{H}_{\dim \overline{\eta}}(\overline{\eta})$ such that for any point $x \in \overline{\eta}$, the image of the class generates $\bar{H}_{\dim \overline{\eta}}(\overline{\eta}, \overline{\eta} - \{x\})$, for some coefficient. The *codimension* q of a singularity type η is the codimension of η in $J^k(m,n)$.

Given a singularity type $\eta \subset J^k(m,n)$, we can induce the subbundle of $J^k(M,N)$ with fiber η, which is denoted by $\eta(M,N)$.

Definition 2.3.2 (singular locus)**.** Let $f \colon M \to N$ be a map between manifolds and $\eta \subset J^k(M,N)$ a singularity type. The *η-singular locus* of f is the subset

$$\eta(f) \coloneqq (j^k f)^{-1}(\eta(M,N)) \subset M.$$

A point of $\eta(f)$ is called an *η-singularity* of f.

For a singularity type $\eta \subset J^k(m,n)$, we say that a map $f \colon M \to N$ is *generic with respect to* η if its k-jet extension is transverse to all strata of $\overline{\eta}$. Notice the following implications for any singularity type $\eta \subset J^k(m,n)$ and any map $f \colon M \to N$:

$$\begin{aligned} f \text{ is locally stable } &\implies f \text{ is generic with respect to } \eta \\ &\implies \overline{\eta(f)} \text{ forms a Whitney stratified cycle of } M, \end{aligned}$$

where $\overline{\eta(f)}$ is the closure of $\eta(f)$ in M.

Theorem–Definition 2.3.3 (Thom polynomial, real C^∞- and $\mathbb{Z}_2$-coefficient version [Tho56, HK56/57])**.** *Consider the case where $\mathbb{K} = \mathbb{R}$. Let $\eta \subset J^k(m,n)$ be a singularity type of codimension q. Then there exists a unique homogeneous polynomial*

$$\mathrm{tp}(\eta) \in \mathbb{Z}_2[x_1, \ldots, x_m, y_1, \ldots, y_n]$$

of degree q ($\deg x_i = i$, $\deg y_j = j$) which satisfies the following condition: for any closed manifold M of dimension m, any manifold N of dimension n, and any map $f \colon M \to N$ generic with respect to η, it holds that

$$[\overline{\eta(f)}]^* = \mathrm{tp}(\eta)(w_1(M), \ldots, w_m(M), f^*w_1(N), \ldots, f^*w_n(N)) \in H^q(M; \mathbb{Z}_2),$$

where the left hand side is the Poincaré dual of the homology class of the cycle $\overline{\eta(f)}$. The polynomial $\mathrm{tp}(\eta)$ is called the Thom polynomial *of η (with $\mathbb{Z}_2$-coefficient).*

For a Lie group G, let $EG \to BG$ denote the universal principal G-bundles. Also, for a space V equipped with an action of G, let $BV \coloneqq EG \times_G V \to BG$ denote the universal bundle with fiber V and structure group G (the Borel construction).

Proof of Theorem–Definition 2.3.3. We regard the jet bundle $J^k(M,N)$ as a vector bundle whose structure group is $\mathrm{O}(m) \times \mathrm{O}(n)$ by the reduction of the structure group (see Remark 2.1.5). Take its classifying map

$$\kappa\colon M \times N \to B(\mathrm{O}(m) \times \mathrm{O}(n)) = B\mathrm{O}(m) \times B\mathrm{O}(n).$$

Note that this is homotopic to the classifying map of $T^*M \times TN$.

Let $\eta \subset J^k(m,n)$ be a singularity type of codimension q. By Definition 2.3.1, η forms the subbundle $B\eta \subset BJ^k(m,n)$ and its Poincaré dual $[B\overline{\eta}]^* \in H^q(BJ^k(m,n);\mathbb{Z}_2)$. Since $J^k(m,n)$ is contractible, the pull-back

$$H^q(BJ^k(m,n);\mathbb{Z}_2) \leftarrow H^q(B\mathrm{O}(m) \times B\mathrm{O}(n);\mathbb{Z}_2)$$

is an isomorphism. Furthermore, it is well-known that

$$H^*(B\mathrm{O}(m) \times B\mathrm{O}(n);\mathbb{Z}_2) \cong H^*(B\mathrm{O}(m);\mathbb{Z}_2) \otimes H^*(B\mathrm{O}(n);\mathbb{Z}_2) = \mathbb{Z}_2[x_1,\ldots,x_m,y_1,\ldots,y_n],$$

where x_i (resp. y_j) is the i-th (resp. j-th) Stiefel–Whitney class of the universal vector bundle $B\mathbb{R}^m \to B\mathrm{O}(m)$ (resp. $B\mathbb{R}^n \to B\mathrm{O}(n)$). Hence, the class $[B\overline{\eta}]^* \in H^q(BJ^k(m,n);\mathbb{Z}_2)$ corresponds to a unique polynomial P in $\mathbb{Z}_2[x_1,\ldots,x_m,y_1,\ldots,y_n]$. It is easy to show that P is the desired $\mathrm{tp}(\eta)$ by chasing following diagrams:

$$\begin{tikzcd}
 & J^k(M,N) \arrow[r] \arrow[d] & BJ^k(m,n) \arrow[d] \\
M \arrow[ur, "j^k f"] \arrow[r, "(\mathrm{id},f)"'] & M \times N \arrow[r, "\kappa"'] & B\mathrm{O}(m) \times B\mathrm{O}(n),
\end{tikzcd}$$

$$\begin{tikzcd}
 & H^q(J^k(M,N);\mathbb{Z}_2) \arrow[dl, "(j^k f)^*"'] & H^q(BJ^k(m,n);\mathbb{Z}_2) \arrow[l] \\
H^q(M;\mathbb{Z}_2) & H^q(M \times N;\mathbb{Z}_2) \arrow[l, "(\mathrm{id},f)^*"] \arrow[u, "\cong"] & H^q(B\mathrm{O}(m) \times B\mathrm{O}(n);\mathbb{Z}_2). \arrow[l, "\kappa^*"] \arrow[u, "\cong"]
\end{tikzcd}$$

□

Remark 2.3.4. Theorem 2.3.3 holds also for the following cases:

- $\mathbb{K} = \mathbb{C}$ and the coefficient of (co)homology is $\mathbb{Z}$. In this case, the variables should be changed to Chern classes.
- $\mathbb{K} = \mathbb{R}$ and the coefficient of (co)homology is $\mathbb{Z}$ if M, N are both orientable and η is so-called coorientable in $J^k(m,n)$. In this case, the variables should be changed to Pontrjagin classes and integral Stiefel–Whitney classes (cf. [MS74, Problem 15-C]).

Remark 2.3.5 ($\mathcal{K}$-equivalence and Thom polynomials)**.** There is a group larger than $\mathcal{A}$, which is denoted by $\mathcal{K}$. Thom polynomials are well computed for $\mathcal{K}$-invariant singularity types using algebraic techniques [Rim01, FR02]. In fact, for every $\mathcal{K}$-invariant type η and every map $f\colon M \to N$, the Thom polynomial $\mathrm{tp}(\eta)(f)$ is expressed by a polynomial of variables $w_i(f) \in H^i(M;\mathbb{Z}_2)$, where $w_i(f)$ is the i-th part of the quotient

$$w(f) \coloneqq \frac{w(f^*TN)}{w(TM)} = \frac{1 + f^*w_1(N) + f^*w_2(N) + \cdots}{1 + w_1(M) + w_2(M) + \cdots}.$$

In the oriented or complex case, the Thom polynomial is expressed in a similar way.

It is also known that for two stable singularities f and g, their $\mathcal{K}$-equivalence is equivalent to their $\mathcal{A}$-equivalence (Mather's classification theorem [MatIV]). Note that for singularity types η and $\zeta \subset J^k(m,n)$ such that ζ is $\mathcal{K}$-invariant and $\overline{\eta} = \overline{\zeta}$, it holds that $\mathrm{tp}(\eta) = \mathrm{tp}(\zeta)$.

We end this chapter recalling two computational results which will be used in Chapter 3. To state them, we recall the *Thom–Boardman singularity types*. First, we define

$$\Sigma^i \coloneqq \{j^k f(0) \mid \dim \operatorname{Ker} j^1 f(0) = i\} \subset J^k(m,n).$$

This subset forms a singularity type of codimension $i(i + |n-m|)$. Second, we recall that there is a singularity type $\Sigma^{i,j} \subset J^k(m,n)$ which satisfies the following property: for every Σ^i-transverse map $f\colon M^m \to N^n$ (hence, $\Sigma^i(f) \subset M$ forms a submanifold), it holds that

$$\Sigma^{i,j}(f) = \Sigma^j(f|_{\Sigma^i(f)}).$$

Types $\Sigma^{i,j,k,\cdots}$ are defined similarly. See [Boa67, GG80] for the direct definition.

Theorem 2.3.6 ([Tho56, Ron71, FR02])**.** *Let M be a closed oriented 4-manifold, N an oriented 4-manifold, and $f\colon M \to N$ a locally stable C^∞ map. Then it holds that*

$$\mathrm{tp}(\Sigma^2)(f) = p_1(f) = f^*p_1(N) - p_1(M) \in H^4(M;\mathbb{Z}).$$

In particular, if $N = \mathbb{R}^4$, then it holds that

$$\mathrm{tp}(\Sigma^2)(f) = -p_1(M) \in H^4(M;\mathbb{Z}), \quad \textit{i.e.,} \quad \#\Sigma^2(f) = -3\sigma(M) \in \mathbb{Z},$$

where $\#\Sigma^2(f)$ is the algebraic number of Σ^2-singularities of f.

Remark 2.3.7. In Example 2.2.4, only the two singularity types $\mathrm{I}_{2,2}$ and $\mathrm{II}_{2,2}$ are of corank 2 at the origin. Therefore, the closure of the disjoint union $\mathrm{I}_{2,2} \cup \mathrm{II}_{2,2}$ in $J^k(4,4)$ coincides with that of Σ^2.

Theorem 2.3.8 ([Szű00])**.** *Let M be a closed oriented 4-manifold and $f\colon M \to \mathbb{R}^5$ a locally stable C^∞ map. Then it holds that*

$$\mathrm{tp}(\Sigma^{1,1})(f) = -p_1(M) \in H^4(M;\mathbb{Z}), \quad \textit{i.e.,} \quad \#\Sigma^{1,1}(f) = -3\sigma(M) \in \mathbb{Z},$$

where $\#\Sigma^{1,1}(f)$ is the algebraic number of $\Sigma^{1,1}$-singularities of f.

Chapter 3

Immersions of 3-manifolds

In this chapter, we review the classification of immersions up to regular homotopy. In particular, we focus on immersions of 3-manifolds into 4- and 5-spaces, and recall their complete invariants and formulae. See the original papers and [Pin18] for more details.

3.1 Smale–Hirsch theory

Given manifolds M, N, let $\mathrm{Imm}(M, N)$ denote the space of all immersions of M into N, endowed with the C^∞ topology.

Definition 3.1.1 (regular homotopicity)**.** Two immersions $f_0, f_1 \colon M \looparrowright N$ are said to be *regularly homotopic* if there exists a path $\{f_t\}_{t\in[0,1]}$ in $\mathrm{Imm}(M, N)$. Such a path is called a *regular homotopy* between f_0 and f_1.

Let $\mathrm{Imm}[M, N] \coloneqq \pi_0(\mathrm{Imm}(M, N))$ denote the set of all *regular homotopy classes* of immersions of M into N.

Let us begin with immersions of the spheres. For integers $m \leqslant n$, let $V_{n,m}$ denote the Stiefel manifold — the space of all orthonormal m-frames in $\mathbb{R}^n$.

Definition 3.1.2 (Smale invariant)**.** The *Smale invariant* is a map

$$\Omega \colon \mathrm{Imm}[S^m, \mathbb{R}^n] \to \pi_m(V_{n,m})$$

defined as follows. Let $f \colon S^m \looparrowright \mathbb{R}^n$ be an immersion. Let $D^m_\pm \subset S^m$ denote closed northern and southern hemispheres, respectively, and put $f_\pm \coloneqq f|_{D^m_\pm}$. We assume that f_- is the standard inclusion. Now take a parallelization $\tau \colon TD^m_+ \to D^m_+ \times \mathbb{R}^m$ of D^m_+, and consider the representation matrix

$$A_{\tau,f_+} \colon D^m_+ \to (\mathbb{R}^n)^m, \quad x \mapsto \begin{bmatrix} df_x(e_1) & \cdots & df_x(e_m) \end{bmatrix},$$

where e_i is the tangent vector on D^m_+ that corresponds to the i-th standard vector via τ. By orthonormalizing A_{τ,f_+}, we have the map

$$\varphi_{\tau,f_+} \colon D^m_+ \to V_{n,m}.$$

Then define

$$\Omega(f)\colon S^m \to V_{n,m}, \quad x \mapsto \begin{cases} \varphi_{\tau,f_+}(x) & (x \in D^m_+), \\ \varphi_{\tau,f_-}(r(x)) & (x \in D^m_-), \end{cases}$$

where $r\colon \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the reflection on the last coordinate. Its homotopy class is also denoted by $\Omega(f)$, and is called the *Smale invariant* of f.

Theorem 3.1.3 ([Sma59b])**.** *The Smale invariant Ω is a group isomorphism, where the group structure on $\mathrm{Imm}[S^k, \mathbb{R}^n]$ is given by the connected sum.*

This result includes several classical results.

Example 3.1.4 (plane curves)**.** The classification of immersions $S^1 \looparrowright \mathbb{R}^2$ is obtained by

$$\mathrm{Imm}[S^1, \mathbb{R}^2] \cong \pi_1(V_{2,1}) = \pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}.$$

In this case, the Smale invariant is equal to the rotation number minus one. This recovers the work of Whitney–Graustein [Whi37].

Example 3.1.5 (sphere eversion)**.** The classification of immersions $S^2 \looparrowright \mathbb{R}^3$ is obtained by

$$\mathrm{Imm}[S^2, \mathbb{R}^3] \cong \pi_2(V_{3,2}) = \pi_2(\mathrm{SO}(3)) = 0.$$

It follows that the standard inclusion $S^2 \hookrightarrow \mathbb{R}^3$ and its reflection are regularly homotopic. It has been pointed out by Smale himself [Sma59a]. (cf. [FM80] for an explicit procedure of eversion.)

Theorem 3.1.3 was generalized to the following statement. Let $\mathrm{Mon}(TM, TN)$ denote the space of all fiberwise injective homomorphisms between the tangent bundles TM and TN, endowed with the compact open topology. We also put $\mathrm{Mon}[TM, TN] \coloneqq \pi_0(\mathrm{Mon}(TM, TN))$.

Theorem 3.1.6 ([Hir59])**.** *Let M, N be manifolds such that $\dim M < \dim N$ unless M is open. Then the natural injection*

$$\mathrm{Imm}(M, N) \hookrightarrow \mathrm{Mon}(TM, TN), \quad f \mapsto df$$

is weak homotopy equivalence. In particular, the induced map

$$\mathrm{Imm}[M, N] \to \mathrm{Mon}[TM, TN]$$

is a bijection.

Example 3.1.7 (parallelizable case)**.** Consider the case where the source manifold M is *parallelizable* (i.e., the tangent bundle TM is trivial) and the target manifold N is the Euclidean space $\mathbb{R}^n$. Then we have that

$$\mathrm{Imm}[M, N] = \mathrm{Mon}[M \times \mathbb{R}^m, \mathbb{R}^n \times \mathbb{R}^n] = [M, \mathbb{R}^n \times V_{n,m}] = [M, V_{n,m}],$$

where each equality means bijectivity.

In the following, we will consider immersions of oriented 3-manifolds. Let M^3 be a closed connected oriented 3-manifold. It is well-known that M^3 is parallelizable, so we can deduce the following results.

- Immersions of M^3 into 6-space are classified by $[M^3, V_{6,3}] \cong \pi_3(V_{6,3}) \cong \mathbb{Z}_2$, since $V_{6,3}$ is 2-connected. Given an immersion $f\colon M^3 \to \mathbb{R}^6$, the corresponding $\mathbb{Z}_2$-value is called the *self-intersection number* of f in [Whi44].
- For any integer $n \geqslant 7$, immersions of M^3 into n-space are classified by $[M^3, V_{n,3}] = \{*\}$, since $V_{n,3}$ is $(n-4)$-connected. This means all immersions are mutually regularly homotopic.
- Moreover, M^3 admits no immersions into 3-space by the inverse mapping theorem.

The remaining non-trivial cases are immersions into 4- and 5-spaces.

3.2 Conventions and facts

For our later use, we choose generators of the homotopy groups $\pi_3(\mathrm{SO}(4))$ and $\pi_3(\mathrm{SO}(5))$.

By Theorem 3.1.3, we have that

$$\mathrm{Imm}[S^3, \mathbb{R}^4] \cong \pi_3(V_{4,3}) \cong \pi_3(\mathrm{SO}(4)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

We fix the last isomorphism as follows, referring to [Ste51, ET11, Kin15, NP15]. We identify $\mathbb{R}^4$ with the vector space $\mathbb{H}$ of all quaternions by identifying standard basis e_1, e_2, e_3, e_4 with $1, i, j, k$, respectively. We also regard S^3 as the set of all unit quaternions. For $x, y \in \mathbb{H}$, let $x \cdot y$ denote their quaternionic product. Now we consider the principal SO(3)-bundle $\pi\colon \mathrm{SO}(4) \to S^3$, where $\pi(R) = R(e_1)$ for any $R \in SO(4)$. Then the map

$$\sigma\colon S^3 \to \mathrm{SO}(4); \quad \sigma(x)(y) = x \cdot y$$

forms a section of π. Hence, $\pi\colon \mathrm{SO}(4) \to S^3$ is trivial, i.e., $\mathrm{SO}(4) \cong S^3 \times \mathrm{SO}(3)$. Furthermore, the map

$$\rho\colon S^3 \to \mathrm{SO}(4); \quad \rho(x)(y) = x \cdot y \cdot x^{-1}$$

forms a universal covering map over the fiber $\mathrm{SO}(3) \subset \mathrm{SO}(4)$. Therefore, we have that

$$\pi_3(\mathrm{SO}(4)) \cong \pi_3(S^3 \times \mathrm{SO}(3)) \cong \pi_3(S^3) \oplus \pi_3(\mathrm{SO}(3)) = \mathbb{Z}[\sigma] \oplus \mathbb{Z}[\rho] \cong \mathbb{Z} \oplus \mathbb{Z},$$

where the last isomorphism is the standard identification.

Again, by Theorem 3.1.3, we have that

$$\mathrm{Imm}[S^3, \mathbb{R}^5] \cong \pi_3(V_{5,3}) \cong \pi_3(\mathrm{SO}(5)) \cong \mathbb{Z}.$$

We fix the last isomorphism by choosing the generator of $\pi_3(\mathrm{SO}(5))$ to be

$$\iota \circ \sigma\colon S^3 \to \mathrm{SO}(4) \hookrightarrow \mathrm{SO}(5),$$

where $\iota\colon \mathrm{SO}(4) \hookrightarrow \mathrm{SO}(5)$ is the standard inclusion (see also Remark 3.4.6 below).

We note several results.

Proposition 3.2.1 (cf. [Hug92, Lemma 2.5]). *Let $g\colon S^3 \looparrowright \mathbb{R}^4$ be an immersion and $r\colon S^3 \to S^3$ an orientation-reversing diffeomorphism. Then it holds that*

$$\Omega(g \circ r) = -\Omega(g) + (-2, 1).$$

Proposition 3.2.2. *Let $f\colon S^3 \looparrowright \mathbb{R}^5$ be an immersion and $r\colon S^3 \to S^3$ an orientation-reversing diffeomorphism. Then it holds that*

$$\Omega(f \circ r) = -\Omega(f).$$

Let $j\colon \mathbb{R}^4 \hookrightarrow \mathbb{R}^5$ denote the standard inclusion. We consider the operation

$$j_*\colon \mathrm{Imm}[S^3, \mathbb{R}^4] \to \mathrm{Imm}[S^3, \mathbb{R}^5], \quad [g] \mapsto [j \circ g],$$

which forms a group homomorphism.

Proposition 3.2.3 ([Ste51, §22.7 and §23.6], see also [Hug92, Lemma 2.4]). *Regard the map j_* as a homomorphism from $\mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$ via our choices of generators. Then it holds that*

$$j_*(a, b) = a + 2b.$$

3.3 Immersions of the 3-sphere into 4-space

Definition 3.3.1 (singular Seifert surface). For an immersion $g\colon S^3 \looparrowright \mathbb{R}^4$, a *singular Seifert surface* of g is a generic (see Remark 2.2.5) map $\hat{g}\colon X^4 \to \mathbb{R}^4$ from a compact oriented 4-manifold X^4 bounded by S^3 satisfying that $\hat{g}|_{S^3} = g$ and has no singularities near the boundary.

Ekholm–Takase presented the following formula based on Hughes' work.

Theorem 3.3.2 ([Hug92, ET11]). *Let $g\colon S^3 \looparrowright \mathbb{R}^4$ be an immersion and $\hat{g}\colon X^4 \to \mathbb{R}^4$ a singular Seifert surface of g. Then the Smale invariant of g is given by*

$$\Omega(g) = \left(D(g) - 1, \frac{3\sigma(X^4) + \#\Sigma^2(\hat{g}) - 2(D(g) - 1)}{4} \right),$$

where $D(g)$ is the normal mapping degree of g and $\#\Sigma^2(\hat{g})$ is the algebraic number of Σ^2-singularities of $\hat{g}$ (see Theorem 2.3.6).

To prove this formula, it is necessary to show the following.

Lemma 3.3.3. *The quantity $3\sigma(X^4) + \#\Sigma^2(\hat{g})$ is independent of the choice of singular Seifert surface $\hat{g}\colon X^4 \to \mathbb{R}^4$ and determined by only the regular homotopy class of g.*

Here, we prove only this lemma, to which Thom polynomial theory is applied.

Proof of Lemma 3.3.3. Let $g_i\colon S^3 \looparrowright \mathbb{R}^4$ be an immersion ($i = 0, 1$), $G\colon S^3 \times [0,1] \to \mathbb{R}^4$ a regular homotopy between g_0 and g_1, and $\widehat{g_i}\colon X_i^4 \to \mathbb{R}^4$ a singular Seifert surface of g_i. Then we define the manifold V^4 and the map G' by

$$V^4 = X_0^4 \cup_\partial (S^3 \times [0,1]) \cup_\partial (-X_1^4),$$

$$G' = g_0 \cup G \cup (-g_1)\colon V^4 \to \mathbb{R}^4.$$

Take a generic perturbation of G' and denote it by G''. Since V^4 is a closed oriented 4-manifold, Theorem 2.3.6 implies that

$$\#\Sigma^2(G'') = -3\sigma(V^4).$$

It is obvious that

$$\#\Sigma^2(G'') = \#\Sigma^2(g_0) - \#\Sigma^2(g_1),$$

and by the Novikov additivity, it holds that

$$-3\sigma(V^4) = -3\sigma(X_0^4) + 3\sigma(X_1^4).$$

Therefore, we have that

$$3\sigma(X_0^4) + \#\Sigma^2(g_0) = 3\sigma(X_1^4) + \#\Sigma^2(g_1),$$

which completes the proof. □

Remark 3.3.4. In fact, the quantity $-(3\sigma(X^4) + \#\Sigma^2(\hat{g}))$ coincides with the so-called *Hirzebruch defect* of the stable framing on S^3 induced by the immersion g. See [KM99] for the definition and [ET11, Proposition 2.7] for the coincidence.

Remark 3.3.5. If the singular Seifert surface $\hat{g}$ can be taken to be an immersion, then the normal mapping degree $D(g)$ is equal to the Euler number $\chi(X^4)$ of the 4-manifold X^4. See [KM99].

Remark 3.3.6. Hughes [Hug92] chose the opposite sign of the generator of $\mathrm{Imm}[S^3, \mathbb{R}^5] \cong \pi_3(\mathrm{SO}(5))$ to Ekholm–Takase's [ET11], and hence to ours. Ekholm–Takase corrected it and obtained the above formula. See [ET11, Remark 2.1].

3.4 Immersions of the 3-sphere into 5-space

Definition 3.4.1 (singular Seifert surface)**.** For an immersion $f\colon S^3 \looparrowright \mathbb{R}^5$, a *singular Seifert surface* of f is a generic map $\hat{f}\colon X^4 \to \mathbb{R}^5$ from a compact oriented 4-manifold X^4 bounded by S^3, which has no singularities near the boundary.

Ekholm–Szűcs presented the following formula based on Hughes–Melvin's work. Recall the convention fixed in §3.2.

Theorem 3.4.2 ([HM85, ES03]). *Let $f\colon S^3 \looparrowright \mathbb{R}^5$ be an immersion and $\hat{f}\colon X^4 \to \mathbb{R}^5$ its singular Seifert surface. Then the Smale invariant of f is given by*

$$\Omega(f) = \frac{3}{2}\sigma(X^4) + \frac{1}{2}\#\Sigma^{1,1}(f),$$

where $\#\Sigma^2(\hat{f})$ is the algebraic number of $\Sigma^{1,1}$-singularities of $\hat{f}$ (see Theorem 2.3.8).

Némethi–Pintér studied the Smale invariant from another viewpoint as follows [NP15]. Let $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic map-germ which is singular only at the origin. Then, for a sufficiently small positive number ε, the preimage $\mathfrak{S} := \Phi^{-1}(S^5_\varepsilon)$ is canonically diffeomorphic to the 3-sphere S^3. Hence, the restriction of Φ to $\mathfrak{S}$ forms an immersion

$$\Phi|_{\mathfrak{S}}\colon S^3 \looparrowright S^5_\varepsilon.$$

Note that its regular homotopy class is independent of the choices of ε and the orientation-preserving diffeomorphism $\mathfrak{S} \cong S^3$.

Theorem 3.4.3 ([NP15]). *Let $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic map-germ which is singular only at the origin. Consider the immersion $\Phi|_{\mathfrak{S}}\colon S^3 \looparrowright S^5$ introduced above. Then the Smale invariant of $\Phi|_{\mathfrak{S}}$ is given by*

$$\Omega(\Phi|_{\mathfrak{S}}) = -C(\Phi),$$

where $C(\Phi)$ is the number of complex Whitney umbrella singularities appearing in a holomorphic stable perturbation of Φ.

Remark 3.4.4. The classification of 3-manifolds into 5-space up to regular homotopy is completely the same as that into the 5-sphere.

Remark 3.4.5. The quantity $C(\Phi)$ is computable in an algebraic way as follows [NP15, §2.2]. Let M_1, M_2, M_3 denote all 2×2-minors of the Jacobi matrix of Φ, and $\mathcal{O}_{\mathbb{C}^2,0}$ the ring of all holomorphic function germs $(\mathbb{C}^2, 0) \to \mathbb{C}$. Also let I denote the ideal of $\mathcal{O}_{\mathbb{C}^2,0}$ generated by M_1, M_2, M_3. Then it holds that

$$C(\Phi) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,0}/I.$$

Remark 3.4.6. Némethi–Pintér also pointed out the ambiguity of the choice of the generator of $\pi_3(\mathrm{SO}(5))$ (e.g., the sign of the Smale invariant), and fixed it. They showed that the sign in Theorems 3.4.2 and 3.4.3 agree each other [NP15, Theorem 9.1.6] (see also [PS24, Appendix A.3]).

3.5 Immersions of 3-manifolds into 5-space

Let M^3 denote an arbitrary oriented 3-manifold (in §3.5.2, we will also assume that M^3 is closed and connected). First, we refine the argument in Example 3.1.7. *We fix* *a parallelization* $\tau\colon TM^3 \to M^3 \times \mathbb{R}^3$ *of* M^3*.* Then for an immersion $f\colon M^3 \looparrowright \mathbb{R}^5$, its differential $df\colon TM^3 \to T\mathbb{R}^5$ is represented as

$$A_{\tau,f}\colon M^3 \to (\mathbb{R}^5)^3, \quad x \mapsto \begin{bmatrix} df_x(e_1) & df_x(e_2) & df_x(e_3) \end{bmatrix},$$

where e_i is the tangent vector on M^3 which corresponds to the i-th standard vector via τ. By orthonormalizing $A_{\tau,f}$, we have the map

$$\varphi_{\tau,f}\colon M^3 \to V_{5,3} \cong \mathrm{SO}(5)/\mathrm{SO}(2).$$

Then the correspondence

$$\mathrm{Imm}[M^3, \mathbb{R}^5] \to [M^3, V_{5,3}], \quad [f] \mapsto [\varphi_{\tau,f}]$$

is a bijection. By homotopy-theoretical arguments based on this bijection, Wu concluded the following. For a class $\chi \in H^2(M^3;\mathbb{Z})$, let $\mathrm{Imm}[M^3, \mathbb{R}^5]_\chi$ denote the set of all regular homotopy classes of immersions of M^3 into $\mathbb{R}^5$ with normal Euler class χ.

Theorem 3.5.1 ([Wu64, Theorem 2], see also [Li82, SST02, Juh05])**.** *For any immersion* $f\colon M^3 \looparrowright \mathbb{R}^5$*, its normal Euler class is of the form* $2C \in H^2(M^3;\mathbb{Z})$*, where* $C \in H^2(M^3;\mathbb{Z})$*. Furthermore, for any* $\chi \in H^2(M^3;\mathbb{Z})$*, there is a bijection*

$$\mathrm{Imm}[M^3, \mathbb{R}^5]_\chi \to \Gamma_2(\chi) \times H^3(M^3;\mathbb{Z})/(2\chi \smile H^1(M^3;\mathbb{Z})),$$

where

$$\Gamma_2(\chi) \coloneqq \{C \in H^2(M^3;\mathbb{Z}) \mid 2C = \chi\}.$$

We explain two components appearing in the right-hand side of this bijection.

3.5.1 Wu invariant

The Wu invariant is the first component of the bijection in Theorem 3.5.1. For its precise definition, we outline the proof of the former assertion in Theorem 3.5.1. For an immersion $f\colon M^3 \looparrowright \mathbb{R}^5$, its normal bundle is obtained as the pull-back of the natural SO(2)-bundle

$$\rho\colon \mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{SO}(2)$$

by $\varphi_{\tau,f}\colon M^3 \to \mathrm{SO}(5)/\mathrm{SO}(2)$. Applying the Gysin exact sequence, we see that the Euler class of ρ is of the form 2Σ, where Σ is a generator of $H^2(V_{5,3};\mathbb{Z}) \cong \mathbb{Z}$. Then, by the naturality of Euler classes, the normal Euler class χ of f coincides with $2(\varphi_{\tau,f})^*(\Sigma)$. Hence we have that $(\varphi_{\tau,f})^*(\Sigma) \in \Gamma_2(\chi)$.

Definition 3.5.2 (Wu invariant [Wu64, Theorem 2], [SST02, Definition 4])**.** Let $f\colon M^3 \looparrowright \mathbb{R}^5$ be an immersion with normal Euler class χ, and $\varphi_{\tau,f}\colon M^3 \to V_{5,3}$ the induced map. Then the *Wu invariant* of f with respect to τ is defined as the second cohomology class

$$c_\tau(f) \coloneqq (\varphi_{\tau,f})^*(\Sigma) \in \Gamma_2(\chi),$$

where Σ is the generator of $H^2(V_{5,3};\mathbb{Z}) \cong \mathbb{Z}$ so that the Euler class of the natural SO(2)-bundle $\rho\colon \mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{SO}(2)$ coincides with 2Σ.

Clearly, the class $c_\tau(f)$ is invariant up to regular homotopy.

Remark 3.5.3. In [SST02], the class $c_\tau(f)$ is simply denoted by $c(f)$ under the contract that a parallelization is fixed. It is important to specify which is the fixed parallelization, since the class $c_\tau(f)$ depends on the choice of τ (cf. below Definition 4 in [SST02, p.18]). In our results, we will choose a certain parallelization (see Definition 5.2.5). In Appendix A, we also discuss how the Wu invariant changes when the parallelization changes.

Remark 3.5.4. For the case where $\chi = 0$, Saeki–Szűcs–Takase gave another expression of the Wu invariant, which differs from ours in Chapter 6. See [SST02, §3] for details.

3.5.2 Smale-type invariant

The second component of the bijection in Theorem 3.5.1 can be identified with the Smale invariant if the 3-manifold M^3 is the 3-sphere. In the following, we review a part of [SST02, §5], which deals with the case where immersions have trivial normal bundle. See [Juh05] for the general case.

Hereafter we assume that the 3-manifold M^3 is connected and closed. We also use the term "singular Seifert surface" in Definition 3.3.1 for an immersion from M^3 to $\mathbb{R}^5$.

Definition 3.5.5 ([SST02, Definition 5])**.** We define an integer $\alpha(M^3)$ to be the dimension of the $\mathbb{Z}_2$-vector space $\mathrm{Tor}(H_1(M^3;\mathbb{Z}))\otimes_{\mathbb{Z}}\mathbb{Z}_2$.

Definition 3.5.6 (Smale-type invariant [SST02, Definitions 5 and 7])**.** Let $f\colon M^3 \looparrowright \mathbb{R}^5$ be an immersion with trivial normal bundle and $\hat{f}\colon X^4 \to \mathbb{R}^5$ its singular Seifert surface. Then we define the *Smale-type invariant* of f by

$$i(f) := \frac{3}{2}(\sigma(X^4) - \alpha(M^3)) + \frac{1}{2}\#\Sigma^{1,1}(\hat{f}) \in \mathbb{Z},$$

where $\#\Sigma^2(\hat{f})$ is the algebraic number of $\Sigma^{1,1}$-singularities of $\hat{f}$.

Saeki–Szűcs–Takase showed that the integer $i(f)$ is well-defined and invariant up to regular homotopy.

Remark 3.5.7. The term $\alpha(M^3)$ is determined by only the topology of M^3, so it is not needed to determine and compare the regular homotopy classes of given immersions. The role of $\alpha(M^3)$ is to make $i(f)$ $\mathbb{Z}$-valued, because the quantity

$$\frac{3}{2}\sigma(X^4) + \frac{1}{2}\#\Sigma^{1,1}(\hat{f})$$

generally takes values in $\mathbb{Z} + \frac{1}{2}$.

They also showed the following theorem, which will be important in Chapter 7.

Theorem 3.5.8 ([SST02, Theorem 6])**.** *Let τ be a parallelization of M^3. Then the correspondence*

$$(c_\tau, i)\colon \mathrm{Imm}[M^3, \mathbb{R}^5]_0 \to \Gamma_2(0) \times \mathbb{Z}$$

is a bijection, i.e., the regular homotopy class of a given immersion with trivial normal bundle is determined by its Wu and Smale-type invariants.

Chapter 4

Simple singularities and immersions

This chapter is devoted to review simple singularities and introduce immersions related to them, which are our main subjects. See [Mil68, Sea06, Kas15, Sea19] for details.

4.1 Links and Milnor fibers

Let $h\colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ with an isolated singularity at the origin. The germ itself or its zero set-germ $(h^{-1}(0), 0) \subset (\mathbb{C}^3, 0)$ are called an *isolated surface singularity in* $\mathbb{C}^3$.

Theorem–Definition 4.1.1 (link [Mil68])**.** *For any sufficiently small positive number ε, the following hold:*

(1) the intersection $K_\varepsilon \coloneqq h^{-1}(0) \cap S^5_\varepsilon$ forms a closed oriented 3-manifold embedded into S^5_ε. Furthermore, the diffeomorphism type and isotopy class of $K_\varepsilon \subset S^5_\varepsilon$ are independent of the choice of ε.

(2) the intersection $h^{-1}(0) \cap B^6_\varepsilon$ is homeomorphic to the cone $\mathrm{Cone}(K_\varepsilon)$.

We call $K = K_\varepsilon$ the link *of the singularity.*

Theorem–Definition 4.1.2 (Milnor fiber [Mil68])**.** *Let ε be the positive number as above. Then for another positive number δ which is sufficiently small with respect to ε, the following hold:*

(1) the argument map $h/|h|\colon S^5_\varepsilon - K_\varepsilon \to S^1$ is a locally trivial fibration. Furthermore, the closure of each fiber forms a compact oriented 4-manifold, whose boundary is the link K_ε;

(2) the restriction $h|_{h^{-1}(S^1_\delta)\cap \mathrm{Int}\, B^6_\varepsilon}\colon h^{-1}(S^1_\delta) \cap \mathrm{Int}\, B^6_\varepsilon \to S^1_\delta$ is a locally trivial fibration;

(3) the above two fibrations are mutually isomorphic.

We call the closure F of the fiber of (1) the Milnor fiber *of h.*

Remark 4.1.3. By (3), F is diffeomorphic to $h^{-1}(\delta) \cap B^6_\varepsilon$, and they are often mutually identified. Notice that $h^{-1}(\delta) \cap B^6_\varepsilon$ admits a natural complex structure as a complex submanifold of $\mathbb{C}^3$. Furthermore, the boundary $\partial F = K$ is isotopic to $h^{-1}(\delta) \cap S^5_\varepsilon$ in S^5_ε.

4.2 Simple singularities

For singularities of function-germs $(\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$, Arnol'd introduced the notion of modality and classified the singularities up to modality 0, 1, and 2 [AGV85]. Singularities of modality 0, which appear at the earliest level in the classification hierarchy, are called *simple singularities*, *Kleinian singularities*, *Du Val singularities*, or *rational double points*. All types of simple singularities in the case where $m = 3$ are listed as follows.

- A_{n-1} $(n \geqslant 2)$: $x^2 + y^2 + z^n$
- D_{n+2} $(n \geqslant 2)$: $x^2 + y^2 z + z^{n+1}$
- E_6: $x^2 + y^3 + z^4$
- E_7: $x^2 + y^3 + yz^3$
- E_8: $x^2 + y^3 + z^5$

Their links and Milnor fibers have rich geometric structures. Let us review specific two.

4.2.1 Quotient manifold structures

Klein showed that for each type of simple singularities, its zero set-germ in $(\mathbb{C}^3, 0)$ is isomorphic to the quotient of $(\mathbb{C}^2, 0)$ by a finite subgroup of SU(2) [Kle56] (see also [Mil75]). The corresponding subgroups are listed as follows.

- A_{n-1} $(n \geqslant 2)$: the cyclic group C_n of order n
- D_{n+2} $(n \geqslant 2)$: the binary dihedral group Dic_n of order $4n$
- E_6: the binary tetrahedral group $2T$, whose order is 24
- E_7: the binary octahedral group $2O$, whose order is 48
- E_8: the binary icosahedral group $2I$, whose order is 120

Let $\Gamma \subset \mathrm{SU}(2)$ be one of these groups. The left action of Γ on $\mathbb{C}^2$ can be restricted to a free action on the 3-sphere $S^3 \subset \mathbb{C}^2$. Then the link K is diffeomorphic to the quotient space $\Gamma \backslash S^3$, i.e., the link K admits the universal covering map $p \colon S^3 \to K$ whose covering degree is the order of the group Γ. Note that by the invariant polynomial theory, holomorphic map-germs $\Phi \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ parametrizing simple singularities are known, e.g.,

$$(s,t) \mapsto \left(\frac{1}{2}(s^n - t^n), \frac{\sqrt{-1}}{2}(s^n + t^n), st\right), \quad (s,t) \mapsto \left(\frac{st}{2}(s^{2n} - t^{2n}), \frac{\sqrt{-1}}{2}(s^{2n} + t^{2n}), s^2t^2\right)$$

for the singularities of types A_{n-1} and D_{n+2}, respectively. These will appear in §4.3.2.

4.2.2 Plumbed manifold structures

For every type of simple singularities, the link admits the plumbed 3-manifold structure corresponding to the weighted dual graph of the minimal resolution of the singularity. The underlying graphs are called *Dynkin diagrams*, which are listed in Figure 4.1.

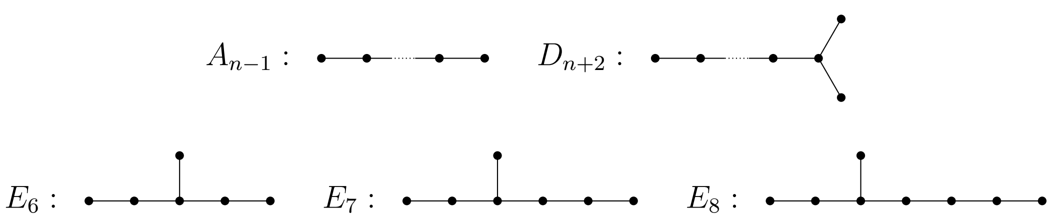

Figure 4.1: Dynkin diagrams of types A-D-E: the number of vertices are corresponding to the indices; weights of vertices and edges are all -2 and 1, respectively.

Let G be one of Dynkin diagrams of types A-D-E, and let $M(G)$ (resp. $X(G)$) denote the plumbed 3- (resp. 4-) manifold corresponding to G. Notice that each component of the plumbing is the unit cotangent bundle UT^*S^2 (resp. the disk cotangent bundle DT^*S^2). For the type of simple singularities corresponding to G, its minimal resolution is diffeomorphic to $X(G)$. Hence, the link K is diffeomorphic to $M(G)$.

As a specific property of simple singularities, the Milnor fiber F is also diffeomorphic to $X(G)$. However, the complex structures on the minimal resolution and the Milnor fiber are quite different in the following sense. Each manifold admits a spine which is expressed as the transverse union of 2-spheres. Spheres in the spine can be taken as complex submanifolds in the minimal resolution (the union of exceptional divisors), while they are totally real submanifolds in the Milnor fiber (the section by real 3-space), where we identify F with the complex manifold $h^{-1}(\delta) \cap B^6_\varepsilon$. This difference will be crucial in §5.3 and §6.3.

4.3 Immersions of links of simple singularities

We list all the immersions studied in this thesis, and known results for them. See also Figure 1.1 in Chapter 1.

4.3.1 Inclusion maps

Let us consider an isolated surface singularity in $\mathbb{C}^3$, its Milnor fiber F, and its link K. By definition, we have inclusion maps

$$\hat{f}\colon F \hookrightarrow S^5_\varepsilon \quad \text{and} \quad f \coloneqq \hat{f}|_K\colon K \hookrightarrow S^5_\varepsilon.$$

Removing one point of S^5_ε missed by $\hat{f}$, we obtain embeddings which are denoted by

$$\hat{f}\colon F \hookrightarrow \mathbb{R}^5 \quad \text{and} \quad f \coloneqq \hat{f}|_K\colon K \hookrightarrow \mathbb{R}^5$$

as well (recall Remark 3.4.4).

4.3.2 Immersions defined by holomorphic map-germs

We recall Némethi–Pintér's construction in §3.4. If the map-germ $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is the parametrization of a given type of simple singularities and K is its link, then the induced immersion is factorized into

$$\Phi|_{\mathfrak{S}} = f \circ p\colon S^3 \to K \hookrightarrow S^5_\varepsilon,$$

where $f\colon K \hookrightarrow S^5_\varepsilon$ is the inclusion map of the link and $p\colon S^3 \to K$ is the universal covering map. Némethi–Pintér applied Theorem 3.4.3 to the following computation.

Theorem 4.3.1 ([NP15]). *Consider any type of simple singularities, its parametrization $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, and the immersion $\Phi|_{\mathfrak{S}} = f \circ p\colon S^3 \looparrowright S^5$. Then the Smale invariant of $\Phi|_{\mathfrak{S}}$ is given by*

$$\Omega(\Phi|_{\mathfrak{S}}) = \begin{cases} -(n^2-1) & \text{for } A_{n-1}\ (n \geqslant 2); \\ -(4n^2+12n-1) & \text{for } D_{n+2}\ (n \geqslant 2); \\ -167 & \text{for } E_6; \\ -383 & \text{for } E_7; \\ -1079 & \text{for } E_8. \end{cases}$$

4.3.3 Immersions associated with Dynkin diagrams of types A-D-E

The following construction is due to Kinjo [Kin15]. We consider an immersion $S^2 \looparrowright \mathbb{R}^4$ with only one positively transverse double point. Its normal Euler class is -2

(cf. [Whi44] or [Her81]), i.e., the disk normal bundle is isomorphic to the disk cotangent bundle $DT^*S^2 = X(A_1)$. Hence, we obtain immersions

$$\hat{g}_{A_1}\colon X(A_1) \looparrowright \mathbb{R}^4 \quad \text{and} \quad g_{A_1} \coloneqq \hat{g}_{A_1}|_{M(A_1)}\colon M(A_1) \looparrowright \mathbb{R}^4.$$

Furthermore, for each Dynkin diagram G of types A-D-E, we construct immersions

$$\hat{g}_G\colon X(G) \looparrowright \mathbb{R}^4 \quad \text{and} \quad g_G \coloneqq \hat{g}_G|_{M(G)}\colon M(G) \looparrowright \mathbb{R}^4$$

by plumbing the copies of $\hat{g}_{A_1}$ according to G. We also pull back the immersions g_G by the universal covering map $p\colon S^3 \to M(G)$. Then we obtain immersions $g_G \circ p\colon S^3 \looparrowright \mathbb{R}^4$.

We will reconstruct (regularly homotope) immersions g_G as nicer ones, so that they fit to almost contact and complex structures (see §5.3).

Remark 4.3.2. To be precise, the immersion Kinjo constructed is $\bar{g}_G \coloneqq g_G \circ r\colon M(G) \looparrowright \mathbb{R}^4$, where $r\colon M(G) \to M(G)$ is an orientation-reversing diffeomorphism. In fact, the 3- (resp. 4-) manifolds Kinjo immersed were constructed by plumbing copies of the circle (resp. disk) bundles over S^2 with Euler class $+2$.

Kinjo applied Theorem 3.3.2 to the following computation.

Theorem 4.3.3 ([Kin15])**.** *Consider the Dynkin diagram G of type A or D, Kinjo's immersion $\bar{g}_G\colon M(G) \looparrowright \mathbb{R}^4$, and the universal covering map $p\colon S^3 \to M(G)$. Then the Smale invariant of $\bar{g}_G \circ p\colon S^3 \looparrowright \mathbb{R}^4$ is given by*

$$\Omega(\bar{g}_G \circ p) = \begin{cases} (n^2 - 1, 0) & \textit{for } A_{n-1}\ (n \geqslant 2); \\ (4n^2 + 12n - 1, 0) & \textit{for } D_{n+2}\ (n \geqslant 2). \end{cases}$$

We can deduce the following from Theorem 4.3.3 and Proposition 3.2.1.

Corollary 4.3.4. *Under the same setup of Theorem 4.3.3, the Smale invariant of $g_G \circ p$ is given by*

$$\Omega(g_G \circ p) = \begin{cases} (-n^2 - 1, 1) & \textit{for } A_{n-1}\ (n \geqslant 2); \\ (-4n^2 - 12n - 1, 1) & \textit{for } D_{n+2}\ (n \geqslant 2). \end{cases}$$

Consequently, we have the following as a corollary to Theorem 4.3.1, Corollary 4.3.3, and Proposition 3.2.3. It was pointed out by [Kin15, NP15].

Corollary 4.3.5. *Consider one type of simple singularities and the corresponding Dynkin diagram G. If the singularity is of type A or D, then two immersions $\Phi|_{\mathfrak{S}} = f \circ p\colon S^3 \looparrowright \mathbb{R}^5$ and $j \circ g_G \circ p\colon S^3 \looparrowright \mathbb{R}^5$ are regularly homotopic.*

We will refine and generalize this fact to the E case in §7.2.2.

Chapter 5

Contact and almost contact structures

In this chapter, we recall the notions of contact and almost contact structures. See [EM02, Gei08] for details. We also make a preparation for our main results.

5.1 Contact structures

Definition 5.1.1 (contact manifold)**.** Let M^{2m+1} be a $(2m+1)$-manifold. A *contact structure* on M^{2m+1} is a maximally non-integrable hyperplane distribution $\xi \subset TM^{2m+1}$, i.e., if ξ is locally defined by a 1-form α as $\xi = \operatorname{Ker}\alpha$, then the $(2m+1)$-form $\alpha \wedge (d\alpha)^m$ nowhere vanishes. We call the pair (M^{2m+1}, ξ) a *contact manifold*. In the case where ξ is cooriented as a hyperplane field in TM^{2m+1}, (M^{2m+1}, ξ) is also said to be *cooriented*.

Remark 5.1.2. For a contact manifold (M^{2m+1}, ξ), a defining 1-form α of ξ induces the symplectic structure $d\alpha|_\xi$ on ξ. Notice that the conformal class of $d\alpha|_\xi$ depends only on ξ. Indeed, for any non-vanishing function f on M^{2m+1}, it holds that $d(f\alpha)|_\xi = f \cdot d\alpha|_\xi$.

Let J_0 denote the standard complex structure on $\mathbb{C}^3$.

Example 5.1.3. Let $S^5_\varepsilon \subset \mathbb{C}^3$ be the 5-sphere with radius $\varepsilon > 0$ and centered at the origin. Then define a hyperplane distribution $\xi_{\mathrm{std}} \subset TS^5_\varepsilon$ as the complex tangency

$$\xi_{\mathrm{std}} = TS^5_\varepsilon \cap J_0(TS^5_\varepsilon).$$

This forms a cooriented contact structure on standardly oriented S^5_ε. We call this the *standard contact structure* on the 5-sphere.

Example 5.1.4. Let K be the link of an isolated surface singularity in $\mathbb{C}^3$. Then define a hyperplane distribution $\xi_{\mathrm{can}} \subset TK$ as the complex tangency

$$\xi_{\mathrm{can}} = TK \cap J_0(TK).$$

This forms a cooriented contact structure on standardly oriented K as well. We call this the *canonical contact structure* on the link K.

Definition 5.1.5 (contact embedding)**.** Let (M^{2m+1}, ξ) and (N^{2n+1}, η) be contact manifolds. A *contact embedding* of (M^{2m+1}, ξ) into (N^{2n+1}, η) is an embedding $f\colon M^{2m+1} \hookrightarrow N^{2n+1}$ satisfying that

$$Tf(M^{2m+1}) \cap \eta|_{f(M^{2m+1})} = df(\xi).$$

Example 5.1.6. The inclusion map $f\colon (K, \xi_{\mathrm{can}}) \hookrightarrow (S^5_\varepsilon, \xi_{\mathrm{std}})$ is a contact embedding.

5.2 Almost contact structures

The definition of almost contact structure is slightly different depending on literature (e.g., [EM02, p.100, §10.1.B], [Gei08, p.66, Definition 2.4.7]). We adopt the following for simplicity of our later arguments.

Definition 5.2.1 (almost contact manifold)**.** Let M^{2m+1} be an oriented $(2m+1)$-manifold. An *almost contact structure* on M^{2m+1} is the pair (ξ, ω) of a cooriented hyperplane distribution $\xi \subset TM^{2m+1}$ and a symplectic form ω on ξ valued in the line bundle TM^{2m+1}/ξ, such that the coorientation of ξ is positive with respect to the orientation ω^m of ξ and the orientation of M^{2m+1}. We call the triplet $(M^{2m+1}; \xi, \omega)$ an *almost contact manifold*.

Example 5.2.2. Let (M^{2m+1}, ξ) be a cooriented contact manifold. We have the hyperplane field ξ and the symplectic structure $d\alpha|_\xi$ for some defining 1-form α of ξ (Remark 5.1.2). Hence every cooriented contact manifold determines an almost contact structure up to conformal equivalence.

We generalize the notion of contact embedding as follows.

Definition 5.2.3 (almost contact embedding)**.** Let $(M^{2m+1}; \xi, \omega)$ and $(N^{2n+1}; \eta, \varpi)$ be almost contact manifolds. An *almost contact embedding* of $(M^{2m+1}; \xi, \omega)$ into $(N^{2n+1}; \eta, \varpi)$ is an embedding $f\colon M^{2m+1} \hookrightarrow N^{2n+1}$ satisfying that

(i) $Tf(M^{2m+1}) \cap \eta|_{f(M^{2m+1})} = df(\xi)$;

(ii) the coorientation of ξ corresponds to that of η via df;

(iii) $(df(\xi), \omega)$ is a symplectic subspace field of $(\eta|_{f(M^{2m+1})}, \varpi')$, where ϖ' is a symplectic structure belonging to the same conformal class as ϖ.

Example 5.2.4. A contact embedding $f\colon (M^{2m+1}, \xi) \hookrightarrow (N^{2n+1}, \eta)$ forms an almost contact embedding. Indeed, take defining 1-forms α and β of ξ and η, respectively. Then $(df(\xi), d\alpha|_\xi)$ is a symplectic subspace field of $(\eta|_{f(M^{2m+1})}, \varpi')$, where ϖ' is a symplectic structure belonging to the same conformal class as $d\beta|_\eta$.

The following notion will help us to investigate properties of Wu invariants of almost contact embeddings (§6.2) and also other immersions (§6.3).

Definition 5.2.5 (almost contact parallelization)**.** Let $(M^3; \xi, \omega)$ be an almost contact manifold such that ξ is trivializable as a 2-plane field. A parallelization $\tau\colon TM^3 \xrightarrow{\cong} M^3 \times \mathbb{R}^3$ of M^3 is said to be an *almost contact parallelization* if the following holds via τ:

(i) the constant vector field e_1 gives the coorientation of ξ;

(ii) the constant vector fields e_2 and e_3 span ξ, and (e_2, e_3) gives the orientation of ξ determined by ω.

Let V be an oriented real vector space of dimension $2m+1$. Here, a *symplectic hyperplane* means the pair (H, Ω) of a cooriented hyperplane H and its symplectic structure Ω such that the coorientation of H is positive with respect to the orientation Ω^m of ξ and the orientation of V. Let $\mathcal{C}(V)$ denote the space of all symplectic hyperplanes in V.

Proposition 5.2.6. *For any oriented real vector space V of dimension $2m+1$, the space $\mathcal{C}(V)$ is homotopy equivalent to $\mathrm{SO}(2m+1)/\mathrm{U}(m)$. Therefore, every almost contact structure on an oriented $(2m+1)$-manifold M^{2m+1} defines a section of the $\mathrm{SO}(2m+1)/\mathrm{U}(m)$-bundle over M^{2m+1} associated to TM^{2m+1}.*

Here the unitary subgroup $\mathrm{U}(m)$ is embedded into the special orthogonal group $\mathrm{SO}(2m)$ as the subgroup $\{A \in \mathrm{SO}(2m) \mid J_0A = AJ_0\}$, where J_0 is the standard complex structure on $\mathbb{R}^{2m} = \mathbb{C}^m$. Also $\mathrm{SO}(2m)$ is embedded into $\mathrm{SO}(2m+1)$ in the standard way.

Remark 5.2.7 (low-dimensional cases)**.** If $m = 1$, then we have that $\mathrm{SO}(3)/\mathrm{U}(1) \cong S^2$. This means, roughly saying, that a symplectic structure on a 2-plane determines only an orientation of the plane up to homotopy. For the case where $m = 2$, we describe the identification $\mathcal{C}(V) \simeq \mathrm{SO}(5)/\mathrm{U}(2)$ in Chapter 6.

5.3 Reconstruction of Kinjo's immersions

At the end of this chapter, we reconstruct the immersion $g_G\colon M(G) \looparrowright \mathbb{R}^4$ introduced in §4.3.3. We also fix an identification between $M(G)$ and the link K of the corresponding singularity. This construction motivates the study of the Wu invariant in §6.3.

5.3.1 Reconstruction for the A_1-case

We consider the *Whitney 2-sphere* [Whi44]

$$w\colon S^2 \looparrowright \mathbb{C}^2, \quad w(x_1, x_2, y) = (1 + y\sqrt{-1})(x_1, x_2).$$

This map has only one positively transverse double point $h_0(0, 0, \pm 1) = (0, 0)$. Moreover, this is a real analytic and totally real immersion with respect to the standard complex structure J_0 on $\mathbb{C}^2$. We take the disk normal bundle of w. Since the normal Euler number of w is -2, the total space is nothing but the disk cotangent bundle $DT^*S^2 = X(A_1)$. Then we have immersions

$$\hat{g}_{A_1}\colon X(A_1) \looparrowright \mathbb{C}^2 \quad \text{and} \quad g_{A_1} = \hat{g}_{A_1}|_{M(A_1)}\colon M(A_1) \looparrowright \mathbb{C}^2.$$

We now induce a complex structure J on $X(A_1)$ by pulling-back J_0, which is the complexification of the zero section S^2. We also define an almost contact structure ξ'_{can} on $M(A_1)$ by

$$\xi'_{\mathrm{can}} = TM(A_1) \cap J(TM(A_1))$$

(we endow ξ'_{can} with the orientation as a complex line field). Note that $\hat{g}_{A_1}$ is holomorphic, and hence it holds that $dg_{A_1}(\xi'_{\mathrm{can}}) \subset T\mathbb{C}^2$ is complex everywhere. We will focus on this property (see §6.3).

5.3.2 Reconstruction for the general case

Let G be one of Dynkin diagrams of types A-D-E. We consider the union of 2-spheres embedded in some Euclidean space according to the following:

- corresponding to each vertex of G, embed one 2-sphere;
- corresponding to each edge of G, arrange two 2-spheres so that they transversely intersect at one point.

Then we construct a map

$$w_G\colon S^2 \cup \cdots \cup S^2 \to \mathbb{C}^2$$

such that each restriction $w_G|_{S^2}$ coincides with the Whitney 2-sphere w after some rotation and parallel shift and is still totally real, and all 2-spheres transversely intersect in $\mathbb{C}^2$. Then taking a regular neighborhood of w_G, we have immersions

$$\hat{g}_G\colon X(G) \looparrowright \mathbb{C}^2 \quad \text{and} \quad g_G = \hat{g}_G|_{M(G)}\colon M(G) \looparrowright \mathbb{C}^2.$$

We now induce a complex structure J on $X(G)$ by pulling-back the standard complex structure J_0 on $\mathbb{C}^2$. We also define an almost contact structure ξ'_{can} on $M(G)$ by

$$\xi'_{\mathrm{can}} = TM(G) \cap J(TM(G))$$

(we endow ξ'_{can} with the orientation as a complex line field). Then the immersion $\hat{g}_G$ is holomorphic, and hence it holds that $dg_G(\xi'_{\mathrm{can}}) \subset T\mathbb{C}^2$ is complex everywhere.

5.3.3 Identification

Proposition 5.3.1. *There exists a diffeomorphism $H\colon M(G) \to K$ such that $dH(\xi'_{\mathrm{can}})$ and ξ_{can} are homotopic as almost contact structures on K.*

Proof. Consider the complex structure on the Milnor fiber F as a complex submanifold of $\mathbb{C}^3$. Then we can find a spine of F, which is the transverse union $S^2 \cup \cdots \cup S^2$ associated with G as above, and all 2-spheres are totally real in F. We take an embedding

$$\iota\colon X(G) \hookrightarrow F$$

such that ι identifies the spines of $X(G)$ and F, and F deformation-retracts to $\iota(X(G))$ by an isotopy $\{h_t\}_{t\in[0,1]}$ on F. Since both complex structures on $X(G)$ and F have their spines as totally real, they are homotopic as almost complex structures. Moreover, along the isotopy, almost complex structures and hence almost contact structures on the boundary are deformed. Thus, the composition $H \coloneqq (h_1)^{-1} \circ \iota$ is what we desired. $\square$

Thus we identify $(M(G), \xi'_{\mathrm{can}})$ with (K, ξ_{can}) as almost contact manifolds via H.

Chapter 6

Geometric properties of the Wu invariant

In this chapter, we state geometric properties of the Wu invariant as the preparation to show our main results. Throughout this section, let $(M^3;\xi,\omega)$ denote an arbitrary closed almost contact 3-manifold.

6.1 Conventions

We introduce the following conventions. Let e_i denote the i-th standard vector of $\mathbb{R}^5$.

Convention 1. We embed SO(2) into SO(5) in the lowest diagonal position, i.e.,

$$\{E_3\}\times \mathrm{SO}(2)\subset \mathrm{SO}(5).$$

We also identify SO(5)/SO(2) with $V_{5,3}$ via the induced map from the action of SO(5) on $V_{5,3}$ at $\begin{bmatrix} e_1 & e_2 & e_3\end{bmatrix}$:

$$\mathrm{SO}(5)\to V_{5,3},\quad \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5\end{bmatrix}\mapsto \begin{bmatrix} v_1 & v_2 & v_3\end{bmatrix}.$$

In addition, let $(\varepsilon_1,\dots,\varepsilon_5)$ be the basis of $(\mathbb{R}^5)^*$ dual to $(e_1,\dots,e_5)$, and let $(\text{-})^*\colon \mathbb{R}^5\to(\mathbb{R}^5)^*$ denote the linear isomorphism sending e_i to ε_i.

Convention 2. We embed U(2) into SO(5) in the lower diagonal position, i.e.,

$$\{1\}\times \mathrm{U}(2)\subset \mathrm{SO}(5).$$

Moreover, considering the standard orientation on $\mathbb{R}^5$, we identify SO(5)/U(2) with $\mathcal{C}(\mathbb{R}^5)$ (up to homotopy equivalence) via the induced map from the action of SO(5) on $\mathcal{C}(\mathbb{R}^5)$ at $(\operatorname{Ker}\varepsilon_1,\varepsilon_2\wedge\varepsilon_3+\varepsilon_4\wedge\varepsilon_5)$:

$$\mathrm{SO}(5)\to\mathcal{C}(\mathbb{R}^5),\quad \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5\end{bmatrix}\mapsto (H,\Omega)=(\operatorname{Ker}(v_1^*),v_2^*\wedge v_3^*+v_4^*\wedge v_5^*),$$

where H is cooriented in the v_1-direction.

6.2 Vanishing for almost contact embeddings

In this subsection, we show the following. The motivative example is that in §4.3.1.

Theorem 6.2.1. *Let* $(\mathbb{R}^5; \eta, \varpi)$ *be* 5*-space endowed with an almost contact structure and*

$$f\colon (M^3; \xi, \omega) \hookrightarrow (\mathbb{R}^5; \eta, \varpi)$$

an almost contact embedding. Then the Wu invariant $c_\tau(f) \in H^2(M^3; \mathbb{Z})$ *vanishes for any almost contact parallelization* τ *of* $(M^3; \xi, \omega)$*.*

Before the proof, here are three remarks.

(a) The triviality of ξ as a 2-plane field is ensured as follows. In general, for an almost contact manifold $(M^{2n-1}; \xi, \omega)$, the hyperplane field ξ admits a complex structure compatible with the symplectic structure ω, which is unique up to homotopy. Hence its first Chern class $c_1(\xi)$ makes sense. Then the following is known:

Theorem 6.2.2 ([Kas16, Theorem 1.3])**.** *If a closed contact manifold* (M^{2n-1}, ξ) *is a contact submanifold of a cooriented contact manifold* (N^{2n+1}, η) *such that* $H^2(N^{2n+1}; \mathbb{Z}) = 0$*, then* $c_1(\xi) = 0$*.*

Notice that Theorem 6.2.2 holds also for almost contact manifolds. Applying this result to the case where $n = 2$, we have the triviality of our ξ as a 2-plane field.

(b) Since the map $f\colon M^3 \hookrightarrow \mathbb{R}^5$ is an embedding, the normal Euler class of f vanishes [MS74, Theorem 11.3]. Hence the Wu invariant of f is a 2-torsion for any parallelization. However, it does not imply the Wu invariant itself vanishes.

(c) Theorem 6.2.1 is related to the following result:

Theorem 6.2.3 ([Kas16, Theorem 1.5])**.** *Let* (M^3, ξ) *be a closed cooriented contact* 3*-manifold with* $c_1(\xi) = 0$*. Then there is a contact structure* η *on* $\mathbb{R}^5$ *such that we can embed* (M^3, ξ) *in* $(\mathbb{R}^5, \eta)$ *as a contact submanifold.*

In the proof of Theorem 6.2.3, such an embedding was constructed so that its Wu invariant vanishes for an almost contact parallelization.

The key to the proof of Theorem 6.2.1 is to consider the quotient map

$$\pi\colon V_{5,3} \cong \mathrm{SO}(5)/\mathrm{SO}(2) \to \mathrm{SO}(5)/\mathrm{U}(2),$$

and the composition

$$\psi_{\tau,f} := \pi \circ \varphi_{\tau,f}\colon M^3 \to \mathrm{SO}(5)/\mathrm{U}(2).$$

We obtain a new expression of the Wu invariant as follows. Recall that $\Sigma \in H^2(V_{5,3}; \mathbb{Z})$ is the generator chosen in Definition 3.5.2.

Lemma 6.2.4. *Let* $f\colon M^3 \looparrowright \mathbb{R}^5$ *be an immersion and* τ *a parallelization of* M^3*. Then the Wu invariant* $c_\tau(f)$ *is equal to the class*

$$(\psi_{\tau,f})^*(\Sigma') \in H^2(M^3; \mathbb{Z}),$$

where Σ' *is the generator of* $H^2(\mathrm{SO}(5)/\mathrm{U}(2); \mathbb{Z}) \cong \mathbb{Z}$ *chosen so that* $\pi^*(\Sigma') = \Sigma$*.*

Proof. It suffices to show that the induced map

$$\pi^*\colon H^2(\mathrm{SO}(5)/\mathrm{U}(2);\mathbb{Z}) \to H^2(\mathrm{SO}(5)/\mathrm{SO}(2);\mathbb{Z})$$

is an isomorphism. This can be shown by applying the Gysin exact sequence to the orientable S^3-bundle $\pi\colon \mathrm{SO}(5)/\mathrm{SO}(2) \to \mathrm{SO}(5)/\mathrm{U}(2)$. $\square$

Remark 6.2.5. There is a diffeomorphism $\mathrm{SO}(5)/\mathrm{U}(2) \cong \mathbb{C}P^3$ [Gei08, §8.1]. Under this identification, the generator Σ' coincides with the Poincaré dual of $\mathbb{C}P^2$ up to sign.

Proof of Theorem 6.2.1. Let us consider the map

$$\varphi_{\tau,f} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} : M^3 \to V_{5,3} \cong \mathrm{SO}(5)/\mathrm{SO}(2)$$

induced from the differential of f and the parallelization τ, where $v_i(x) = df_x(e_i)$ for $i = 1,2,3$. We also consider the quotient map $\pi\colon \mathrm{SO}(5)/\mathrm{SO}(2) \to \mathrm{SO}(5)/\mathrm{U}(2)$ and the composition

$$\psi_{\tau,f} \coloneqq \pi \circ \varphi_{\tau,f}\colon M^3 \to \mathrm{SO}(5)/\mathrm{U}(2).$$

By Lemma 6.2.4, it suffices to show that the map $\psi_{\tau,f}$ is null-homotopic.

By Proposition 5.2.6, the almost contact structure (η,ϖ) is regarded as a smooth map

$$(\eta,\varpi)\colon \mathbb{R}^5 \to \mathcal{C}(\mathbb{R}^5).$$

Then it suffices to show that the lower left triangle in the following diagram commutes, where the maps ^{*1} and ^{*2} are the identification fixed in Conventions 1 and 2.

$$\begin{tikzcd}
M^3 \arrow[r, "\varphi_{\tau,f}"] \arrow[d, "f"] \arrow[drr, "\psi_{\tau,f}"] & V_{5,3} \arrow[r, "^*1"] & \mathrm{SO}(5)/\mathrm{SO}(2) \arrow[d, "\pi"] \\
\mathbb{R}^5 \arrow[r, "(\eta,\varpi)"'] & \mathcal{C}(\mathbb{R}^5) \arrow[r, "^*2"'] & \mathrm{SO}(5)/\mathrm{U}(2)
\end{tikzcd}$$

Since the map f is an almost contact embedding, the distribution $df(\xi) \subset (\eta,\varpi)|_{f(M^3)}$ forms a symplectic subspace field, and the 2-frame (v_2, v_3) is homotopic to a symplectic basis field of $(df(\xi), \varpi|_{df(\xi)})$ (recall Remark 5.2.7). Then for each $x \in M^3$, by the extension of symplectic basis, the symplectic structure $\varpi_{f(x)}$ on $\eta_{f(x)}$ has the form

$$v_2(x)^* \wedge v_3(x)^* + v_4(x)^* \wedge v_5(x)^*$$

up to homotopy, where $v_4(x)$ and $v_5(x)$ are appropriate vectors. Then we have that the restriction $(\eta,\varpi)|_{f(M^3)}$ has the representative

$$\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}$$

in the sense of Convention 2, up to homotopy. By the form of $\varphi_{\tau,f}$ and Convention 1, we find the commutativity of the lower left triangle. This completes the proof. $\square$

6.3 Vanishing for immersions with complex tangency

In this subsection, we show the following. The motivative example is introduced in §4.3.3 and §5.3. Let J_0 be the standard complex structure on $\mathbb{R}^4 = \mathbb{C}^2$:

$$J_0 := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Let $j\colon \mathbb{R}^4 \hookrightarrow \mathbb{R}^5,\ (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4, 0)$ be the standard inclusion.

Theorem 6.3.1. *Assume that ξ is trivial as a 2-plane field. Let*

$$g\colon (M^3; \xi, \omega) \looparrowright (\mathbb{C}^2, J_0)$$

be an immersion satisfying that $dg(\xi) \subset T\mathbb{C}^2$ forms a complex line everywhere. Then the Wu invariant $c_\tau(j \circ g) \in H^2(M^3; \mathbb{Z})$ vanishes for any almost contact parallelization τ of $(M^3; \xi, \omega)$.

To prove this assertion, we consider the map

$$\varphi_{\tau, j\circ g}\colon M^3 \to \mathrm{SO}(5)/\mathrm{SO}(2),$$

and an additional map

$$\Phi_{\tau,g}\colon M^3 \to V_{4,3} \cong \mathrm{SO}(4),$$

which is induced from the differential of g and the parallelization τ. Since the 5th column of the map $\varphi_{\tau, j\circ g}$ is constantly e_5, the relationship between these maps is as follows.

$$\begin{tikzcd} \mathrm{SO}(4) \arrow[r, hook, "\iota"] & \mathrm{SO}(5) \arrow[d, "\rho"] \\ M^3 \arrow[u, "\Phi_{\tau,g}"] \arrow[r, "\varphi_{\tau, j\circ g}"'] & \mathrm{SO}(5)/\mathrm{SO}(2) \end{tikzcd}$$

Here $\iota\colon \mathrm{SO}(4) \hookrightarrow \mathrm{SO}(5)$ is the embedding in the *upper* diagonal position and $\rho\colon \mathrm{SO}(5) \to \mathrm{SO}(5)/\mathrm{SO}(2)$ is the quotient map (recall Convention 1). Then we obtain another expression of the Wu invariant as follows. Consider the usual embedding of $\mathrm{U}(2)$ into $\mathrm{SO}(4)$ defined by $\mathrm{U}(2) = \{A \in \mathrm{SO}(4) \mid J_0 A = A J_0\}$.

Lemma 6.3.2. *Let $g\colon M^3 \looparrowright \mathbb{C}^2$ be an immersion and τ a parallelization of M^3. Then the Wu invariant $c_\tau(j \circ g)$ is equal to*

$$(\Phi_{\tau,g})^*[\mathrm{U}(2)]^* \in H^2(M^3; \mathbb{Z}),$$

where $[\mathrm{U}(2)]^*$ is the Poincaré dual of $\mathrm{U}(2) \subset \mathrm{SO}(4)$.

Proof. Notice that the induced map

$$(\rho \circ \iota)^* \colon H^2(\mathrm{SO}(5)/\mathrm{SO}(2);\mathbb{Z}) \xrightarrow{\rho^*} H^2(\mathrm{SO}(5);\mathbb{Z}) \xrightarrow{\iota^*} H^2(\mathrm{SO}(4);\mathbb{Z}),$$

which is regarded as $\mathbb{Z} \to \mathbb{Z}_2 \to \mathbb{Z}_2$, is a surjection. Indeed, applying the Gysin exact sequence of the orientable $\mathrm{SO}(2)$-bundle ρ, we see that ρ^* is a surjection; applying the Bockstein exact sequence to $\mathrm{SO}(4)$ and $\mathrm{SO}(5)$, we see that ι^* is an isomorphism.

Next, we verify that the Poincaré dual $[\mathrm{U}(2)_J]^*$ generates $H^2(\mathrm{SO}(4);\mathbb{Z}) \cong \mathbb{Z}_2$ as follows. Applying the Gysin exact sequence to the orientable S^3-bundle

$$\pi' \colon \mathrm{SO}(4) \to \mathrm{SO}(4)/\mathrm{SU}(2) \cong \mathbb{R}P^3,$$

we have that $(\pi')^* \colon H^2(\mathrm{SO}(4)/\mathrm{SU}(2);\mathbb{Z}) \to H^2(\mathrm{SO}(4);\mathbb{Z})$ is an isomorphism. Moreover, applying the Gysin exact sequence to the orientable S^1-bundle

$$\pi'' \colon \mathrm{SO}(4)/\mathrm{SU}(2) \to \mathrm{SO}(4)/\mathrm{U}(2) \cong S^2,$$

we have that $(\pi'')^* \colon H^2(S^2;\mathbb{Z}) \to H^2(\mathrm{SO}(4)/\mathrm{SU}(2);\mathbb{Z})$ is a surjection. Hence the generator of $H^2(\mathrm{SO}(4);\mathbb{Z})$ is obtained by

$$(\pi')^* \circ (\pi'')^*[\mathrm{pt.}]^* = [(\pi'' \circ \pi')^{-1}(\mathrm{pt.})]^* = [\mathrm{U}(2)]^*.$$

Thus we see that

$$c_\tau(j \circ g) = (\varphi_{j\circ g})^*(\Sigma) = (\Phi_{\tau,g})^*(\rho \circ \iota)^*(\Sigma) = (\Phi_{\tau,g})^*[\mathrm{U}(2)]^*,$$

which completes the proof. □

Proof of Theorem 6.3.1. Let

$$\Phi_{\tau,g} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} : M^3 \to V_{4,3} \cong \mathrm{SO}(4)$$

denote the map induced from the differential of g and the parallelization τ, where $v_i(x) = dg_x(e_i)$ for $i = 1, 2, 3$, and v_4 is the normal vector field to g.

By Lemma 6.3.2, it suffices to show that $\Phi_{\tau,g}$ does not meet with $\mathrm{U}(2)$. Let $x \in M^3$. By the form of τ, the 2-plane spanned by $v_2(x)$ and $v_3(x)$ coincides with

$$dg_x(\xi_x) \subset T_{g(x)}\mathbb{C}^2 = \mathbb{C}^2.$$

By assumption, this 2-plane is complex with respect to J_0. Hence we observe that the 2-plane spanned by $v_3(x)$ and $v_4(x)$ is not complex with respect to J_0. Now we assume that the point $x \in M^3$ satisfies that $\Phi_{\tau,g}(x) \in \mathrm{U}(2)$, i.e.,

$$J_0 \cdot \Phi_{\tau,g}(x) = \Phi_{\tau,g}(x) \cdot J_0.$$

Then we have that $v_4(x) = J_0 \cdot v_3(x)$. This makes the contradiction to the earlier observation, which completes the proof. □

Chapter 7

Determination of regular homotopy classes of several immersions

In this chapter, we determine the regular homotopy classes of immersions listed in §4.3, as our main result. In particular, we prove Main Theorems I and II.

7.1 Inclusion maps of links of surface singularities into the 5-sphere

We consider inclusion maps

$$\hat{f}\colon F \hookrightarrow \mathbb{R}^5 \quad \text{and} \quad f := \hat{f}|_K \colon K \hookrightarrow \mathbb{R}^5$$

defined in §4.3.1. We also notice that the normal bundle of f is trivial (see (b) of remarks below Theorem 6.2.1). Therefore the Smale-type invariant of f makes sense.

Proof of Main Theorem I. Even if we remove one point of S^5_ε, the map $f\colon (K, \xi_{\mathrm{can}}) \hookrightarrow (\mathbb{R}^5, \xi_{\mathrm{std}}|_{\mathbb{R}^5})$ is still a contact embedding. Then the vanishing of the Wu invariant $c_\tau(f)$ follows from Theorem 6.2.1.

The Smale-type invariant $i(f)$ is computed by choosing the inclusion map $\hat{f}\colon F \hookrightarrow \mathbb{R}^5$ as a singular Seifert surface of f. $\square$

Remark 7.1.1. The integer $\alpha(K)$ is an intrinsic data. Furthermore, the canonical contact structure ξ_{can} on K is characterized as the Milnor fillable structure, which is unique up to contactomorphism [CNP06]. Thus, *the regular homotopy class of the inclusion map is determined only by the diffeomorphism type of the link as an abstract* 3*-manifold and the signature of the Milnor fiber.* This fact will be used in Chapter 8.

Remark 7.1.2. Main Theorem I holds also for the boundary of the Milnor fiber of every non-isolated surface singularity in $\mathbb{C}^3$.

We list the Smale-type invariants $i(f)$ for all types of simple singularities in Table 7.1. Here, (co)homological data are computed from the plumbing manifold structures of $K \cong M(G)$ and $F \cong X(G)$. We note that these data have been studied also in general (cf. [Bri66, Dur78, Ném99, GP24] for $\sigma(F)$ and [ADHSZ03, NS12] for $\alpha(K)$).

Table 7.1: Data of simple singularities and their immersions ($n \geqslant 2$)

type	$H^2(K;\mathbb{Z})$	$\sigma(F)$	$\alpha(K)$	$i(f)$
A_{n-1}	$\mathbb{Z}_n$	$-(n-1)$	1 (n even) 0 (n odd)	$-3n/2$ (n even) $-3(n-1)/2$ (n odd)
D_{n+2}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (n even) $\mathbb{Z}_4$ (n odd)	$-(n+2)$	2 (n even) 1 (n odd)	$-3(n+4)/2$ (n even) $-3(n+3)/2$ (n odd)
E_6	$\mathbb{Z}_3$	-6	0	-9
E_7	$\mathbb{Z}_2$	-7	1	-12
E_8	0	-8	0	-12

7.2 Immersions associated with the Dynkin diagrams of simple singularities

7.2.1 Computations for $j \circ g_G\colon M(G) \looparrowright \mathbb{R}^5$

We consider Kinjo's immersions

$$\hat{g}_G\colon X(G) \looparrowright \mathbb{C}^2 \quad \text{and} \quad g_G \coloneqq \hat{g}_G|_{M(G)}\colon M(G) \looparrowright \mathbb{C}^2$$

which are defined in §4.3.3 and reconstructed in §5.3. Notice that the normal bundle of $j \circ g_G$ is trivial. Therefore the Smale-type invariant of $j \circ g_G$ makes sense.

Proof of Main Theorem II. We consider the almost contact structure ξ'_{can} on $M(G)$, which was defined in §5.3. To show the vanishing of the Wu invariant, we will apply Theorem 6.3.1 to g_G. We verify that ξ'_{can} satisfies the assumptions of Theorem 6.3.1 as follows. Since we have the identification of $(M(G), \xi'_{\text{can}})$ and (K, ξ_{can}) as almost contact manifolds, it holds that $c_1(\xi'_{\text{can}}) = c_1(\xi_{\text{can}})$. We also recall that (K, ξ_{can}) admits a contact embedding into $(\mathbb{R}^5, \xi_{\text{std}}|_{\mathbb{R}^5})$ by the inclusion map f. Then Theorem 6.2.2 ensures that $c_1(\xi_{\text{can}}) = 0$. Hence ξ'_{can} is trivial as a 2-plane field. Furthermore, $dg_G(\xi'_{\text{can}}) \subset T\mathbb{C}^2$ was complex everywhere by construction. Therefore we can apply Theorem 6.3.1 to g_G.

The Smale-type invariant $i(j \circ g_G)$ is computed by choosing the immersion $j \circ \hat{g}_G\colon X(G) \looparrowright \mathbb{R}^5$ as a singular Seifert surface of $j \circ g_G$. Notice that $\sigma(X(G)) = -\#V(G)$. □

7.2.2 Computations for $j \circ g_G \circ p \colon S^3 \looparrowright \mathbb{R}^5$

We obtain the following as a corollary to Main Theorems I and II, which refines and generalizes Corollary 4.3.5.

Corollary 7.2.1. *For each type of simple singularities, its link K, and the corresponding Dynkin diagram G, identify K and $M(G)$ as explained in §5.3.3. Then the inclusion map $f \colon K \hookrightarrow \mathbb{R}^5$ and the immersion $j \circ g_G \colon M(G) \looparrowright \mathbb{R}^5$ have the same Wu and Smale-type invariants. Therefore, these two immersions are regularly homotopic.*

The following is immediate from the above and Theorem 4.3.1.

Corollary 7.2.2. *Consider the same setup as above. Then the Smale invariant of $j \circ g_G \circ p \colon S^3 \looparrowright \mathbb{R}^5$ is given by*

$$\Omega(j \circ g_G \circ p) = \begin{cases} -(n^2-1) & \text{for } A_{n-1} \ (n \geqslant 2); \\ -(4n^2+12n-1) & \text{for } D_{n+2} \ (n \geqslant 2); \\ -167 & \text{for } E_6; \\ -383 & \text{for } E_7; \\ -1079 & \text{for } E_8. \end{cases}$$

We also give an alternative proof of Corollary 7.2.1. Corollary 7.2.1 follows by showing the following assertion and applying it to the Milnor fiber.

Theorem 7.2.3. *Let X^4 be a parallelizable 4-manifold homotopic to a bouquet of the 2-spheres. Then it holds that*

$$\mathrm{Imm}[X^4, \mathbb{R}^5] = \{*\},$$

i.e., all immersions $X^4 \looparrowright \mathbb{R}^5$ are mutually regularly homotopic.

Proof. Using the formula obtained in Example 3.1.7, we have that

$$\begin{aligned} \mathrm{Imm}[F, \mathbb{R}^5] &= [F, V_{5,4}] \\ &= [S^2 \vee \cdots \vee S^2, \mathrm{SO}(5)] \\ &= \pi_2(\mathrm{SO}(5)) \oplus \cdots \oplus \pi_2(\mathrm{SO}(5)), \end{aligned}$$

where each equality means bijectivity. Since $\pi_2(\mathrm{SO}(5)) = 0$, the assertion holds. $\square$

7.2.3 Computations for $g_G \circ p \colon S^3 \looparrowright \mathbb{R}^4$

We also show the following by further argument, which recovers Theorem 4.3.3. The proof is due to G. Pintér.

Theorem 7.2.4. *For each type of simple singularities and the corresponding Dynkin diagram G, the Smale invariant of $\bar{g}_G \circ p$ is given by*

$$\Omega(\bar{g}_G \circ p) = \begin{cases} (n^2-1, 0) & \text{for } A_{n-1} \ (n \geqslant 2); \\ (4n^2+12n-1, 0) & \text{for } D_{n+2} \ (n \geqslant 2); \\ (167, 0) & \text{for } E_6; \\ (383, 0) & \text{for } E_7; \\ (1079, 0) & \text{for } E_8. \end{cases}$$

Proof. Let us denote $\Omega(\bar{g}_G \circ p) = (a, b)$. By Proposition 3.2.3,

$$\Omega(j \circ g_G \circ p) = -\Omega(j \circ \bar{g}_G \circ p) = -(a + 2b).$$

Since the left-hand side is already known by Corollary 7.2.2, it suffices to compute only the integer a. By Theorem 3.3.2, it holds that $a = D(\bar{g}_G \circ p) - 1$, where $D(\bar{g}_G \circ p)$ is the normal mapping degree of $g_G \circ p$. Since the covering degree of p is equal to the order $\#\Gamma$ of the corresponding group Γ (recall §4.2.1), we obtain that

$$a = D(\bar{g}_G \circ p) - 1 = \#\Gamma \cdot D(\bar{g}_G) - 1.$$

Moreover the immersion $g_G \colon M(G) \looparrowright \mathbb{R}^4$ bounds the immersion $\hat{g}_G \colon X(G) \looparrowright \mathbb{R}^4$. Hence

$$D(\bar{g}_G) = \chi(X(G)) = 1 + b_2(X(G)) = 1 + \#V(G)$$

(cf. [KM99, Theorem 2.2(b)]), where $\#V(G)$ is the number of vertices of the corresponding Dynkin diagram G. Therefore we have that

$$a = \#\Gamma \cdot (1 + \#V(G)) - 1.$$

The proof is completed by computing the right-hand side for each case. $\square$

Remark 7.2.5. In contrast to Kinjo's argument in [Kin15], we avoided to construct singular Seifert surfaces and count their singularities. The crucial point in our argument is to focus on almost contact structures on the 3-manifolds and their tangential properties in target spaces, and employ Némethi–Pintér's formula.

We obtain two corollaries of Theorem 7.2.4. The following corresponds to Corollary 4.3.4.

Corollary 7.2.6. *Under the same setup in Theorem 7.2.4, the Smale invariant of $g_G \circ p$ is given by*

$$\Omega(g_G \circ p) = \begin{cases} (-n^2-1, 1) & \text{for } A_{n-1} \ (n \geqslant 2); \\ (-4n^2-12n-1, 1) & \text{for } D_{n+2} \ (n \geqslant 2); \\ (-169, 1) & \text{for } E_6; \\ (-385, 1) & \text{for } E_7; \\ (-1081, 1) & \text{for } E_8. \end{cases}$$

The following is immediate from the computation. However, the author does not know any geometric reason for this phenomenon.

Corollary 7.2.7. *Among immersions $g_G \circ p \colon S^3 \looparrowright \mathbb{R}^4$ for all A-D-E cases, only two immersions $g_{D_{17}} \circ p$ and $g_{E_8} \circ p$ are regularly homotopic.*

Here notice that the corresponding assertion for immersions $\Phi|_{\mathfrak{S}} = f \circ p$, which are indeed regularly homotopic to $j \circ g_G \circ p$, is already known by Theorem 4.3.1.

Chapter 8

Links of surface singularities having the same diffeomorphism type

As we noted in Remark 7.1.1, given two links, their inclusions are regularly homotopic if and only if the two links are diffeomorphic as abstract 3-manifolds and the signatures of their Milnor fibers coincide. Then the following problem arises.

Problem III. *Find the pair (K, K') of links of two different isolated surface singularity in $\mathbb{C}^3$ which are mutually diffeomorphic as abstract 3-manifolds and (resp. but) whose inclusion maps $f\colon K \hookrightarrow S^5_\varepsilon$, $f'\colon K' \hookrightarrow S^5_\varepsilon$ are (resp. are not) regularly homotopic.*

In fact, both cases are possible. In this chapter, we will find such examples in two ways. Notice that *non-regular homotopicity* implies *non-isotopicity*.

8.1 Brieskorn singularities

For integers $a_i \geqslant 2$ $(i = 1, 2, 3)$, let $K(a_1, a_2, a_3)$ and $F(a_1, a_2, a_3)$ denote the link and Milnor fiber of the holomorphic function-germ $x^{a_1} + y^{a_2} + z^{a_3}$.

First, we give pairs of singularities whose links are mutually diffeomorphic as abstract 3-manifolds. We recall the following fact.

Theorem 8.1.1 ([Mil75, Theorem 7.3])**.** *If integers $a_i \geqslant 2$ $(i = 1, 2, 3)$ and m satisfy that*

$$\operatorname{lcm}(a_1, a_2) = \operatorname{lcm}(a_2, a_3) = \operatorname{lcm}(a_3, a_1) = m,$$

then the link $K(a_1, a_2, a_3)$ is diffeomorphic to the total space of the S^1-bundle $E_{g,e}$ over an oriented closed surface of genus

$$g = 1 - \frac{a_1a_2 + a_2a_3 + a_3a_1 - a_1a_2a_3}{2m}$$

and with Euler class

$$e = -\frac{a_1a_2a_3}{m^2}.$$

Corollary 8.1.2. *If $p, q \geqslant 2$ are coprime integers, then $K(p, q, pq)$ is diffeomorphic to $E_{(p-1)(q-1)/2,-1}$.*

Example 8.1.3. Both $K(2, 7, 14)$ and $K(3, 4, 12)$ are diffeomorphic to $E_{3,-1}$.

Example 8.1.4. Both $K(2, 9, 18)$ and $K(3, 5, 15)$ are diffeomorphic to $E_{4,-1}$. This pair was given by Milnor [Mil75, p.220, Example 2].

Example 8.1.5. All of $K(2, 13, 26)$, $K(3, 7, 21)$, and $K(4, 5, 20)$ are diffeomorphic to $E_{6,-1}$.

Second, we compute the signatures of the Milnor fibers of the singularities which appeared above. Let $\sigma(a_1, a_2, a_3)$ denote the signature of the Milnor fiber $F(a_1, a_2, a_3)$. We recall the following formula.

Theorem 8.1.6 ([Bri66], see also [Ném22, p.263, Example 6.9.14])**.** *It holds that*

$$\sigma(a_1, a_2, a_3) = \sum_{k_1=1}^{a_1-1} \sum_{k_2=1}^{a_2-1} \sum_{k_3=1}^{a_3-1} \operatorname{sgn} \sin \left(\pi \sum_{j=1}^{3} \frac{k_j}{a_j} \right).$$

Example 8.1.7. $\sigma(2, 7, 14) = -48$, $\sigma(3, 4, 12) = -40$.

Example 8.1.8. $\sigma(2, 9, 18) = -80$, $\sigma(3, 5, 15) = -64$.

Example 8.1.9. $\sigma(2, 13, 26) = -168$, $\sigma(3, 7, 21) = -128$, $\sigma(4, 5, 20) = -120$.

Therefore, the above are examples of mutually diffeomorphic links whose inclusion maps into the 5-sphere are not regularly homotopic.

8.2 Singularities constructed from Zariski pairs

From M. Oka's study on so-called Zariski pairs of links [Oka23a, Oka23b], we can find an example answering to Problem III. The notion of Zariski pair of curves was introduced by Artal Bartolo [Art94]. Its first example (C, C') and the defining polynomial of C were given by Zariski [Zar29, Zar37], and the explicit defining polynomial of C' was given by Oka [Oka92] (see Example 8.2.4).

Definition 8.2.1 (Zariski pair of curves)**.** Let $C, C' \subset \mathbb{C}P^2$ be projective curves of the same degree. Let $S(C), S(C')$ denote the singular sets of C, C', respectively. The pair (C, C') is said to be a *Zariski pair* if there are regular neighborhoods $\nu(C)$ and $\nu(C')$ such that the triples $(\nu(C), C, S(C))$ and $(\nu(C'), C', S(C'))$ are diffeomorphic, while the triples $(\mathbb{C}P^2, C, S(C))$ and $(\mathbb{C}P^2, C', S(C'))$ are not homeomorphic.

Definition 8.2.2 (Zariski pair of links)**.** We consider a Zariski pair of projective curves

$$C\colon f_d(x,y,z)=0, \quad C'\colon g_d(x,y,z)=0$$

of degree d. We assume that the singular points of C and C' are Newton non-degenerate. We also consider the polynomials

$$f(x,y,z)=f_d(x,y,z)+z^{d+m}, \quad g(x,y,z)=g_d(x,y,z)+z^{d+m}$$

for an integer $m \geqslant 1$, and their links K_f and K_g. Then we call the pair (K_f, K_g) a *Zariski pair of links*.

Remark 8.2.3. Each of f and g has an isolated singularities at the origin, because of adding the term z^{d+m} (this operation is called an *isolation*). Furthermore, f and g have the same zeta function [Oka23b], and hence the same Milnor number.

Example 8.2.4 ([Oka23a, §4])**.** We consider the two polynomials

$$\begin{aligned} f_6(x,y,z) &= (x^2+y^2+z^2)^3+(x^3+y^3+z^3)^2, \\ g_6(x,y,z) &= -\frac{215z^6}{64}+\frac{51xz^5}{16}+\frac{63x^2z^4}{16}-\frac{3x^3z^3}{2}-\frac{9x^4z^2}{4} \\ &\quad +3x^5z+x^6-\frac{41yz^5}{4}+8xyz^4+10x^2yz^3-4x^4yz-\frac{571y^2z^4}{48} \\ &\quad +\frac{22xy^2z^3}{3}+\frac{47x^2y^2z^2}{6}-x^4y^2-\frac{190y^3z^3}{27}+\frac{8xy^3z^2}{3} \\ &\quad +\frac{8x^2y^3z}{3}-\frac{85y^4z^2}{36}+\frac{xy^4z}{3}+\frac{x^2y^4}{3}-\frac{4y^5z}{9}-\frac{y^6}{27}, \end{aligned}$$

which form a Zariski pair. Note that g_6 was given in [Oka92]. We also consider the polynomials

$$f(x,y,z)=f_6(x,y,z)+z^7, \quad g(x,y,z)=g_6(x,y,z)+z^7.$$

It was shown that f and g have the same dual resolution graph. In particular, their links are diffeomorphic as abstract 3-manifolds.

Oka asked whether these links are ambient-diffeomorphic in the 5-sphere, or equivalently, their inclusion maps into the 5-sphere are isotopic [Oka23b, p.1257, below Theorem 25]. We do not give the answer to this question. However, it turns out that the inclusion maps are *regularly homotopic* by the following fact.

Theorem 8.2.5 ([Dur78, Theorem 1.7], see also [Sea06, p.130, 9.2 Theorem])**.** *Let V be an isolated surface singularity in $\mathbb{C}^3$, $\widetilde{V}$ its resolution, F its Milnor fiber. Then the signature of F is given by*

$$\sigma(F)=-\frac{1}{3}(2\mu-1)+K^2+b_2(\widetilde{V})+2b_1(\widetilde{V}),$$

where μ is the Milnor number and K is the canonical divisor of $\widetilde{V}$.

Furthermore, he showed the following, which generalizes the above example.

Theorem 8.2.6 ([Oka23b, Theorem 25]). *If (C, C') is a Zariski pair of irreducible projective curves with simple singularities, then the two links defined by the procedure in Definition 8.2.2 are diffeomorphic.*

More strongly, he showed their dual resolution graphs are the same. Therefore, the inclusion maps of these links are regularly homotopic.

Chapter 9

Further problems

In this chapter, we present further problems related to our current results.

Problem IV. *Let K be the link of any isolated surface singularity in $\mathbb{C}^3$ and τ an almost contact parallelization along the canonical contact structure ξ_{can}. Suppose that $\Gamma_2(0) \neq 0$ in $H^2(K;\mathbb{Z})$. Construct an immersion $f\colon K \looparrowright \mathbb{R}^5$ satisfying that $c_\tau(f) \neq 0$.*

Such a construction seems unknown even in the case where the singularity is of type A_1. In this case, K is diffeomorphic to $\mathbb{R}P^3$, UT^*S^2, and $\mathrm{SO}(3)$.

Problem V. *Let K be the link of any isolated surface singularity in $\mathbb{C}^3$ and $f\colon K \hookrightarrow S^5_\varepsilon$ the inclusion. Find a nice immersion $g\colon K \looparrowright \mathbb{R}^4$ such that the push-forward $j \circ g$ is regularly homotopic to f, like Kinjo's ones for simple singularities.*

Note that since its normal bundle is trivial, by Hirsch's lemma, f is regularly homotopic to an immersion whose image is contained in 4-space. Furthermore, if we have an immersion of the Milnor fiber into 4-space, then its boundary gives an example of g by Theorem 7.2.3.

Problem VI. *In Corollary 7.2.6, the second component of the Smale invariant $\Omega(g_G \circ p)$ is always 1. Clarify the geometric reason for it.*

It will be crucial to give the geometric interpretation of the second component of

$$\Omega\colon \mathrm{Imm}[S^3, \mathbb{R}^4] \to \pi_3(\mathrm{SO}(4)) \cong \pi_3(S^3) \oplus \pi_3(\mathrm{SO}(3)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Problem VII. *As we saw in Corollary 7.2.7, among immersions $g_G \circ p\colon S^3 \looparrowright \mathbb{R}^4$ for all A-D-E cases, only two immersions $g_{D_{17}} \circ p$ and $g_{E_8} \circ p$ are regularly homotopic. Clarify the geometric reason for it.*

Problem VIII. *Determine whether the following assertion is true or not: for any two Brieskorn singularities determined by different triples $(a_1, a_2, a_3) \neq (b_1, b_2, b_3)$, if their links $K(a_1, a_2, a_3)$ and $K(b_1, b_2, b_3)$ are mutually diffeomorphic as abstract 3-manifolds, then their Milnor fibers have different signatures $\sigma(a_1, a_2, a_3) \neq \sigma(b_1, b_2, b_3)$, i.e., the inclusion maps of the links into the 5-sphere are not regularly homotopic.*

This is arisen from §8.1. The author made the numerical experiment to generate such pairs of singularities, but has not obtained any example with the same signatures of their Milnor fibers yet. This problem is already reduced the problem studying the formula for $\sigma(a_1, a_2, a_3)$ given in Theorem 8.1.6.

Problem IX. *Find the pair (K, K') of links of two different isolated surface singularity in $\mathbb{C}^3$ which are mutually diffeomorphic as abstract 3-manifolds and whose inclusion maps $f\colon K \hookrightarrow S^5_\varepsilon$, $f'\colon K' \hookrightarrow S^5_\varepsilon$ are also isotopic.*

This is stronger than Problem III. The possibility still remains that the pairs of links that appeared in §8.2 are such examples.

Finally, we pose the following problem.

Problem X. *Establish the theory of counting singularities of prescribed types, on maps which are fixed on boundaries or submanifolds.*

As noted in Chapter 3, formulae for invariants of immersions are derived using Thom polynomial theory. In a private communication with Ohmoto, Takase pointed out that his works [Tak07, Tak12] can be interpreted as a relative version of Thom polynomials. This interpretation seems very natural, and the development and formal establishment of the theory are expected. However, no such work has yet been published. The author is currently preparing an article to address this issue [Tan25b].

Appendix A

Geometric properties of the Wu invariant, II

In Chapter 6, we investigated the property of the Wu invariant with respect to an almost contact parallelization. In this appendix, we determine how the Wu invariant changes when the parallelization is replaced by a different one. We also discuss the realization problem of a given cohomology class as the Wu invariant.

A.1 Statements and Setup

Let M^3 be an oriented 3-manifold and $f\colon M^3 \looparrowright \mathbb{R}^5$ an immersion. We show the following two.

Proposition A.1.1. *For two parallelizations $\tau_0, \tau_1\colon TM^3 \xrightarrow{\cong} M^3 \times \mathbb{R}^3$ of M^3, it holds that*

$$c_{\tau_1}(f) = c_{\tau_0}(f) + \beta(d(\tau_0, \tau_1)),$$

where $d(\tau_0, \tau_1) \in H^1(M^3; \pi_1(\mathrm{SO}(3))) \cong H^1(M^3; \mathbb{Z}_2)$ is the difference class between τ_0 and τ_1 (see below) and $\beta\colon H^1(M^3; \mathbb{Z}_2) \to H^2(M^3; \mathbb{Z})$ is the Bockstein homomorphism.

Proposition A.1.2. *For the normal Euler class $\chi \in H^2(M^3; \mathbb{Z})$ of f and any class $c \in \Gamma_2(\chi)$, there exists a parallelization τ of M^3 such that $c_\tau(f) = c$.*

To prove these, we make a bit preparation. We consider the homogeneous space $G_{5,3} \coloneqq \mathrm{SO}(5)/\{\mathrm{SO}(3) \times \mathrm{SO}(2)\}$, the oriented Grassmannian. Then we have the natural quotient $V_{5,3} \to G_{5,3}$, the Gauss map $\gamma_f\colon M^3 \to G_{5,3}$ of the immersion $f\colon M^3 \looparrowright \mathbb{R}^5$, and the following pull-back diagram.

$$\begin{tikzcd} (\gamma_f)^*\pi \arrow[r, dashed, "\widetilde{\gamma}_f"] \arrow[d, dashed] & V_{5,3} \arrow[d, "\pi"] \\ M^3 \arrow[r, "\gamma_f"] & G_{5,3} \end{tikzcd}$$

Notice that there is one-to-one correspondence

$$\{\text{parallelizations of } M^3\} \longleftrightarrow \{\text{sections of } (\gamma_f)^*\pi\},$$

which is given by $\varphi_{\tau,f} = \widetilde{\gamma}_f \circ s$ for a parallelization τ and the corresponding section s. Also notice that the bundle $(\gamma_f)^*\pi$ is trivializable, since this is a principal SO(3)-bundle and has a section.

Let $d(\tau_0, \tau_1) \in H^1(M^3; \pi_1(\mathrm{SO}(3))) \cong H^1(M^3; \mathbb{Z}_2)$ denote the *difference class* between two parallelizations τ_0 and τ_1 of M^3 over the 1-skeleton of M^3, as sections of $(\gamma_f)^*\pi$.

A.2 Proof of Proposition A.1.1

We trivialize the bundle $(\gamma_f)^*\pi$ by $\varphi_{\tau_0,f}$, i.e., so that τ_0 corresponds to the constant section. Then $\widetilde{\gamma}_f \colon M^3 \times \mathrm{SO}(3) \to V_{5,3}$ is of the form

$$\widetilde{\gamma}_f(x, A) = \varphi_{\tau_0,f} \cdot A.$$

Furthermore, the other parallelization τ_1 of M^3 forms the section

$$s_{\tau_1} \colon M^3 \to M^3 \times \mathrm{SO}(3), \quad s_{\tau_1}(x) = (x, g_{10}(x)),$$

where $g_{10} \colon M^3 \to \mathrm{SO}(3)$ is the transition function from τ_0 to τ_1 on the bundle TM^3. Then the following diagram commutes.

$$\begin{tikzcd}
M^3 \times \mathrm{SO}(3) \arrow[r, "\widetilde{\gamma}_f"] & V_{5,3} \arrow[d, "\pi"] \\
M^3 \arrow[u, "s_{\tau_1}"] \arrow[ur, "\varphi_{\tau_1,f}"] \arrow[r, "\gamma_f"] & G_{5,3}
\end{tikzcd}$$

We can verify that $c_{\tau_1}(f) = (s_{\tau_1})^* \circ (\widetilde{\gamma}_f)^*(\Sigma)$ from the diagram. Since it holds that

$$(\widetilde{\gamma}_f)^*(\Sigma) = (c_{\tau_0}(f), \iota^*(\Sigma)) \quad \text{and} \quad (s_{\tau_1})^*(u, t) = u + (g_{10})^*(t),$$

where $\iota \colon \mathrm{SO}(3) \hookrightarrow V_{5,3}$ is the inclusion as the fiber of $V_{5,3} \to G_{5,3}$, we have that

$$c_{\tau_1}(f) = c_{\tau_0}(f) + (g_{10})^* \circ \iota^*(\Sigma).$$

To conclude Proposition A.1.1, we claim the following two. Let $\kappa \in H^2(\mathrm{SO}(3); \mathbb{Z}) \cong \mathbb{Z}_2$ and $\lambda \in H^1(\mathrm{SO}(3); \mathbb{Z}_2) \cong \mathbb{Z}_2$ denote the generator, respectively.

Claim 1. *The restriction $\iota^* \colon H^2(V_{5,3}, \mathbb{Z}) \to H^2(\mathrm{SO}(3); \mathbb{Z})$ is a surjection, i.e., $\iota^*(\Sigma) = \kappa$.*

Proof. We decompose the inclusion $\iota\colon \mathrm{SO}(3) \hookrightarrow V_{5,3}$ into three maps as follows:

$$\iota = \rho \circ \iota'\colon \mathrm{SO}(3) \hookrightarrow \mathrm{SO}(5) \to V_{5,3},$$

where $\iota'\colon \mathrm{SO}(3) \hookrightarrow \mathrm{SO}(5)$ is the inclusion and $\rho\colon \mathrm{SO}(5) \to V_{5,3}$ is the natural quotient. First, the restriction $(\iota')^*\colon H^2(\mathrm{SO}(5);\mathbb{Z}) \to H^2(\mathrm{SO}(3);\mathbb{Z})$ is an isomorphism. Indeed, the 2-skeletons of SO(3) and SO(5) are the same. (In fact, both groups are isomorphic to $\mathbb{Z}_2$.) Second, the pull-back $\rho^*\colon H^2(V_{5,3};\mathbb{Z}) \to H^2(\mathrm{SO}(5);\mathbb{Z})$ is a surjection. Indeed, applying the Gysin exact sequence to the orientable SO(2)-bundle ρ, we have that

$$H^2(V_{5,3};\mathbb{Z}) \xrightarrow{\rho^*} H^2(\mathrm{SO}(5);\mathbb{Z}) \to H^1(V_{5,3};\mathbb{Z}) = 0.$$

Therefore $\iota^* = (\iota')^* \circ \rho^*$ is a surjection. □

Claim 2. *It holds that $(g_{10})^*(\lambda) = d(\tau_0, \tau_1) \in H^1(M^3;\mathbb{Z}_2)$.*

Proof. We regard $(g_{10})^*(\lambda) \in H^1(M^3;\mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(M^3), \mathbb{Z}_2)$. Then this homomorphism detects a loop $\gamma \in \pi_1(M^3)$ whose image $(g_{10})_*(\gamma) \in \pi_1(\mathrm{SO}(3))$ is the generator $[\mathrm{SO}(2)] \in \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$. Hence $(g_{10})^*(\lambda)$ is the obstruction to homotope g_{10} to the constant map on the 1-skeleton of M^3. Therefore $(g_{10})^*(\lambda)$ coincides with the difference class $d(\tau_0, \tau_1)$. □

The equality of Claim 2 is that of the upper row of the following diagram.

$$\begin{tikzcd}
H^1(\mathrm{SO}(3);\mathbb{Z}_2) \cong \mathbb{Z}_2 \arrow[r, "\tau^*"] \arrow[d, "\beta"] & H^1(M^3;\mathbb{Z}_2) \arrow[d, "\beta"] \\
H^2(\mathrm{SO}(3);\mathbb{Z}) \cong \mathbb{Z}_2 \arrow[r, "\tau^*"] & H^2(M^3;\mathbb{Z})
\end{tikzcd}$$

Taking down the equality by Bockstein homomorphisms, we have that

$$(g_{10})^*(\kappa) = \beta(d(\tau_0, \tau_1)) \in H^2(M^3;\mathbb{Z}).$$

Then Claims 1 and 2 prove Proposition A.1.1.

A.3 Proof of Proposition A.1.2

This proposition is an immediate conclusion of the following and the surjectivity of the Bockstein homomorphism $\beta\colon H^1(M^3;\mathbb{Z}_2) \to \Gamma_2(0)$.

Claim 3 (cf. [Ste51, §33.9])**.** *Fix a parallelization τ_0 of M^3. For any class $c' \in H^1(M^3;\mathbb{Z}_2)$, there exists a parallelization τ_1 such that $c' = d(\tau_0, \tau_1)$.*

Proof. We regard $c' \in H^1(M^3;\mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(M^3), \mathbb{Z}_2)$, and will find a map from M^3 to SO(3) which becomes a transition function from τ_0 to a desired τ_1. Let $\mathrm{sk}_i M^3$ denote the i-th skeleton of M^3. We make a skeletonwise construction as follows. On the 1-skeleton of M^3, we define a map $g'_{01}\colon \mathrm{sk}_1 M^3 \to \mathrm{SO}(3)$ so that for each simple loop γ in $\mathrm{sk}_1 M^3$,

- if $c'(\gamma) = 1$, then $g'_{01}|_\gamma \colon \gamma \to \mathrm{SO}(3)$ is a homeomorphism onto $\mathrm{SO}(2)$;
- if $c'(\gamma) = 0$, then $g'_{01}|_\gamma \colon \gamma \to \mathrm{SO}(3)$ is null-homotopic.

We can extend g'_{01} to a map $g_{01} \colon M^3 \to \mathrm{SO}(3)$, since each of homotopy groups

$$\pi_1(\mathrm{sk}_2\mathrm{SO}(3), \mathrm{sk}_1\mathrm{SO}(3)) = \pi_1(\mathbb{R}P^2, \mathrm{SO}(2)), \quad \pi_2(\mathrm{SO}(3), \mathrm{sk}_2\mathrm{SO}(3)) = \pi_2(\mathrm{SO}(3), \mathbb{R}P^2)$$

vanishes. We now define τ_1 so that g_{01} is the transition function from τ_0 to τ_1. Then it holds that $c' = d(\tau_0, \tau_1)$ by construction. $\square$

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