博士学位論文

On progress of Miyanishi conjecture

（宮西予想の進展について）

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２０２６年３月

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Abstract

We consider conditions when endomorphisms of varieties become automorphisms. For example, there is a remarkable theorem, called Ax-Grothendieck theorem, which states that any injective endomorphism of a variety is bijective. Over an algebraically closed field of characteristic zero, bijectivity of endomorphisms of varieties implies that the endomorphisms are automorphisms, thus Ax-Grothendieck theorem gives one of the conditions we consider. There is also a conjecture, called Miyanishi conjecture, which claims that for any endomorphism of a variety over an algebraically closed field of characteristic zero, if it is injective outside a closed subset of codimension at least 2, then it is an automorphism. We have studied this conjecture, and published some papers. In this thesis, we describe our published results, and add some new results which are related to those. Note that our early works contain some wrong proofs and miss some assumptions, however we fix those in this thesis. One of our approaches is using birational geometry, and nobody seems to have ever used it for Miyanishi conjecture. We obtain some conditions when Miyanishi conjecture holds, which are arising from the perspective of minimal model program. Another of our approaches is an application of the work by I. Biswas and N. Das. Recently, I. Biswas and N. Das proved that any endomorphism which satisfies the conditions of Miyanishi conjecture induces an automorphism of the smooth locus of the variety with some conditions. We apply this result to analyze the endomorphisms in Miyanishi conjecture, and obtain the result describing relations between the exceptional locus of endomorphisms and the singular loci of varieties.

Contents

1 Introduction

We consider conditions when endomorphisms of varieties become automorphisms. There is a remarkable theorem by J. Ax, called Ax-Grothendieck theorem, proved in [3].

Theorem 1.1 ([3, Theorem]). *Let A and B be schemes such that A is of finite type over B. For any endomorphism φ of A over B, if φ is injective, then φ is bijective.*

The following generalization of Theorem 1.1 is conjectured by M. Miyanishi in [9].

Conjecture 1.2 (Miyanishi conjecture, [9]). *Let $\varphi : X \to X$ be an endomorphism of a variety X over an algebraically closed field of characteristic zero, and let Y be a closed subset of X which is of codimension at least 2. If φ is injective on $X \setminus Y$, then φ is an automorphism.*

There are some known cases as described in Section 2.2, for example affine, complete, and smooth cases. One of our approaches to Conjecture 1.2 is using birational geometry, and nobody seems to have ever used birational geometric approaches. Birational geometry gives some new results, and we have treated those in [1][2]. Also, we use the results of [5] to analyze endomorphisms as in Conjecture 1.2. In this thesis, we describe the results in [1][2], and add some new results which are related to those. Note that our early works [1][2] contain some wrong proofs and miss some assumptions, however we fix those in this thesis. There are four main theorems. Three of these are obtained by birational geometric approaches, and the other is an application of [5].

Here are our main theorems. Theorem 1.4, 1.6 are new results not appearing in [1][2], Theorem 1.3 is the correction of a part of [2, Proposition 3.2], and Theorem 1.5 generalizes and fixes a part of [2, Theorem 1.7].

Theorem 1.3. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, and X a nonempty open subset of $\overline{X}$ with $\operatorname{codim}(\overline{X} \setminus X) \geq 2$. Assume that $\overline{X}$ is Fano, that is, the canonical divisor $K_{\overline{X}}$ of $\overline{X}$ is anti-ample. If $\overline{X}$ has log terminal singularities, then Conjecture 1.2 holds for X. Moreover, φ in Conjecture 1.2 induces an automorphism of $\overline{X}$.*

Theorem 1.4. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety and suppose it is a minimal model. Let X be a nonempty open subset of $\overline{X}$. Then, Conjecture 1.2 holds for X.*

Theorem 1.5. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety and suppose that there is the canonical model $\overline{X}^{\mathrm{can}}$ birational to $\overline{X}$. Let X be a nonempty open subset of $\overline{X}$. Then, Conjecture 1.2 holds for X*

Theorem 1.6. *Let $\varphi : X \to X$ be an endomorphism of a normal variety X and Y a closed subset of X. Assume that φ is injective on $X \setminus Y$, $\operatorname{codim} Y \geq 2$, and one of the followings holds.*

(1) φ is surjective,

(2) X is $\mathbb{Q}$-factorial,

(3) X is locally a complete intersection.

Then, birational transform by φ gives a permutation of irreducible components of $\operatorname{Sing}(X)$ preserving codimension.

2 Preliminaries

2.1 Notations

We work over an algebraically closed field k of characteristic zero. Varieties mean separated integral schemes of finite type over k, and curves mean 1-dimensional varieties. In this subsection, we clarify some terminologies and notations.

In Section 3, we consider Conjecture 1.2 for quasi-projective varieties X which are open dense subsets of some projective varieties $\overline{X}$. Then, we may regard endomorphisms of X as birational self-maps of $\overline{X}$.

Definition 2.1. *Let $\overline{X}$ be a projective variety. Birational self-maps of $\overline{X}$ are called birational automorphisms of $\overline{X}$. Birational automorphisms of $\overline{X}$ form a group $\operatorname{Bir}(\overline{X})$, called the birational automorphism group of $\overline{X}$. Automorphisms of $\overline{X}$ also form a group $\operatorname{Aut}(\overline{X})$, called the automorphism group of $\overline{X}$.*

When we consider birational maps, the maps need not to be defined on closed subsets. However, in some sense, we can map some closed subsets to closed subsets.

Definition 2.2. *Let $f : X \dashrightarrow Y$ be a birational map of projective varieties, and A an irreducible closed subset of X. If f is an isomorphism on some dense open subsets A' of A, then we call the closure of $f(A')$ in Y the birational transform of A by f, and it is denoted by $f_*(A)$. Note that $f_*(A)$ is independent of a choice of A'. If f is merely defined on some dense open subsets A' of A, then we call the closure of $f(A')$ in Y the proper image of A by f. The proper image of A by f is also independent of a choice of A'.*

Remark 2.3. *In Definition 2.2, let f be a birational morphism of projective varieties. If $\dim f(A) = \dim A$, then the birational transform and the proper* *image of A by f are $f(A)$, and it coincides with the support of the push-forward of A by f as cycles. Note that the push-forward of A by f as cycles is usually denoted by $f_*(A)$. On the other hand, if $\dim f(A) < \dim A$, then the birational transform of A by f is not defined, the proper image is $f(A)$, and the push-forward as cycles is zero.*

Birational transform and proper image by birational maps is not defined for closed subsets contained in the indeterminacy loci. However, we can define push-forward and pull-back for numerical classes of 1-cycles, and Weil divisors by birational maps of $\mathbb{Q}$-factorial normal varieties. To see this, we begin by defining push-forward of divisors and pull-back of numerical classes of 1-cycles by birational morphisms from normal varieties to $\mathbb{Q}$-factorial normal varieties.

Definition 2.4. *Let $f : X \to Y$ be a birational morphism of normal varieties with Y $\mathbb{Q}$-factorial. Since we assume that Y is $\mathbb{Q}$-factorial, we can extend push-forward of Weil divisors to Cartier $\mathbb{Q}$-divisors and Cartier $\mathbb{R}$-divisors so that the push-forwards are the same sort of divisors as the original divisors.*

For any curve C on Y, we define the pull-back $f^*[C]$ of its numerical class $[C]$ by f so that the following holds.

$$\text{For any } [D] \in NS_{\mathbb{R}}(X), ([D], f^*[C]) = ([f_*D], [C]),$$

where $NS(X)$ is the Néron-Severi group of X, and $NS_{\mathbb{R}}(X) = NS(X) \otimes \mathbb{R}$. We extend this definition linearly for any 1-cycle, $\mathbb{Q}$-1-cycle, and $\mathbb{R}$-1-cycle.

Definition 2.5. *Let $f : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial normal projective varieties. Take a resolution of indeterminacy $(\widetilde{X}, p, q)$ of f, that is, $\widetilde{X}$ is a smooth projective variety, p and q are birational proper morphisms which satisfy $q = f \circ p$ as birational maps.*

1. *For any Weil prime divisor D on X, we define its push-forward by f as $f_\# D = q_* p^* D$. We extend this definition linearly for any Weil $\mathbb{Q}$-divisor, Cartier divisor, Cartier $\mathbb{Q}$-divisor, and Cartier $\mathbb{R}$-divisor so that the resultant is the same sort of divisors as D.*
2. *For any Weil prime divisor D on Y, we define its pull-back by f as $f^\# D = (f^{-1})_\# D = p_* q^* D$. We extend this definition as (1).*
3. *For any numerical class of curves $[C]$ on X, we define its push-forward by f as $f_\#[C] = q_* p^*[C]$. We extend this definition linearly for any numerical class of 1-cycle, $\mathbb{Q}$-1-cycle, and $\mathbb{R}$-1-cycle.*
4. *For any numerical class of curves $[C]$ on Y, we define its pull-back by f as $f^\#[C] = (f^{-1})_\#[C] = p_* q^*[C]$. We extend this definition as (3).*

There are some important properties of push-forward and pull-back of 1-cycles and divisors by birational maps of $\mathbb{Q}$-factorial normal projective varieties.

Proposition 2.6. *Let $f : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial normal projective varieties. The followings hold.*

(1) Push-forward and pull-back of 1*-cycles and divisors by* f *are independent of the choice of a resolution of indeterminacy of* f*.*

(2) $f_\#$ *on* $NS_{\mathbb{R}}(X)$ *is the dual map of* $f^\#$ *on* $N_1(Y)$*, that is, projection formula holds. Similarly,* $f_\#$ *on* $N_1(X)$ *is the dual map of* $f^\#$ *on* $NS_{\mathbb{R}}(Y)$*. Note that* $N_1(X)$ *is the* $\mathbb{R}$*-vector space generated by numerical classes of* 1*-cycles on* X*.*

Proof. (1) For two resolutions of indeterminacy of f, we can take another resolution of indeterminacy of f which dominates two given resolutions. Precisely, if we have the following commutative diagrams

$$\begin{tikzcd}
& \widetilde{X}_1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & \\
X \arrow[rr, dashed, "f"] & & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
& \widetilde{X}_2 \arrow[dl, "p_2"'] \arrow[dr, "q_2"] & \\
X \arrow[rr, dashed, "f"] & & Y,
\end{tikzcd}$$

then we can take $(\widetilde{X}_3, p_3, q_3)$ with $P : \widetilde{X}_3 \to \widetilde{X}_1$ and $Q : \widetilde{X}_3 \to \widetilde{X}_2$ which makes the following diagram commute.

$$\begin{tikzcd}
& & \widetilde{X}_3 \arrow[dl, "P"'] \arrow[dr, "Q"] \arrow[dd, "q_3"] \arrow[ddll, bend right, "p_3"'] \arrow[ddrr, bend left, "p_3"] & & \\
& \widetilde{X}_1 \arrow[dl, "p_1"'] \arrow[dr, "q_1"] & & \widetilde{X}_2 \arrow[dl, "q_2"'] \arrow[dr, "p_2"] & \\
X \arrow[rr, dashed, "f"] & & Y & & X. \arrow[ll, dashed, "f"']
\end{tikzcd}$$

By this diagram, for any Weil divisor D on X, we have

$$(q_1)_*(p_1)^*D = (q_1)_*P_*P^*(p_1)^*D = (q_3)_*(p_3)^*D.$$

This implies that the definition of push-forwards of Weil divisors by birational maps is independent of the choice of a resolution of indeterminacy of f. The other assertions are similar.

(2) Let $(\widetilde{X}, p, q)$ be a resolution of indeterminacy of f. For any $[D] \in NS_{\mathbb{R}}(X)$, $[C] \in N_1(Y)$, we have

$$(f_\#[D], [C]) = (q_*p^*[D], [C]) = (p^*[D], q^*[C]) = ([D], p_*q^*[C]) = ([D], f^\#[C]).$$

This implies that projection formula holds, and the other assertions are similar. □

Remark 2.7. *Push-forwards of numerical classes of* 1*-cycles and divisors by birational maps are not generalizations of birational transforms of ones by birational maps. In fact, let* D *be a Weil prime divisor on* X *which is not contracted by* f*. Note that* f *is defined on some dense open subsets of* D *since* X *is normal. Then, we can also consider the birational transform* $f_*(D)$ *of* D *by* f*.* $f_*(D)$

is also a Weil prime divisor, however $f_{\#}(D)$ may have other components arising from exceptional prime divisors of f^{-1}, thus these may not coincide. This phenomenon can also occur for push-forwards of numerical classes of 1*-cycles.*

By Remark 2.7, we also see that effectiveness of numerical classes of 1-cycles is not preserved by push-forward by birational maps, and it is similar for divisors. The following propositions give us conditions when effectiveness of numerical class of 1-cycles is preserved by push-forward by birational maps.

Proposition 2.8. *Let $f : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial normal projective varieties, and $(\widetilde{X}, p, q)$ a resolution of indeterminacy of f. For any curve C on X, if C is not contained in $p(\mathrm{Ex}(p))$, then $p^*[C]$ is effective. Hence, $f_{\#}[C]$ is effective.*

Proof. We may assume that p is a composite of successive blowups with smooth centers. We prove the assertion by induction on the number of blowups.

When p is a single blowup, we consider a $\mathbb{Q}$-1-cycle $A = p_*^{-1}C - ((D_1, p_*^{-1}C)/(D_1, E_1))E_1$ on $\widetilde{X}$, where D_1 is the exceptional divisor of the blowup, and E_1 is an exceptional curve of the blowup. This cycle is effective and intersects with D_1 trivially. For any $[D] \in NS_{\mathbb{R}}(\widetilde{X})$, since $p^*p_*[D] - [D]$ is in the kernel of p_*, it is some multiple of $[D_1]$. Thus, we have

$$([D], [A]) = (p^*p_*[D], [A]) = (p_*[D], p_*[A]) = (p_*[D], [C]).$$

This implies that $[A]$ is equal to $p^*[C]$, and hence $p^*[C]$ is effective.

When the number of blowups is 2, we take an exceptional curve E_1 of the first blowup so that it is not contained in the second blowup center. Then, we can prove the assertion by the above argument. By repeating this argument, we can prove the assertion for general cases. $\square$

Proposition 2.9. *Let $f : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial normal projective varieties, and $(\widetilde{X}, p, q)$ a resolution of indeterminacy of f. For any curve C on X, if C is not contained in the indeterminacy locus of f, then $f_{\#}[C]$ is effective.*

Proof. We may assume that p is a composite of successive blowups with smooth centers. We construct a 1-cycle on $\widetilde{X}$ which generates the numerical class $p^*[C]$.

For the first blowup $p_1 : X_1 \to X$, let C_1 be a curve on X_1 which satisfies $p_1(C_1) = C$, and set $d_1 = \deg(C_1/C)$. We consider a $\mathbb{Q}$-1-cycle $A_1 = C_1/d_1 - ((D_1, C_1)/(D_1, E_1)d_1)E_1$ on X_1, where D_1 is the exceptional divisor of p_1, and E_1 is an exceptional curve of p_1. This cycle intersects with D_1 trivially, thus $[A_1] = (p_1)^*[C]$ as in the proof of Proposition. Note that we may take E_1 so that $p_1(E_1)$ is not contained in the indeterminacy locus of f if $p_1(D_1)$ is not.

For the second blowup $p_2 : X_2 \to X_1$, let C_2 be a curve on X_2 which satisfies $p_2(C_2) = C_1$, and set $d_2 = \deg(C_2/C_1)$. We may assume that E_1 is not contained in the center of p_2. We consider a $\mathbb{Q}$-1-cycle $A_2 = C_2/d_2 + (p_2)_*^{-1}E_1 - ((D_2, C_2)/(D_2, E_2)d_2 + (D_2, (p_1)_*^{-1}E_1)/(D_2, E_2))E_2$ on X_2, where D_2 is the exceptional divisor of p_2, and E_2 is an exceptional curve of

p_2. This cycle intersects with D_2 trivially, thus $[A_2] = (p_1)^*[A_1]$ as in the proof of Proposition. Note that we may take E_2 so that $(p_1 \circ p_2)(E_2)$ is not contained in the indeterminacy locus of f if $(p_2 \circ p_1)(D_2)$ is not.

By repeating this construction, we finally obtain a 1-cycle $\widetilde{A}$ on $\widetilde{X}$ of the form $\widetilde{C}/\widetilde{d}+a_1\widetilde{E}_2+\cdots+a_n\widetilde{E}_n$, where $\widetilde{C}$ is a curve on $\widetilde{X}$ which satisfies $p(\widetilde{C}) = C$, $\widetilde{d} = \deg(\widetilde{C}/C)$, n is the number of blowups, a_i is a rational number, and $\widetilde{E}_i$ is the birational transform of E_i on $\widetilde{X}$ for $i = 1, 2, \ldots, n$. By construction, $[\widetilde{A}] = p^*[C]$, and a_i can be negative only if C is contained in $(p_i \circ \cdots \circ p_1)(E_i)$. Thus, if a_i is negative, then $p(\widetilde{E}_i)$ is not contained in the indeterminacy locus of f, hence $q_*\widetilde{E}_i = 0$ by $q = f \circ p$. Therefore, we deduce $f_\#[C] = q_*p^*[C] = q_*[\widetilde{A}]$ is effective. $\square$

On considering effectiveness of push-forwards of Weil prime divisors by birational maps, we should compare the set of p-exceptional prime divisors with the set of q-exceptional prime divisors. There is an important notion introduced in [12], called rational contractions. C. Casagrande gives a useful characterization of rational contractions in [6] as follows.

Proposition 2.10 ([6, Remark 2.2]). *Let $f : X \dashrightarrow Y$ be a birational map of normal projective varieties with X $\mathbb{Q}$-factorial. Since f is birational, we have nonempty open subsets $U \subset X$ and $V \subset Y$ such that f induces an isomorphism between U and V. We also take a resolution of indeterminacy $(\widetilde{X}, p, q)$ of f. Then, the followings are equivalent.*

(1) f is a rational contraction,

(2) $\operatorname{codim}(Y \setminus V) \geq 2$,

(3) every p-exceptional prime divisor is also q-exceptional.

Since we will only use (2) and (3) in Proposition 2.10, we do not mention the definition of rational contractions.

Remark 2.11. *In the settings of Proposition 2.10, we note that $\widetilde{X}$ also gives a resolution of $f^{-1} : Y \dashrightarrow X$. By the equivalence of* (2) *and* (3) *in Proposition 2.10, if* $\operatorname{codim}(X \setminus U) \geq 2$ *and* $\operatorname{codim}(Y \setminus V) \geq 2$, *then the set of p-exceptional prime divisors coincides with the set of q-exceptional prime divisors.*

By using the notion of rational contractions, we have the following proposition on effectiveness of push-forwards of Weil prime divisors by birational maps.

Proposition 2.12. *Let $f : X \dashrightarrow Y$ be a birational map of $\mathbb{Q}$-factorial normal projective varieties. If f is a rational contraction, then for any Weil prime divisor D of X, $f_\#[D]$ is effective.*

Proof. Let $(\widetilde{X}, p, q)$ be a resolution of indeterminacy of f. Since f is a rational contraction, p-exceptional prime divisors are also q-exceptional by Proposition 2.10 (3). This implies $f_\# D = q_*p^*D = q_*p_*^{-1}D$ as p^*D is the sum of $p_*^{-1}D$ and some p-exceptional divisor. Hence, the assertion holds. $\square$

Next, we recall outcomes of minimal model program(MMP, in short). We may obtain minimal models, the canonical model, or Mori fiber spaces by MMP. See [16] for minimal model program.

Definition 2.13 ([16, Definition 3.50]). *Let $f : X \dashrightarrow X'$ be a birational map of $\mathbb{Q}$-factorial normal projective varieties. We consider the following conditions.*

(1) X has log canonical singularities,

(2) X has log terminal singularities,

(3) f^{-1} contracts no divisors,

(4) The canonical class $K_{X'}$ of X' is nef,

(5) $K_{X'}$ is ample,

(6) for any f-exceptional prime divisor D, we have $a(X, D) \leq a(X', D)$,

(7) for any f-exceptional prime divisor D, we have $a(X, D) < a(X', D)$,

where $a(X, D)$ is the discrepancy of D over X.

If (2), (3), (4), (7) *are satisfied, then we call X' a minimal model of X.*

If (1), (3), (5), (6) *are satisfied, then we call X' the canonical model of X.*

Definition 2.14. *Let $f : X \to S$ be a contraction of an extremal ray of a $\mathbb{Q}$-factorial normal projective variety X. We say X is a Mori fiber space over S if $\dim X > \dim S$. Moreover, if $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ holds, then we call X a birationally superrigid Mori fiber space.*

Definition 2.15. *Let X be a $\mathbb{Q}$-factorial normal projective variety. We say X is a Fano variety if the anti-canonical class $-K_X$ of X is ample. Note that, in Definition 2.14, if $\dim S = 0$, then X is a Fano variety.*

Minimal models and canonical models have uniqueness in some sense as follows.

Proposition 2.16 ([16, Theorem 3.52]). *Let X be a $\mathbb{Q}$-factorial normal projective variety.*

(1) Let $f : X \dashrightarrow X'$ and $g : X \dashrightarrow X''$ be two minimal models of X. Then, X' and X'' are isomorphic in codimension 1, that is, the set of exceptional prime divisors of f coincides with that of g.

(2) Canonical models of X are isomorphic to each other.

2.2 Known results

S. Kaliman showed that Conjecture 1.2 is true in the case that X is affine or complete in [13], and N. Das showed that Conjecture 1.2 is true in other several cases, especially when X is smooth, in [7].

S. Kaliman used the following two useful lemmas in [13].

Lemma 2.17 ([13, Lemma 2])**.** *If Conjecture 1.2 is true for normal varieties, then it is true for all varieties X.*

Lemma 2.18 ([13, Lemma 3])**.** *Let $Z = X \setminus \varphi(X \setminus Y)$. Then, Z is a closed subset of X and $\dim Y = \dim Z$ if $Z \neq \emptyset$.*

N. Das used the following lemma in [7].

Lemma 2.19 ([7, Lemma 2.1])**.** *Any endomorphism which satisfies the conditions of Conjecture 1.2 is birational.*

Remark 2.20. *Actually, the proof of Lemma 2.19 in [7] implies that any injective dominant morphism of varieties over k is birational.*

Due to Remark 2.20, in many situations, we can use a variant of Zariski's main theorem as follows.

Theorem 2.21 ([17, §9, Original form])**.** *Let $f : X \to Y$ be a birational morphism of varieties. If f has finite fibers and Y is normal, then f is an open embedding.*

Combining Remark 2.20 and Theorem 2.21, we have the following proposition.

Proposition 2.22 ([1, Proposition 1.11])**.** *Let $f : X \to Y$ be a bijective morphism of varieties. If X is normal, then f is an isomorphism.*

Proof. By Remark 2.20, f is birational. Thus, by Theorem 2.21, f is an open embedding. Since f is bijective, the image of f is Y, hence, f is an isomorphism. □

Since we may assume that X is normal by Lemma 2.17, it is enough to show that φ is injective by Theorem 1.1 and Proposition 2.22. Moreover, by Theorem 2.21, we may prove that there does not exist any curve contracted by φ.

In [5], I. Biswas and N. Das showed that any endomorphism as in Conjecture 1.2 induces an automorphism of the smooth locus under a certain condition as follows.

Theorem 2.23 ([5, Theorem 1.3])**.** *Let $\varphi : X \to X$ be an endomorphism of a normal variety X, and let Y be a closed subset of X. We take a positive integer c such that $\operatorname{codim} Y \geq c$, and set $W = X \setminus \operatorname{Sing} X$. Assume that φ is injective on $X \setminus Y$, and one of the followings holds.*

(1) $c = 2$, *and φ is surjective,*

(2) $c = 2$*, and* X *is* $\mathbb{Q}$*-factorial,*

(3) $c = 2$*, and* X *is locally a complete intersection,*

(4) $c = 3$.

Then, we have $\varphi(W) \subset W$*, and* $\varphi|_W : W \to W$ *is an isomorphism.*

To prove Conjecture 1.2, we can replace the endomorphisms with some iterate as follows.

Lemma 2.24 ([1, Lemma 3.2])**.** *We use the same notation in Conjecture 1.2. For any positive integer* m*, let* $Y_m = Y \cup \varphi^{-1}(Y) \cup \cdots \cup (\varphi^{m-1})^{-1}(Y)$*. Then,* φ^m *is injective on* $X \setminus Y_m$*, and* $\operatorname{codim} Y_m \geq 2$*. In particular, we may replace* φ *with* φ^m *to prove Conjecture 1.2.*

Proof. Since φ is injective on $X \setminus Y$ and $\operatorname{codim} Y \geq 2$, the claims are obvious. $\square$

3 Birational geometric argument

In this section, we consider Conjecture 1.2 for quasi-projective varieties X. Since quasi-projective varieties have projective varieties as ambient spaces, say $\overline{X}$ for X, we can use properties of $\overline{X}$.

3.1 The conditions on pull-backs of the canonical divisors

Take X and $\overline{X}$ as above, and let Y be a closed subset of X of codimension at least 2, and φ an endomorphism of X. Assume that φ is injective on $X \setminus Y$. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. Also assume $\overline{X}$ has a sequence of divisorial contractions $\overline{X} = \overline{X}_0 \to \overline{X}_1 \to \cdots \to \overline{X}_n$. Let f_i be the morphism from $\overline{X}$ to $\overline{X}_i$ for any $i \in \{0, 1, \ldots, n\}$. In these settings, we consider the following conditions.

($\dagger$) For some resolutions of indeterminacy $(\widetilde{X}, p, q)$ of $\overline{\varphi}$, we have$(f_i \circ p)^* K_{\overline{X}_i} = (f_i \circ q)^* K_{\overline{X}_i}$ in $NS_{\mathbb{R}}(\widetilde{X})$ for any $i \in \{0, 1, \ldots, n\}$.

($\dagger'$)There exists a resolutions of indeterminacy $(\widetilde{X}, p, q)$ of $\overline{\varphi}$ such that for any i and prime divisor $\widetilde{E}$ on $\widetilde{X}$ which is both $(f_i \circ p)$-exceptional and $(f_i \circ q)$-exceptional, the coefficient of $\widetilde{E}$ in $(f_i \circ p)^* K_{\overline{X}_i}$ coincides with that in $(f_i \circ q)^* K_{\overline{X}_i}$.

In some arguments in [1][2], we used ($\dagger$) or ($\dagger'$) without mention, however we should not have done so since this may not hold. In this thesis, we assume ($\dagger$) or ($\dagger'$) properly and fix arguments in [1][2] which contain something wrong.

Proposition 3.1. *If* ($\dagger$) *holds for* φ*, then* ($\dagger$) *also holds for any iterate of* φ*.*

Proof. We prove that if (†) holds for φ, then (†) also holds for φ^2. It is similar for general iterates.

Take a resolution of indeterminacy $(\widetilde{X}, p, q)$ of $\overline{\varphi}$ which makes (†) hold for φ. We also take a resolution of indeterminacy (X', p', q') of $q^{-1} \circ p : \widetilde{X} \dashrightarrow \widetilde{X}$, that is, we have the following commutative diagram for any i.

$$\begin{tikzcd}
 & & X' \arrow[dl, "p'"'] \arrow[dr, "q'"] & & \\
 & \widetilde{X} \arrow[dl, "p"'] \arrow[dr, "q"] & & \widetilde{X} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\overline{X} \arrow[rr, dashed, "\overline{\varphi}"] \arrow[d, "f_i"] & & \overline{X} \arrow[rr, dashed, "\overline{\varphi}"] \arrow[d, "f_i"] & & X \arrow[d, "f_i"] \\
\overline{X}_i & & \overline{X}_i & & \overline{X}_i .
\end{tikzcd}$$

Then, we have

$$\begin{aligned}
(f_i \circ p \circ p')^* K_{\overline{X}_i} &= (p')^*(f_i \circ p)^* K_{\overline{X}_i} = (p')^*(f_i \circ q)^* K_{\overline{X}_i} = (q')^*(f_i \circ p)^* K_{\overline{X}_i} \\
&= (q')^*(f_i \circ q)^* K_{\overline{X}_i} = (f_i \circ q \circ q')^* K_{\overline{X}_i}
\end{aligned}$$

in $NS_{\mathbb{R}}(X')$. Thus, (†) holds for φ^2. □

Remark 3.2. *Note that (†) is stronger than (†′), and (†) is closed under iteration of φ as Proposition 3.1, while (†′) does not seem closed under iteration of φ. Thus, when we use (†′), we may assume (†′) for an appropriate iterate of φ according to the situation.*

Proposition 3.3. *Suppose that $\overline{X}_n$ is one of the followings.*

(1) A minimal model,

(2) a canonical model,

(3) a birationally superrigid Mori fiber space.

Then, $(f_n \circ p)^* K_{\overline{X}_n} = (f_n \circ q)^* K_{\overline{X}_n}$ in $NS_{\mathbb{R}}(\widetilde{X})$, a part of (†), holds.

Proof. Let $\overline{\varphi}_n$ be the birational automorphism of $\overline{X}_n$ induced by $\overline{\varphi}$.

(1) By [14, Theorem 1], $\overline{\varphi}_n$ factors into the composition of flops. See [14, Theorem 1] for definition of flops, and the definition makes the assertion hold.

(2)(3) Since $\overline{\varphi}_n$ is an automorphism automatically, we have

$$(f_n \circ p)^* K_{\overline{X}_n} = (f_n \circ p)^* (\overline{\varphi}_n)^* K_{\overline{X}_n} = (\overline{\varphi}_n \circ f_n \circ p)^* K_{\overline{X}_n} = (f_n \circ q)^* K_{\overline{X}_n}$$

in $NS_{\mathbb{R}}(\widetilde{X})$. □

Here, we explain an example where (†) seems to hold. We consider one of the cases in Proposition 3.3, and try to prove (†) by induction on n. For $n = 0$, the assertion is a special case of Proposition 3.3. Assume that (†) holds for $n-1$, in particular, we assume $(f_1 \circ p)^* K_{\overline{X}_1} = (f_1 \circ q)^* K_{\overline{X}_1}$ in $NS_{\mathbb{R}}(\widetilde{X})$. Let E_1 be the exceptional prime divisor of f_1. Then, we have

$$(f_1 \circ p)^* K_{\overline{X}_1} = p^*(f_1)^* K_{\overline{X}_1} = p^*(K_{\overline{X}} - a(E_1, \overline{X}_1)),$$

$$(f_1 \circ q)^* K_{\overline{X}_1} = q^*(f_1)^* K_{\overline{X}_1} = q^*(K_{\overline{X}} - a(E_1, \overline{X}_1))$$

in $NS_{\mathbb{R}}(\widetilde{X})$. Thus, if $p^* E_1 = q^* E_1$, then $p^* K_{\overline{X}} = q^* K_{\overline{X}}$. Also assume $\operatorname{codim}(\overline{X} \setminus X) \geq 2$. Then, by Proposition 2.16(1) for the minimal model case, or the birational automorphism of $\overline{X}_n$ induced by $\overline{\varphi}$ being an automorphism for the canonical model case and the birationally superrigid Mori fiber space case, we have that $\overline{\varphi}$ acts the set of exceptional prime divisors of f_n as a permutation. Since the set of exceptional prime divisors of f_n is a finite set, we may assume $\overline{\varphi}_* E_1 = E_1$ by replacing $\overline{\varphi}$ with some iterate. Then, $p^* E_1 = q^* E_1$ seems to hold, however we have not been able to show that yet.

3.2 An easy case

By using push-forward of 1-cycles by birational maps, we easily obtain the following result.

Proposition 3.4. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, X a nonempty open subset of $\overline{X}$, Y a closed subset of X of codimension at least 2, and φ an endomorphism of X. If φ is injective on $X \setminus Y$, then there does not exist any curve C on Y such that C is also closed in $\overline{X}$ and $\varphi(C)$ is a point.*

Proof. Suppose such a C exists. We regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. We consider the push-forward of $[C]$ by φ. By the proof of Proposition 2.9, we see that $\varphi_{\#}[C]$ is zero. This and projection formula imply that C intersects with pull-backs of any divisors by $\overline{\varphi}$ trivially. We also know that C does not meet divisors in $\overline{X} \setminus X$, and pull-backs of prime divisors by $\overline{\varphi}$ are of the form $D + E$, where D is a prime divisor meeting X or 0, E is a divisor contained in $\overline{X} \setminus X$. Since any divisor meeting X appears in pull-backs of some divisors by $\overline{\varphi}$, we deduce that C intersects with any divisors trivially. This contradicts that $\overline{X}$ is projective. □

3.3 For outcomes of MMP

As seen in Section 2.1, there are three types of outcomes of MMP, minimal models, canonical models, and Mori fiber spaces. Since canonical models and birationally superrigid Mori fiber spaces have only automorphisms as birational automorphisms, we have the following proposition.

Proposition 3.5 (Proposition 3.1, [2]). *Let $\overline{X}$ be a complete variety, and X be a nonempty open subset of $\overline{X}$. If* $\mathrm{Bir}(\overline{X}) = \mathrm{Aut}(\overline{X})$ *holds, then Conjecture 1.2 holds for X.*

In particular, Conjecture 1.2 holds for X which is a nonempty open subset of a canonical model, or a birationally superrigid Mori fiber space.

Proof. We may regard the endomorphism φ of X as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. By assumption, any birational automorphism of $\overline{X}$ is an automorphism of $\overline{X}$, thus $\overline{\varphi}$ is an automorphism of $\overline{X}$. In particular, φ is an automorphism of X. □

A typical type of Mori fiber spaces is Fano varieties. Since Fano varieties are $\mathbb{Q}$-factorial, we can use a very powerful tool, intersection theory.

Proposition 3.6 (The correction of [1, Theorem 1.3]). *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, and X a nonempty open subset of $\overline{X}$ with* $\mathrm{codim}(\overline{X} \setminus X) \geq 2$. *If the canonical divisor $K_{\overline{X}}$ of $\overline{X}$ is ample or anti-ample, then Conjecture 1.2 holds for X under* (†).

In particular, Conjecture 1.2 holds for X which is an open subset of a Fano variety with the condition $\mathrm{codim}(\overline{X} \setminus X) \geq 2$ *under* (†).

Proof. We take a resolution of indeterminacy $(\widetilde{X}, p, q)$ of $\overline{\varphi}$. Suppose that φ is not an automorphism. Then, there is a curve C on X such that $\varphi(C)$ is a point. Let $\overline{C}$ be the closure of C in $\overline{X}$, and $\widetilde{C}$ be a curve on $\widetilde{X}$ such that $p(\widetilde{C}) = \overline{C}$. Note that $q(\widetilde{C})$ is a point. By (†), we have $p^*K_{\overline{X}} = q^*K_{\overline{X}}$ in $NS_{\mathbb{R}}(\widetilde{X})$. Then, we have

$$0 = (K_{\overline{X}}, q_*\widetilde{C}) = (q^*K_{\overline{X}}, \widetilde{C}) = (p^*K_{\overline{X}}, \widetilde{C}) = (K_{\overline{X}}, p_*\widetilde{C}) = \deg(\widetilde{C}/\overline{C})(K_{\overline{X}}, \overline{C}).$$

Since $\deg(\widetilde{C}/\overline{C})$ is a positive integer, we have $(K_{\overline{X}}, \overline{C}) = 0$, however this contradicts that $K_{\overline{X}}$ is ample or anti-ample. □

Remark 3.7. *We can consider Proposition 3.6 without* (†). *If $K_{\overline{X}}$ is ample, then $\overline{X}$ often becomes a canonical model, thus we may apply Proposition 3.5. For the case that $K_{\overline{X}}$ is anti-ample, we can prove one of our main theorems as follows.*

Theorem 3.8. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, and X a nonempty open subset of $\overline{X}$ with* $\mathrm{codim}(\overline{X} \setminus X) \geq 2$. *Assume that $\overline{X}$ is Fano, that is, the canonical divisor $K_{\overline{X}}$ of $\overline{X}$ is anti-ample. If $\overline{X}$ has log terminal singularities, then Conjecture 1.2 holds for X. Moreover, φ in Conjecture 1.2 induces an automorphism of $\overline{X}$.*

Proof. By [4, Corollary 1.3.2], $\overline{X}$ is a Mori dream space. Then, we have a bijection between the set of rational contractions of $\overline{X}$ and the set of faces of $\mathrm{Nef}(\overline{X})$, the cone of nef divisors on $\overline{X}$. Precisely, for any rational contraction $f : \overline{X} \dashrightarrow X'$ of $\overline{X}$, we have a face $f^{\#}(\mathrm{Nef}(X'))$ of $\mathrm{Nef}(\overline{X})$. In particular, for two distinct rational contractions $f : \overline{X} \dashrightarrow X'$ and $g : \overline{X} \dashrightarrow X''$, the interior of $f^{\#}(\mathrm{Nef}(X'))$ does not meet the interior of $g^{\#}(\mathrm{Nef}(X''))$. See [6, Section 3] for this fact and Mori dream spaces. Note that for two rational contractions $f : \overline{X} \dashrightarrow X'$ and $g : \overline{X} \dashrightarrow X''$, f is identified with g if there is an isomorphism $h : X' \to X''$ such that $g = h \circ f$. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. By Proposition 2.10 and the assumption $\mathrm{codim}(\overline{X} \setminus X) \geq 2$, $\overline{\varphi}$ is a rational contraction. Moreover, we have $\overline{\varphi}^*(-K_{\overline{X}}) = -K_{\overline{X}}$ and $-K_{\overline{X}}$ is in the interior of $\mathrm{Nef}(\overline{X}) = (\mathrm{id}_{\overline{X}})^{\#}\mathrm{Nef}(\overline{X})$ since it is ample. Then, the interior of $(\mathrm{id}_{\overline{X}})^{\#}\mathrm{Nef}(\overline{X})$ intersect with the interior of $\overline{\varphi}^{\#}\mathrm{Nef}(\overline{X})$. Thus, we have $\mathrm{id}_{\overline{X}} = \overline{\varphi}$ as rational contractions, that is, there is an isomorphism $h : \overline{X} \to \overline{X}$ such that $\overline{\varphi} = h \circ \mathrm{id}_{\overline{X}}$. Hence, $\overline{\varphi}$ is an automorphism of $\overline{X}$. In particular, φ is an automorphism of X. □

For birational automorphisms of minimal models, there is an important result in [14], which states that birational automorphisms of minimal models are factored into sequences of flops. By using this result, we obtain one of our main theorems as follows.

Theorem 3.9. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety and suppose it is a minimal model. Let X be a nonempty open subset of $\overline{X}$, Y a closed subset of X of codimension at least 2, and φ an endomorphism of X. If φ is injective on $X \setminus Y$, then φ is an automorphism of X.*

To prove Theorem 3.9, we begin with preparing a lemma and a proposition.

Lemma 3.10. *We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. Let $(\widetilde{X}, p, q)$ be a resolution of indeterminacy of φ. For any divisor $\overline{H}$ on $\overline{X}$, we have $(p^*\overline{\varphi}_*^{-1}\overline{H})|_{p^{-1}(X)} = (q^*\overline{H})|_{p^{-1}(X)}$.*

Proof. Let $q|^X_{p^{-1}(X)}$ denote the morphism $p^{-1}(X) \to X$ induced by q. Note that on $p^{-1}(X)$, we have $q = \varphi \circ p$ as morphisms. Thus, we have

$$\begin{aligned}(q^*\overline{H})|_{p^{-1}(X)} &= (q|^X_{p^{-1}(X)})^*(\overline{H}|_X) = (\varphi \circ p|^X_{p^{-1}(X)})^*(\overline{H}|_X) = (p|^X_{p^{-1}(X)})^*\varphi^*(\overline{H}|_X) \\ &= (p|^X_{p^{-1}(X)})^*((\overline{\varphi}_*^{-1}\overline{H})|_X) = (p^*\overline{\varphi}_*^{-1}\overline{H})|_{p^{-1}(X)}.\end{aligned}$$

□

According to the proof of [14, Theorem 1], we can obtain $\overline{\varphi}$ by running minimal model program for $(\overline{X}, \overline{D})$, where $\overline{D}$ is the birational transform of some effective ample divisor $\overline{H}$ on $\overline{X}$ by $\overline{\varphi}^{-1}$, that is, $\overline{D} = \overline{\varphi}_*^{-1}\overline{H}$. Moreover, divisorial contractions cannot appear in this minimal model program since birational automorphisms of minimal models are isomorphisms in codimension 1.

Proposition 3.11. *Let C be a curve on X, and $\overline{C}$ the closure of C in $\overline{X}$. Then, we have $(\overline{D}, \overline{C}) \geq 0$. In particular, we have $(K_{\overline{X}} + \overline{D}, \overline{C}) \geq 0$ since $K_{\overline{X}}$ is nef.*

Proof. Let $\widetilde{C}$ be a curve on $\widetilde{X}$ such that $p(\widetilde{C}) = \overline{C}$. By definition of flops in [14], we have $p^*K_{\overline{X}} = q^*K_{\overline{X}}$. In addition, we have $\mathrm{supp}(p^*\overline{D} - q^*\overline{H}) \cap p^{-1}(X) = \emptyset$ by Lemma 3.10. Thus, in $NS_{\mathbb{R}}(\widetilde{X})$, we have

$$p^*(K_{\overline{X}} + \overline{D}) - q^*(K_{\overline{X}} + \overline{H}) = \sum_{\widetilde{E}} a(\widetilde{E})\widetilde{E},$$

where $\widetilde{E}$ runs over the exceptional prime divisors of p not intersecting $p^{-1}(X)$, and each $a(\widetilde{E})$ is the difference $a(\widetilde{E}, \overline{X}, \overline{H}) - a(\widetilde{E}, \overline{X}, \overline{D})$ between log discrepancies of $\widetilde{E}$. Since $\overline{\varphi} : (\overline{X}, \overline{D}) \dashrightarrow (\overline{X}, \overline{H})$ is obtained by minimal model program, $p^*(K_{\overline{X}} + \overline{D}) - q^*(K_{\overline{X}} + \overline{H})$ is effective, that is, each $a(\widetilde{E})$ is nonnegative. Furthermore, for each $\widetilde{E}$, we have $(\widetilde{E}, \widetilde{C}) \geq 0$ since $\widetilde{C}$ intersects with $p^{-1}(X)$ but $\widetilde{E}$ does not. Thus, we have

$$0 \leq (p^*(K_{\overline{X}}+\overline{D})-q^*(K_{\overline{X}}+\overline{H}), \widetilde{C}) = (p^*K_{\overline{X}}-q^*K_{\overline{X}}, \widetilde{C})+(\overline{D}, p_*\widetilde{C})-(\overline{H}, q_*\widetilde{C}).$$

Since we have $p^*K_{\overline{X}} = q^*K_{\overline{X}}$ and $\overline{H}$ is ample, the above inequality implies

$$0 \leq (\overline{D}, p_*\widetilde{C}) = \deg(\widetilde{C}/\overline{C})(\overline{D}, \overline{C}).$$

As $\deg(\widetilde{C}/\overline{C})$ is a positive integer, we have $(\overline{D}, \overline{C}) \geq 0$. □

Proof of Theorem 3.9. Let $(\overline{X}, \overline{D}) = (\overline{X}_0, \overline{D}_0) \dashrightarrow (\overline{X}_1, \overline{D}_1) \dashrightarrow \cdots \dashrightarrow (\overline{X}_n, \overline{D}_n) = (\overline{X}, \overline{H})$ be a minimal model program factoring $\overline{\varphi}$. For $i \in \{1, 2, \cdots, n\}$, g_i denotes the i-th step of this program. By Proposition 3.11, the exceptional locus $\mathrm{Ex}(g_1)$ of g_1 does not intersect with X, and similarly, we have $\mathrm{Ex}(g_i) \cap (g_{i-1} \circ g_{i-2} \circ \cdots \circ g_1)(X) = \emptyset$. This implies that φ is injective on X. □

3.4 Descent along divisorial contractions

There are two types of birational maps appearing in MMP, divisorial contractions and flips. Divisorial contractions are morphisms and go well with intersection theory. In this subsection, we extend some results in Section 3.3 to ones admitting divisorial contractions.

The following is an extension of Proposition 3.6.

Proposition 3.12. *Let X be a nonempty open subset of a $\mathbb{Q}$-factorial normal projective variety $\overline{X}$. Suppose that $\overline{X}$ has canonical singularities and $\overline{X}$ has a sequence of divisorial contractions $\overline{X} = \overline{X}_0 \to \overline{X}_1 \to \cdots \to \overline{X}_n$. Let Y be a closed subset of X of codimension at least 2, and φ an endomorphism of X. If φ is injective on $X \setminus Y$ and $K_{\overline{X}_n}$ is ample or anti-ample, then φ is an automorphism of X under (†).*

In particular, Conjecture 1.2 holds for X when $\overline{X}_n$ is a Fano variety and (†) holds.

Proof. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. Let $(\widetilde{X}, p, q)$ be a resolution of indeterminacy of $\overline{\varphi}$ which makes (†) hold. Assume that φ is not an automorphism, then there exists a curve C in X such that C is contracted by φ. Let $\overline{C}$ be the closure of C in $\overline{X}$, and $\widetilde{C}$ a curve in $\widetilde{X}$ such that $p(\widetilde{C}) = \overline{C}$. Note that $q_*\widetilde{C} = 0$. Let f_1 be the morphism $\overline{X} \to \overline{X}_1$, and $\overline{E}_1$ be its exceptional prime divisor. By (†), we have $p^*K_{\overline{X}} = q^*K_{\overline{X}}$. Then, we have

$$0 = (K_{\overline{X}}, q_*\widetilde{C}) = (q^*K_{\overline{X}}, \widetilde{C}) = (p^*K_{\overline{X}}, \widetilde{C}) = (K_{\overline{X}}, p_*\widetilde{C}) = \deg(\widetilde{C}/\overline{C})(K_{\overline{X}}, \overline{C}).$$

Since $\deg(\widetilde{C}/\overline{C})$ is a positive integer, we have $(K_{\overline{X}}, \overline{C}) = 0$. This implies that f_1 does not contract $\overline{C}$. Set $\overline{C}_1 = f_1(\overline{C})$. We also have $(f_1 \circ p)^*K_{\overline{X}_1} = (f_1 \circ q)^*K_{\overline{X}_1}$ by (†). Since $(f_1)^*K_{\overline{X}_1} = K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1$, we have

$$\begin{aligned} 0 &= (K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, q_*\widetilde{C}) = (q^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1), \widetilde{C}) \\ &= (p^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1), \widetilde{C}) = (K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, p_*\widetilde{C}) \\ &= \deg(\widetilde{C}/\overline{C})(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, \overline{C}). \end{aligned}$$

Since $\deg(\widetilde{C}/\overline{C})$ is a positive integer, we have $(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, \overline{C}) = 0$. Thus, we have $a(\overline{E}_1, \overline{X}_1)(\overline{E}_1, \overline{C}) = 0$. This implies $(K_{\overline{X}_1}, \overline{C}_1) = 0$.

We repeat this operation, and we finally obtain the curve $\overline{C}_n = f_n(\overline{C})$ in $\overline{X}_n$, where f_n is the morphism $\overline{X} \to \overline{X}_n$. By construction, $\overline{C}_n$ satisfies $(K_{\overline{X}_n}, \overline{C}_n) = 0$, however this contradicts that $K_{\overline{X}_n}$ is ample or anti-ample. □

The following is an extension of Proposition 3.5, and also the correction of [2, Theorem 1.7]. Note that the proof in [2] is invalid without some additional conditions like (†). However, by observing the proof of [2, Theorem 1.7], we see that it is enough for making the proof valid to assume a weaker condition than (†) described later.

Proposition 3.13 (The correction of [2, Theorem 1.7])**.** *Let X be a nonempty open subset of a $\mathbb{Q}$-factorial normal projective variety $\overline{X}$. Suppose that $\overline{X}$ has canonical singularities and $\overline{X}$ has a sequence of divisorial contractions $\overline{X} = \overline{X}_0 \to \overline{X}_1 \to \cdots \to \overline{X}_n$. Let Y be a closed subset of X of codimension at least 2, and φ an endomorphism of X. If φ is injective on $X \setminus Y$, $\mathrm{Bir}(\overline{X}_n) = \mathrm{Aut}(\overline{X}_n)$ holds, and (†′) holds for an appropriate iterate of φ as described later, then φ is an automorphism.*

In particular, Conjecture 1.2 holds for X when $\overline{X}_n$ is the canonical model and (†′) holds for an appropriate iterate of φ as described later.

We follow [2, Section 3.2]. We set

$$EPD(f)_X = \{\textit{an exceptional prime divisor } \overline{E} \textit{ of } f \mid \overline{E} \cap X \neq \emptyset\}.$$

We may regard the endomorphism φ of X as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. Since we suppose $\mathrm{Bir}(\overline{X}_n) = \mathrm{Aut}(\overline{X}_n)$, the operation of taking birational transform by $\overline{\varphi}$ acts $EPD(f)_X$ as a permutation. Thus, we assume that the action is trivial by replacing φ with some iterate by Lemma 2.24. For any $i = 0, 1, \ldots, n$, let f_i be the morphism $\overline{X} \to \overline{X}_i$. We assume (†′) for φ after replacing φ to make the action is trivial.

Let $\overline{D}_1, \cdots, \overline{D}_m$ be the prime divisors which appear in the irreducible decomposition of $\overline{X} \setminus X$. Note that if $\mathrm{codim}(\overline{X} \setminus X)$ is at least 2, then we may ignore this notation since such a prime divisor does not exist.

We take a resolution of indeterminacy $(\widetilde{X}, p, q)$ of $\overline{\varphi}$ which makes $(\dagger')$ hold. Note that the set of p-exceptional prime divisors does not necessarily coincide with the set of q-exceptional prime divisors since $\overline{\varphi}$ and $\overline{\varphi}^{-1}$ may contract some divisors which are contained in $\overline{X} \setminus X$. However, we can prove the following lemmas.

Lemma 3.14 ([2, Lemma 3.3]). *We set*

$$P = \{\textit{an exceptional prime divisor of } p\} \setminus \{q_*^{-1}\overline{D}_1, \cdots, q_*^{-1}\overline{D}_m\},$$

$$Q = \{\textit{an exceptional prime divisor of } q\} \setminus \{p_*^{-1}\overline{D}_1, \cdots, p_*^{-1}\overline{D}_m\}.$$

Then, we have $P = Q$.

Proof. First of all, we explain the definitions of P and Q. If $\overline{\varphi}$ contracts some $\overline{D}_i$, then q must contract $p_*^{-1}\overline{D}_i$ since $q = \overline{\varphi} \circ p$. Thus, the set of q-exceptional prime divisors contains birational transforms by p of prime divisors which are contracted by $\overline{\varphi}$. Birational transforms of prime divisors by p cannot be p-exceptional prime divisors, thus we have defined Q as above, and it is similar for P.

Take $\widetilde{D} \in P$, and suppose it is not a q-exceptional prime divisor. Then, $q(\widetilde{D})$ is a prime divisor intersecting with X, thus it is not contracted by $\overline{\varphi}^{-1}$. However, this contradicts that $\widetilde{D}$ is a p-exceptional prime divisor by $p = \overline{\varphi}^{-1} \circ q$. Since it is clear that $\widetilde{D}$ is not a birational transform of prime divisor by p, we have $P \subset Q$. Similarly, we have $Q \subset P$. □

Lemma 3.15 ([2, Lemma 3.4]). *Let A be the set of $\overline{\varphi}$-exceptional prime divisors, and B be the set of $\overline{\varphi}^{-1}$-exceptional prime divisors. Then, we have $\#A = \#B$.*

Proof. Note that A and B are contained in $\{\overline{D}_1, \cdots, \overline{D}_m\}$. By $\overline{\varphi}^{-1} \circ \overline{\varphi} = \mathrm{id}_{\overline{X}}$, for two distinct elements $\overline{D}, \overline{E}$ of $\{\overline{D}_1, \cdots, \overline{D}_m\} \setminus A$, their birational transforms $\overline{\varphi}_*\overline{D}$ and $\overline{\varphi}_*\overline{E}$ by $\overline{\varphi}$ are two distinct elements of $\{\overline{D}_1, \cdots, \overline{D}_m\} \setminus B$. This implies $\#A \geq \#B$. Similarly, we have $\#B \geq \#A$. □

For easy notation, we may suppose $A = \{\overline{D}_1, \cdots, \overline{D}_l\}$. Note that if $A = \emptyset$, then we set $l = 0$. Thus, we have

$$\{\textit{an exceptional prime divisor of } q\} \setminus Q = \{p_*^{-1}\overline{D}_1, \cdots, p_*^{-1}\overline{D}_l\}.$$

Also, we can write

$$\{\textit{an exceptional prime divisor of } p\} \setminus P = \{q_*^{-1}\overline{E}_1, \cdots, q_*^{-1}\overline{E}_l\},$$

where $\{\overline{E}_1, \cdots, \overline{E}_l\} \subset \{\overline{D}_1, \cdots, \overline{D}_m\}$.

By pulling back $K_{\overline{X}}$ by p and q, we have the following equalities in the Néron-Severi group of $\widetilde{X}$.

$$K_{\widetilde{X}} = p^* K_{\overline{X}} + \sum_i a(\widetilde{D}_i, \overline{X}) \widetilde{D}_i,$$

$$K_{\widetilde{X}} = q^* K_{\overline{X}} + \sum_j a(\widetilde{E}_j, \overline{X}) \widetilde{E}_j.$$

In these equalities, $\widetilde{D}_i$ runs over all p-exceptional prime divisors, and $\widetilde{E}_j$ runs over all q-exceptional prime divisors.

Here, we explain about $(\dagger')$. In general, note that even if some $\widetilde{D}_i$ equals some $\widetilde{E}_j$ as closed subsets of $\widetilde{X}$, they may differ as divisors over $\overline{X}$, thus $a(\widetilde{D}_i, \overline{X}) \neq a(\widetilde{E}_j, \overline{X})$ may occur. However, $(\dagger')$ implies that such a thing cannot occur.

By Lemma 3.14 and $(\dagger')$, we have

$$p^* K_{\overline{X}} - q^* K_{\overline{X}} = \sum_{j=1}^{l} a(p_*^{-1}\overline{D}_j, \overline{X}) p_*^{-1}\overline{D}_j - \sum_{i=1}^{l} a(q_*^{-1}\overline{E}_i, \overline{X}) q_*^{-1}\overline{D}_i$$

in $NS_{\mathbb{R}}(\widetilde{X})$.

From now on, we suppose that φ is not an automorphism. Then, there is a curve C on X such that $\varphi(C)$ is a point. The following lemmas are essential to prove Proposition 3.13.

Lemma 3.16 ([2, Proposition 3.5]). *Let $\overline{C}$ be the closure of C in $\overline{X}$. Then, we have $(K_{\overline{X}}, \overline{C}) \geq 0$.*

Proof. Let $\widetilde{C}$ be a curve on $\widetilde{X}$ such that $p(\widetilde{C}) = \overline{C}$. Note that $q(\widetilde{C})$ is a point of X. Since we assume that $\overline{X}$ has canonical singularities, and we have $(p^* K_{\overline{X}}, \widetilde{C}) = \deg(\widetilde{C}/\overline{C})(K_{\overline{X}}, \overline{C})$ and $(q^* K_{\overline{X}}, \widetilde{C}) = 0$ by projection formula, it suffices to show $(p_*^{-1}\overline{D}_j, \widetilde{C}) \geq 0$ and $(q_*^{-1}\overline{E}_i, \widetilde{C}) \leq 0$ for any $i,\ j \in \{1, \cdots, l\}$.

For any $j \in \{1, \cdots, l\}$, since $\overline{C}$ is not contained in $\overline{D}_j$, we have $\widetilde{C} \not\subset p_*^{-1}\overline{D}_j$. This implies $(p_*^{-1}\overline{D}_j, \widetilde{C}) \geq 0$.

For any $i \in \{1, \cdots, l\}$, since $q(\widetilde{C}) \cap \overline{E}_i$ is empty, we have $\widetilde{C} \cap q_*^{-1}\overline{E}_i = \emptyset$. This implies $(q_*^{-1}\overline{E}_i, \widetilde{C}) = 0$. Note that we need only non-positiveness of the intersection numbers here, but we need also vanishing of the intersection numbers later. □

Lemma 3.17 (The correction of [2, Proposition 3.6]). *Let $\overline{E}_1$ be the exceptional prime divisor of f_1. If $\overline{E}_1 \in EPD(f)_X$, then we have $a(\overline{E}_1, \overline{X}_1)(\overline{E}_1, \overline{C}) \leq 0$.*

Proof. By pulling back $(f_1)^* K_{\overline{X}_1} = K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1$ by p and q respectively, we have the following equalities in $NS_{\mathbb{R}}(\widetilde{X})$.

$$K_{\widetilde{X}} - a(\overline{E}_1, \overline{X}_1) p_*^{-1}\overline{E}_1 = p^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1) + \sum_i a(\widetilde{D}_i, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1)\widetilde{D}_i,$$

$$K_{\widetilde{X}} - a(\overline{E}_1, \overline{X}_1) q_*^{-1}\overline{E}_1 = q^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1) + \sum_j a(\widetilde{E}_j, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1)\widetilde{E}_j.$$

In these equalities, $\widetilde{D}_i$ runs over all p-exceptional prime divisors, and $\widetilde{E}_j$ runs over all q-exceptional prime divisors. We have replaced φ by some iterate, we may assume $p_*^{-1}\overline{E} = q_*^{-1}\overline{E}$. By ($\dagger'$), we have

$$\begin{aligned} & (f_1 \circ p)^* K_{\overline{X}_1} - (f_1 \circ q)^* K_{\overline{X}_1} = p^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1) - q^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1) \\ = & \sum_{j=1}^{l} a(p_*^{-1}\overline{D}_j, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1)) p_*^{-1}\overline{D}_j - \sum_{i=1}^{l} a(q_*^{-1}\overline{E}_i, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1)) q_*^{-1}\overline{D}_i. \end{aligned}$$

We know $(p^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1), \widetilde{C}) = \deg(\widetilde{C}/\overline{C})(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, \overline{C})$, $(q^*(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1), \widetilde{C}) = 0$, $(p_*^{-1}\overline{D}_j, \widetilde{C}) \geq 0$ for any $j \in \{1, \cdots, l\}$, and $(q_*^{-1}\overline{E}_i, \widetilde{C}) = 0$ for any $i \in \{1, \cdots, l\}$, thus we have

$$(K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, \overline{C}) = \sum_{j=1}^{l} a(p_*^{-1}\overline{D}_j, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1)(p_*^{-1}\overline{D}_j, \widetilde{C}).$$

Since $a(\overline{E}_1, \overline{X}_1)$ is nonnegative by $\overline{X}$ having canonical singularities, [16, Lemma 2.27] tells us $a(p_*^{-1}\overline{D}_j, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1) \geq a(p_*^{-1}\overline{D}_j, \overline{X})$ for any $j \in \{1, \cdots, l\}$. Then, we have

$$\begin{aligned} (K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, \overline{C}) &= \sum_{j=1}^{l} a(p_*^{-1}\overline{D}_j, \overline{X}, -a(\overline{E}_1, \overline{X}_1)\overline{E}_1)(p_*^{-1}\overline{D}_j, \widetilde{C}) \\ &\geq \sum_{j=1}^{l} a(p_*^{-1}\overline{D}_j, \overline{X})(p_*^{-1}\overline{D}_j, \widetilde{C}) = (K_{\overline{X}}, \overline{C}). \end{aligned}$$

Thus, we have

$$(K_{\overline{X}}, \overline{C}) \leq (K_{\overline{X}} - a(\overline{E}_1, \overline{X}_1)\overline{E}_1, \overline{C}) = (K_{\overline{X}}, \overline{C}) - a(\overline{E}_1, \overline{X}_1)(\overline{E}_1, \overline{C}).$$

This implies $a(\overline{E}_1, \overline{X}_1)(\overline{E}_1, \overline{C}) \leq 0$. □

Remark 3.18 (Remark 3.7, [2])**.** *We clarify conditions which we exactly need for proving Lemma 3.16 and 3.17 except for ($\dagger'$). In fact, we do not need that $\overline{\varphi}$ is defined on a dense open subset of $\overline{C}$. What we need for proving Proposition 3.16 are that $p(\widetilde{C}) = \overline{C}$, $q(\widetilde{C})$ is a point, and $\overline{\varphi}$, $\overline{\varphi}^{-1}$ do not contract any divisors intersecting X. For proving Proposition 3.17, we also need that the exceptional prime divisor is invariant for taking birational transform by some iterate of $\overline{\varphi}$.*

Proof of Proposition 3.13. By Lemma 3.16, $\overline{C}$ is not contracted by f_1, thus we can take a curve $\overline{C}_1 = f_1(\overline{C})$ in $\overline{X}_1$. We show $(K_{\overline{X}_1}, \overline{C}_1) \geq 0$.

Let $\overline{E}_1$ be the exceptional prime divisor of f_1. If $\overline{E}_1$ is an element of $EPD(f)_X$, then by pulling back $K_{\overline{X}_1}$ by f_1, we have

$$K_{\overline{X}_0} = (f_1)^* K_{\overline{X}_1} + a(\overline{E}, \overline{X}_1)\overline{E}$$

in $NS_{\mathbb{R}}(\overline{X}_0)$. This implies

$$\begin{aligned} \deg(\overline{C}/\overline{C}_1)(K_{\overline{X}_1}, \overline{C}_1) &= (K_{\overline{X}_1}, (f_1)_*\overline{C}) = ((f_1)^*K_{\overline{X}_1}, \overline{C}) \\ &= (K_{\overline{X}_0}, \overline{C}) - a(\overline{E}, \overline{X}_1)(\overline{E}, \overline{C}) \geq 0 \end{aligned}$$

by Lemma 3.16 and 3.17. Since $\deg(\overline{C}/\overline{C}_1)$ is a positive integer, we have $(K_{\overline{X}_1}, \overline{C}_1) \geq 0$.

If $\overline{E}_1$ is not an element of $EPD(f)_X$, then by Lemma 3.16 and Remark 3.18, we can show $(K_{\overline{X}_1}, \overline{C}_1) \geq 0$.

Thus, we can take a curve $\overline{C}_2 = f_2(\overline{C})$ in $\overline{X}_2$. We repeat this argument again and again, and finally, we can take a curve $\overline{C}_n = f_n(\overline{C})$ in $\overline{X}_n$. Let $\overline{\varphi}_n$ be the birational automorphism of $\overline{X}_n$ which satisfies $\overline{\varphi}_n \circ f = f \circ \overline{\varphi}$. Since we suppose $\mathrm{Bir}(\overline{X}_n) = \mathrm{Aut}(\overline{X}_n)$, $\overline{\varphi}_n$ is actually an automorphism of $\overline{X}_n$, in particular, it does not contract $\overline{C}_n$. Hence, $\overline{C}$ is not contracted by $\overline{\varphi}_n \circ f$, but this contradicts that $\overline{C}$ is contracted by $f \circ \overline{\varphi}$. □

Although we have considered (†) or (†′), we can prove one of main theorems as follows which does not need (†), (†′), or intersection theoretic argument.

Theorem 3.19. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety and suppose that there is the canonical model $\overline{X}^{\mathrm{can}}$ birational to $\overline{X}$. Let X be a nonempty open subset of $\overline{X}$. Then, Conjecture 1.2 holds for X*

Proof. It is well-known that $\mathrm{Bir}(\overline{X}) \cong \mathrm{Aut}(\overline{X}^{\mathrm{can}})$ holds and $\mathrm{Aut}(\overline{X}^{\mathrm{can}})$ is a finite group. We may regard the endomorphism φ of X as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. Since $\mathrm{Bir}(\overline{X})$ is a finite group, there exists a positive integer n such that $\overline{\varphi}^n = \mathrm{id}_{\overline{X}}$ as birational maps. As φ is an endomorphism of X, $\overline{\varphi}^n$ is defined on X and agrees with φ^n on X as morphisms. This implies $\varphi^n = \mathrm{id}_X$, hence φ is an automorphism of X. □

Remark 3.20. *Note that we only need* $\mathrm{Bir}(\overline{X})$ *being a finite group in the proof of Theorem 3.19.*

3.5 Sequences of divisorial contractions to minimal models

We consider Conjecture 1.2 under the following settings. Let X be a nonempty open subset of a $\mathbb{Q}$-factorial normal projective variety $\overline{X}$. Suppose that $\overline{X}$ has canonical singularities and $\overline{X}$ has a sequence of divisorial contractions $\overline{X} = \overline{X}_0 \to \overline{X}_1 \to \cdots \to \overline{X}_n$ with $\overline{X}_n$ a minimal model. Let $\overline{\varphi}_i$ be the birational automorphism of $\overline{X}_i$ induced by $\overline{\varphi}$ for any $i \in \{0, 1, \ldots, n\}$. We can imitate the same argument as the proof of Proposition 3.12 or 3.13 with (†) or (†′) respectively. The difference is that the existence of $\overline{C}_n$ or $(K_{\overline{X}_n}, \overline{C}_n) = 0$ do not yield any contradiction. When any $\overline{\varphi}_i$ is defined on some nonempty open subsets of $\overline{C}_i$, Conjecture 1.2 holds for X by applying Theorem 3.9. Thus, we consider conditions when $\overline{\varphi}_i$ is defined on some nonempty open subsets of $\overline{C}_i$ by using push-forward of numerical classes of 1-cycles by birational maps as in Definition 2.5. Note that we do not need to assume (†) or (†′) except in Proposition 3.25.

Lemma 3.21. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, $\overline{\varphi}$ a birational automorphism of $\overline{X}$, R an extremal ray of divisorial type, and E the exceptional divisor of the contraction of R. If $\overline{\varphi}$ does not contract E, and $\overline{\varphi}_{\#}R$ is an extremal ray, then $\overline{\varphi}_{\#}R$ is of divisorial type.*

Proof. The union of birational transforms of general curves in R by $\overline{\varphi}$ is dense in the birational transform of E by $\overline{\varphi}$. Since $\overline{\varphi}_{\#}R$ is an extremal ray and generated by birational transforms of general curves in R, we have that the contraction of $\overline{\varphi}_{\#}R$ is a divisorial contraction. □

Proposition 3.22. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety with canonical singularities and non-negative Kodaira dimension. Let X be a nonempty open subset of $\overline{X}$, Y a closed subset of X of codimension at least 2, and φ an endomorphism of X. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. We suppose that φ is injective on $X \setminus Y$, and push-forwards of extremal rays of divisorial type by $\overline{\varphi}$ are also extremal if the exceptional divisors of the corresponding contractions meet X. Then, for any extremal ray R of divisorial type, there exists some positive integer n such that $(\overline{\varphi}_{\#})^n R = R$ if the exceptional divisor of the contraction of R meets X.*

Proof. Let R be an extremal ray such that the exceptional divisor E of the contraction of R meets X. We regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. Since $\overline{\varphi}$ does not contract any prime divisor meeting X, the birational transform E' of E is also a prime divisor meeting X. By Lemma 3.21, we see that E' is the exceptional divisor of the contraction of $\overline{\varphi}_{\#}R$. When we take another R, we obtain another $\overline{\varphi}_{\#}R$ by the injectivity of φ on $X \setminus Y$. Thus, we obtain an injective self-map of the set of such extremal rays R. By [10, Corollary 1.2], $\overline{X}$ has finitely many extremal rays of divisorial type. In particular, the set of such extremal rays is a finite set. Hence, there exists some positive integer n such that $(\overline{\varphi}_{\#})^n R = R$. □

Proposition 3.23. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, $f : \overline{X} \to \overline{X}_1$ a divisorial contraction corresponding to an extremal ray R with exceptional prime divisor E, X a nonempty open subset of $\overline{X}$, Y a closed subset of X of codimension at least 2, and φ an endomorphism of X. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. We suppose that φ is injective on $X \setminus Y$, and either E does not meet X or $\overline{\varphi}_{\#}R = R$. Then, the birational automorphism $\overline{\varphi}_1$ of $\overline{X}_1$ induced by $\overline{\varphi}$ is defined on $f(X)$.*

Proof. If E does not meet X, then the assertion is trivial, thus we may suppose E meets X and $\overline{\varphi}_{\#}R = R$. By the proof of Proposition 3.22, $\overline{\varphi}_{\#}R = R$ makes push-forwards of curves contracted by f by $\overline{\varphi}$ defined. By rigidity lemma and fibers of f being rationally chain connected, it is enough to show that for any intersecting two curves C_1 and C_2 in the same fiber of f, these push-forwards by $\overline{\varphi}$ also intersect. Let $(\widetilde{X}, p, q)$ be a resolution of indeterminacy of φ, and $\widetilde{C}_1$, $\widetilde{C}_2$ be curves on $\widetilde{X}$ such that $p(\widetilde{C}_1) = C_1$ and $p(\widetilde{C}_2) = C_2$. Although $\widetilde{C}_1$ and $\widetilde{C}_2$ may not intersect, there is a chain of curves contracted by p connecting these curves. $\overline{\varphi}_{\#}R = R$ implies this chain is contracted to a point by q, thus $q(\widetilde{C}_1)$ and $\widetilde{C}_2)$ intersect. Therefore, $\overline{\varphi}_{\#}C_1$ and $\overline{\varphi}_{\#}C_2$ intersect as desired. $\square$

Remark 3.24. *In the latter case of Proposition 3.23, we have also shown that $\overline{\varphi}_1$ is defined on $f(E)$.*

Combining Proposition 3.22 and 3.23, we have the following condition as we consider.

Proposition 3.25. *Let X be a nonempty open subset of a $\mathbb{Q}$-factorial normal projective variety $\overline{X}$. Suppose that $\overline{X}$ has canonical singularities and $\overline{X}$ has a sequence of divisorial contractions $\overline{X} = \overline{X}_0 \to \overline{X}_1 \to \cdots \to \overline{X}_n$ with $\overline{X}_n$ a minimal model. For $i \in \{1, 2, \dots, n\}$, let f_i be the morphism $\overline{X} \to \overline{X}_i$, and $\overline{E}_i$ the exceptional divisor of i-th step of the given sequence of divisorial contractions. For each i, we assume one of the followings.*

(1) $\overline{E}_i$ does not meet $X_i = f_{i-1}(X)$,

(2) the push-forward of any extremal ray of $\overline{X}_i$ corresponding to some divisorial contraction contracting $\overline{E}_i$ is also extremal.

Moreover, we assume that $(\dagger')$ holds for an appropriate iterate of φ as described later. Then, Conjecture 1.2 holds for X.

Proof. By Lemma 2.24, Proposition 3.22, we may replace φ with some iterate and assume any extremal ray of $\overline{X}_i$ corresponding to some divisorial contraction contracting $\overline{E}_i$ is preserved by push-forward by $\overline{\varphi}$. After replacing φ, we assume $(\dagger')$ holds for φ. By Proposition 3.23, $\overline{\varphi}_n$ is defined on a nonempty open subset of $\overline{X}_n$ containing $f_n(X)$. Then, we can apply Theorem 3.9 and we have that φ is an automorphism of X. $\square$

These argument are similar for extremal rays of fiber type and we obtain results of the same kind except for Proposition 3.25. We describe those without proof. Note that the corresponding result to [10, Corollary 1.2] is [8, Theorem A.1].

Lemma 3.26. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, $\overline{\varphi}$ a birational automorphism of $\overline{X}$, and R an extremal ray of fiber type. If $\overline{\varphi}_{\#}R$ is an extremal ray, then $\overline{\varphi}_{\#}R$ is of fiber type.*

Proposition 3.27. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety with log terminal singularities. Let X be a nonempty open subset of $\overline{X}$, Y a closed subset of X of codimension at least 2, and φ an endomorphism of X. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. We suppose that φ is injective on $X \setminus Y$, and push-forwards of extremal rays of fiber type are also extremal. Then, for any extremal ray R of fiber type, there exists some positive integer n such that $(\overline{\varphi}_{\#})^n R = R$.*

Proposition 3.28. *Let $\overline{X}$ be a $\mathbb{Q}$-factorial normal projective variety, $f : \overline{X} \to \overline{X}_1$ a contraction of fiber type corresponding to an extremal ray R, X a nonempty open subset of $\overline{X}$, Y a closed subset of X of codimension at least 2, and φ an endomorphism of X. We may regard φ as a birational automorphism $\overline{\varphi}$ of $\overline{X}$. We suppose that φ is injective on $X \setminus Y$ and $\overline{\varphi}_{\#}R = R$. Then, there uniquely exists the birational automorphism $\overline{\varphi}_1$ of $\overline{X}_1$ induced by $\overline{\varphi}$ defined on $\overline{X}_1$.*

4 Dimensional argument

In this section, we discuss relations between the exceptional locus of endomorphisms and the singular locus of varieties by using Theorem 2.23 and dimensional argument under settings of Conjecture 1.2.

4.1 A lemma and easy cases

We begin by proving a useful lemma for considering Conjecture 1.2 following [2, Section 2.1].

Lemma 4.1. *Suppose one of the conditions in Theorem 2.23 holds. Then, we can replace Y to make $Y,\ Z \subset \mathrm{Sing}(X)$ with the condition that φ is injective on $X \setminus Y$.*

To prove Lemma 4.1, we use the following fact.

Theorem 4.2 ([17, III. 9. Proposition 1])**.** *Let X be a factorial variety, and $f : X' \to X$ be a birational morphism from a variety X'. Then, there exists an open subset U of X with the following conditions.*

(1) $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is an isomorphism,

(2) for any irreducible component E of $X' \setminus f^{-1}(U)$, we have $\dim E = \dim X - 1$, and $\dim \overline{f(E)} \leq \dim X - 2$.

Proof of Lemma 4.1. Let A be the set of $x \in X$ such that the irreducible component of $\varphi^{-1}(\{\varphi(x)\})$ which contains x is of dimension at least 1. Then, we have $A \subset Y$, A is a closed subset of X by [11, Exercise II. 3.22(d)], and $\varphi|_{X\setminus A} : X \setminus A \to X$ has finite fibers. By Theorem 2.21, $\varphi|_{X\setminus A}$ is an open embedding, thus we can replace Y with A.

Now, we apply Theorem 4.2 to $\varphi|_{\varphi^{-1}(W)} : \varphi^{-1}(W) \to W$, where W is the smooth locus $X \setminus \mathrm{Sing}\, X$ of X. Since $\varphi|_{\varphi^{-1}(W)}$ does not contract any divisor, we have $\varphi(Y) \subset \mathrm{Sing}(X)$ by Theorem 4.2. By Theorem 2.23, we know $\varphi(W) \subset W$, thus we have $Y \subset \mathrm{Sing}(X)$. Also, this implies $Z \subset \mathrm{Sing}(X)$. □

By Lemma 4.1, we find it easy to prove Conjecture 1.2 in some cases as follows.

Proposition 4.3. *If one of the followings holds, then Conjecture 1.2 holds for X.*

(1) $\dim \mathrm{Sing}(X) \leq 0$*,*

(2) $\dim X \leq 2$*,*

(3) φ *is surjective.*

Proof. We replace Y with A as in the proof of Lemma 4.1.

(1) By Lemma 4.1, we have $\dim Y \leq 0$. This implies that φ has finite fibers, thus φ is an automorphism.

(2) We may assume X is normal by Lemma 2.17. Then, we have $\operatorname{codim} \mathrm{Sing}(X) \geq 2$. Since we suppose $\dim X \leq 2$, we have $\dim \mathrm{Sing}(X) \leq 0$, thus φ is an automorphism by (1).

(3) Since we have replaced Y with A as in the proof of Lemma 4.1, for all y in Y, $\varphi^{-1}(\{\varphi(y)\})$ is of dimension at least 1. By the definition of Z and surjectivity of φ, we have $\varphi(Y) = Z$. If Z is not empty, then we have $\dim Y > \dim Z$, but this contradicts Lemma 2.18. ☐

4.2 On the induced permutation of the irreducible components of the singular locus

We extend [2, Theorem 1.4] as follows.

Proposition 4.4. *Let* $\varphi : X \to X$ *be an endomorphism of a normal variety* X *and* Y *a closed subset of* X*. Assume that* φ *is injective on* $X \setminus Y$*,* $\operatorname{codim} Y \geq 2$*, and one of the followings holds.*

(1) φ *is surjective,*

(2) X *is* $\mathbb{Q}$*-factorial,*

(3) X *is locally a complete intersection.*

Then, we can replace Y *so that* $Y \subset \mathrm{Sing}(X)$*,* $\operatorname{codim} \mathrm{Sing}(X) < \operatorname{codim} Y$*, and* φ *is injective on* $X \setminus Y$*. Note that* $Z = X \setminus \varphi(X \setminus Y)$ *becomes a subset of* $\mathrm{Sing}(X)$*.*

Proof. By [2, Lemma 2.1], we may assume $Y,\ Z \subset \mathrm{Sing}(X)$. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X which is an isomorphism outside $\mathrm{Sing}(X)$. Note that any irreducible component of $\mathrm{Sing}(X)$ is the center of some exceptional prime divisors of π. Since we know $\operatorname{codim} Y = \operatorname{codim} Z$, it is enough to show that for any exceptional prime divisor E of π, if $\pi(E)$ is contained in Z, then we have $\operatorname{codim} \pi(E) > \operatorname{codim} \mathrm{Sing}(X)$. Let $\widetilde{\varphi}$ be the birational automorphism of $\widetilde{X}$ induced by φ, and $\widetilde{\varphi}^{-1}$ its inverse. Since $\widetilde{\varphi}^{-1}$ is defined in codimension 1, we can consider the birational transform F of E by $\pi \circ \widetilde{\varphi}^{-1}$. Since the inverse image of Z by φ equals Y, we may assume $F \subset Y$. We have

$$\operatorname{codim} \pi(E) = \operatorname{codim} \varphi(F) \geq \operatorname{codim} F \geq \operatorname{codim} Y = \operatorname{codim} Z,$$

and two inequalities cannot be equality at the same time. In fact, if $\operatorname{codim} F = \operatorname{codim} Y$, then F is an irreducible component of Y, thus F is contracted by φ. Hence, we have $\operatorname{codim} \pi(E) > \operatorname{codim} Z \geq \operatorname{codim} \operatorname{Sing}(X)$. $\square$

Proposition 4.4 implies the following corollary.

Corollary 4.5. *In the settings of Theorem 2.23, suppose that* $\dim X = 3$*, and one of the conditions in Theorem 2.23 holds. Then, Conjecture 1.2 holds for* X*.*

Proof. By Proposition 4.4, we may assume $\operatorname{codim} Y \geq 3$. This implies $\dim Y \leq 0$ since we consider a threefold X, thus φ has finite fibers. Hence φ is an automorphism of X. $\square$

By using the argument in the proof of Proposition 4.4, we have one of our main results as follows.

Theorem 4.6. *Let* $\varphi : X \to X$ *be an endomorphism of a normal variety* X *and* Y *a closed subset of* X*. Assume that* φ *is injective on* $X \setminus Y$*,* $\operatorname{codim} Y \geq 2$*, and one of the followings holds.*

(1) φ *is surjective,*

(2) X *is* $\mathbb{Q}$*-factorial,*

(3) X *is locally a complete intersection.*

Then, birational transform by φ *gives a permutation of irreducible components of* $\operatorname{Sing}(X)$ *preserving codimension.*

Proof. Let S^i be the set of the irreducible components of $\operatorname{Sing}(X)$ whose elements are of codimension i in X for $i \in [\operatorname{codim} \operatorname{Sing}(X), \dim X]$. Set $c = \operatorname{codim} \operatorname{Sing}(X)$.

By Proposition 4.4, we may assume that any element s of S^c is not contained in Y. Thus, the birational transform of s by φ is defined and an element of S^c again, and this correspondence is injective. Since S^c is a finite set, we obtain a permutation of S^c.

Next, we consider S^{c+1}. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X which is an isomorphism outside $\operatorname{Sing}(X)$. For any $s \in S^{c+1}$, there exists an exceptional prime divisor E of π whose center is s. Let $\widetilde{\varphi}$ be the birational automorphism of $\widetilde{X}$ induced by φ, and $\widetilde{\varphi}^{-1}$ its inverse. Since $\widetilde{\varphi}^{-1}$ is defined in codimension 1, we can consider the proper image F of E by $\pi \circ \widetilde{\varphi}^{-1}$. We claim that F is an irreducible component of $\operatorname{Sing}(X)$. First, note that we know $F \subset \operatorname{Sing}(X)$ by φ inducing an automorphism of $X \setminus \operatorname{Sing}(X)$ by [5, Theorem 1.3]. Second, we have $\operatorname{codim} F \leq \operatorname{codim} s = c + 1$. Third, if F is not an irreducible component of $\operatorname{Sing}(X)$, then there exists an element of S^c which contains s. However, we have shown that birational transform of elements of S^c by φ are defined and elements of S^c again, thus the proper image of F by φ cannot be s, which is an irreducible component of $\operatorname{Sing}(X)$. Hence, we can show our claim. Moreover, we can also show $F \in S^{c+1}$. Thus, for any $s \in S^{c+1}$, we can take $F \in S^{c+1}$ whose birational transform by φ is s, and this correspondence is injective. Since S^{c+1} is a finite set, we obtain a permutation of S^{c+1}. Note that we have constructed the correspondence like taking inverse image by φ, however the inverse permutation gives the correspondence like taking direct image by φ.

Inductively, we can obtain a permutation of S^i for all i. This implies the assertion. □

Acknowledgements

The author thanks Seidai Yasuda, my supervisor, for his invaluable guidance. The author also thanks research room members, especially Hisato Matsukawa, for their advice on the content of this paper. This work was supported by JST SPRING, Grant Number JPMJSP2119.

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