Doctoral Thesis

A multi-microlocal study of strongly regular subanalytic sheaves

(強正則な subanalytic 層の多重超局所的研究)

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Abstract

Honda, Prelli and Yamazaki define multi-microlocalizations of subanalytic sheaves to extend and unify various types of theories of specializations and microlocalizations. This article aims to further develop the theory.

Chapter 1 aims to further extend the theory of multi-microlocalizations of subanalytic sheaves by exploring advanced topics. One is a stalk formula for multi-microlocalized Hom functors and we compute some examples in multi-microlocal settings. Secondly we construct the Sato triangle in the context of multi-microlocal analysis. Ultimately we get the Sato triangle for the multi-microlocalized Hom functors.

In Chapter 2, we define the notions of strong regularity of subanalytic sheaves, and prove an estimate of supports and microsupports of multi-microlocalizations for strongly regular subanalytic sheaves. We apply the results to the subanalytic sheaves of Whitney and temperate holomorphic solutions of regular $\mathcal{D}$-modules. We prove initial value problems for multi-microlocal objects such as a solution sheaves of Majima's strongly asymptotically developable functions. We establish division theorems with coefficients of temperate and Whitney multi-microfunctions for regular $\mathcal{D}$-modules. Ultimately, we show multi-microlocal Bochner's tube theorem.

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Contents

Introduction

Important analytical objects such as asymptotically developable functions and temperate distributions do not form classical sheaves, therefore conventional methods of [12] cannot be applied directly. New concepts such as ind-sheaves and subanalytic sheaves have been introduced and studied to handle these objects [13, 20]. In [8], Honda, Prelli and Yamazaki constructed the theory of multi-microlocalizations of subanalytic sheaves along a certain family of submanifolds. This unifies various types of specializations and microlocalizations [25, 4] which extend the theory. Their framework allows us to handle multi-microlocal objects with growth conditions such as a sheaf of Majima's asymptotically developable functions functorially (see [18, 5]). This paper aims to further extend the theory by exploring advanced topics.

This article consists of two parts. The following is an overview of each part.

In Chapter 1, our objectives are to develop tools in multi-microlocal analysis of subanalytic sheaves. We derive a stalk formula for multi-microlocalized Hom functors in general multi-microlocal cases for subanalytic sheaves and also establish Sato triangle within the framework of multi-microlocal analysis. Note that Chapter 1 is a modified version of the paper [23].

Chapter 1 is organized as follows: In Section 1.1, we assume knowledge of the derived category of sheaves and summarize the basic concepts and definitions related to sheaves on subanalytic sites and multi-microlocal analysis as explained in [20], [22] and [8]. Also we would like to give supplementary facts which follow immediately from the past results but could not be found in the literature. These subsections provide a foundation for the rest of the paper, which builds upon this material to explore more advanced topics and ideas.

In Section 1.2, we derive a stalk formula for multi-microlocalized Hom functors in general multi-microlocal cases for subanalytic sheaves. A significant challenge in obtaining a stalk formula for subanalytic sheaves is the absence of a microlocal cut-off functor. We circumvent this issue by considering constructible sheaves as the first argument of the Hom-functors. In the following section we prepare some lemmas to obtain more explicit formula for an easy computation in particular multi-microlocal settings. Finally we compute some examples in specific multi-microlocal settings. As is well known in the classical theory, a stalk formula is crucial for the microlocal analysis of sheaves (see [12]). We believe that the stalk formula for multi-microlocalized Hom functors proved in this paper will be instrumental in the future.

In Section 1.3, our objective is to establish Sato triangle within the framework of multi-microlocal analysis. To achieve this, we first prepare Uchida triangle in subanalytic sheaf theory. Next we need to multi-microlocalize Uchida triangle in subanalytic sheaf theory. The next step involves multi-microlocalizing Uchida triangle, which poses a challenge because proper direct image functors do not generally map conic subanalytic sheaves to conic subanalytic sheaves.

We address this difficulty by proving several lemmas, ultimately leading to the formulation of Sato triangle for the *μhom* functor in multi-microlocalizations. We conclude this section with a simple example in a multi-microlocal setting to illustrate the theoretical developments. Experts recognize Uchida triangle and Sato triangle as fundamental tools for the microlocal analysis of sheaves. They are particularly suited for solving microlocal Cauchy problems, boundary value problems and various other issues. Our results extend these concepts within the theory of multi-microlocalizations, opening up possibilities for addressing problems in more general settings.

We now explain the background of Chapter 2. The notion of microsupport for ind-sheaves was defined in [14]. And its functorial properties was studied in [19]. In [22], Prelli obtained an estimate of supports of microlocalizations for subanalytic sheaves by means of the microsupport in terms of Kashiwara and Schapira [14]. After some study we found that it is necessary to strengthen the microlocal conditions to find the estimate of supports (or microsupports) of multi-microlocalizations for subanalytic sheaves. In this article, we introduce the possible solutions to find an estimate of supports and microsupport of multi-microlocalizations for subanalytic sheaves.

One of the possible solutions is introducing the notion of stronger regularity than that of Kashiwara and Schapira's notion introduced in [14]. In [19], Martins proved that the regularity of ind-sheaves in the sense of [14] has a nice relation with regular $\mathcal{D}$-modules along an involutive submanifold in the sense of Kashiwara and Oshima [11]. We found that strengthening the regularity is still valid to have the natural relation with the regular $\mathcal{D}$-modules. Applying this fact, we deduce some properties of multi-microlocal analytic objects including a Majima's strongly asymptotically developable solution of regular $\mathcal{D}$-modules. Let us mention many authors studied temperate and Whitney solution to regular $\mathcal{D}$-modules [16, 29] using Andronikof's $THom$ and Colin's $\overset{w}{\otimes}$.

With the estimations in hand, we prove inverse image and multi-microlocalization theorem. Ultimately, we obtain division theorems with coefficients of temperate and Whitney multi-microfunctions for regular $\mathcal{D}$-modules. As an application, we show Bochner's tube theorem for solution sheaves of strongly asymptotically developable functions. This is multi-microlocal counterparts of the results obtained in [1], [3], [28]. Let us mention the related papers [26], [27] of this topic.

Chapter 2 is organized as follows: In Section 2.1, we introduce the notion of strong regularity and study its fundamental properties. We show a partial Abelian property of strong regularity. We discuss various solutions to regular $\mathcal{D}$-modules in the sense of [11] satisfy strong regularity.

In Section 2.2, we obtain a formula for estimations of microsupports of multi-microlocalizations. We begin by an estimate of the supports and consider some easy consequence of the results. And we prove the microsupport estimate using the techniques developed in [14]. As a simple application we show the microlocal isomorphism of subanalytic sheaves under a non-characteristic condition.

In Section 2.3, restricting ourselves in the case of normal crossing type, we prove inverse and multi-microlocalization theorems. This is a multi-microlocal generalization of the [12, Theorem 6.7.1]. The multi-conic structure of multi-microlocalizations plays a crucial role. Namely, [8]'s multi-normal cones behave different from the [12]'s classical normal cone. At first we discuss the inverse and multi-microlocalization theorem of classical sheaf case. Secondly, we apply the result in Section 2.2 to prove the theorem in the case of strongly regular subanalytic sheaf.

In Section 2.4, we give an application to $\mathcal{D}$-modules theory in multi-microlocal analysis. By making use of the triangle obtained in Chapter 1, we first prove the initial value problem for multi-microlocal objects including Majima's strongly asymptotically developable solutions to $\mathcal{D}$-modules regular along an involutive subbundle. Lastly our goal is to obtain division theorems with coefficients of temperate and Whitney multi-microfunctions for regular $\mathcal{D}$-modules. This is thought of an initial value problem for temperate and Whitney multi-microfunctions for regular $\mathcal{D}$-modules. In particular, by applying the division theorem we obtain multi-microlocal Bochner's tube theorem.

Chapter 1

The stalk formula for multi-microlocal Hom functors and the multi-microlocal Sato triangle

1.1 Preliminaries.

In this section we recall some basic notions. References are made to [20] , [22] and [13] for the theory of sheaves on subanalytic sites, and [8], [5] and [6] for the theory of multi-microlocalizations.

1.1.1 Theory of subanalytic sheaves.

X will be a real analytic manifold and k a field. We assume all manifolds are analytic, finite-dimensional and countable at infinity. Denote by $\mathrm{Op}(X_{sa})$ (resp. $\mathrm{Op}^c(X_{sa})$) the category of open (resp. relatively compact open) subanalytic subsets of X. One endows $\mathrm{Op}(X_{sa})$ with the following topology: $\mathrm{Cov}(U) \subset \mathrm{Op}(X_{sa})$ is a covering of $U \in \mathrm{Op}(X_{sa})$ if for any compact subset K of X there exists a finite subset S_0 of $\mathrm{Cov}(U)$ such that $K \cap \bigcup_{V \in S_0} V = K \cap U$. We will call X_{sa} the subanalytic site. Let $\mathrm{Mod}(k_{X_{sa}})$ denote the category of sheaves on X_{sa}.

Let $\mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$ be the abelian category of $\mathbb{R}$-constructible sheaves on X, and consider its subcategory $\mathrm{Mod}^c_{\mathbb{R}\text{-}c}(k_X)$ consisting of sheaves whose support is compact. Denote $\rho : X \to X_{sa}$ the natural morphism of sites. $\rho_* : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_{X_{sa}})$ is fully faithful and left exact.

Notation 1.1.1. Since the functor ρ_* is fully faithful and exact on $\mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$ we identify $\mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$ with the image in $\mathrm{Mod}(k_{X_{sa}})$. When there is no risk of confusion we will write F instead of $\rho_* F$, for $F \in \mathrm{Mod}_{\mathbb{R}\text{-}c}(k_X)$.

Let X, Y be two real analytic manifolds, and let $f : X \to Y$ be a real analytic map. We get external operations f_* and f^{-1} which are always defined for sheaves on Grothendieck topologies. Let Z be a locally closed subanalytic set. As in classical sheaf theory we may define Γ_Z and $(\cdot)_Z$. We may define the functor of proper direct image $f_{!!}$ for subanalytic sheaves. If f is proper on $\mathrm{supp}(F)$ then $f_*F \simeq f_{!!}F$, in this case $f_{!!}$ commutes with ρ_*.

We denote by $D(k_{X_{sa}})$ the derived category of $\mathrm{Mod}(k_{X_{sa}})$ and its full subcategory consisting of bounded (resp. bounded below, resp. bounded above) complexes is denoted by $D^b(k_{X_{sa}})$ (resp. $D^+(k_{X_{sa}})$, resp. $D^-(k_{X_{sa}})$). As usual we denote by $D^b_{\mathbb{R}\text{-}c}(k_X)$ (resp. $D^b_{\mathbb{R}\text{-}c}(k_{X_{sa}})$) the full subcategory of $D^b(k_X)$ (resp. $D^b(k_{X_{sa}})$) consisting of objects with $\mathbb{R}$-constructible cohomology. We would like to state the result which immediately follows from [20, Proposition 2.4.8].

Proposition 1.1.2. *Let $F \in D^+(k_{Y_{sa}})$, and let $f : X \to Y$ be a closed embedding. Then $Rf_{!!}f^{-1}F \simeq F_X$.*

Remark 1.1.3. Unlike the classical case, this isomorphism does not hold for an open embedding in general.

The functor $Rf_{!!}$ admits a right adjoint, denoted by $f^!$. A similar consequence as in [13, Proposition 5.3.8] follows as shown below.

Proposition 1.1.4. *Let $G \in D^b(k_{X_{sa}})$, $F \in D^+(k_{X_{sa}})$. There is a natural isomorphism $f^! \,\mathrm{R}\mathscr{H}om(G, F) \simeq \mathrm{R}\mathscr{H}om(f^{-1}G, f^!F)$.*

1.1.2 Multi-microlocalization.

Let $\tau_i : E_i \to Z$ $(1 \le i \le \ell)$ be vector bundles over Z, and let E_i^* be the dual bundle of E_i. We denote by $\wedge_i$ and $\vee_i$ the Fourier-Sato transformations and the inverse Fourier-Sato transformations on E_i respectively. Moreover we denote $\wedge_i^*$ and $\vee_i^*$ the Fourier-Sato and the inverse Fourier-Sato transformation on E_i^* respectively. Set $E := E_1 \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell$ and $E^* := E_1^* \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell^*$. Let $\tau : E \to Z$ be the canonical projection. Set $P_i' := \{(\eta, \xi) \in E_i \underset{Z}{\times} E_i{}^*; \langle \eta, \xi \rangle \le 0\}$.

$$P' := P_1' \underset{Z}{\times} \cdots \underset{Z}{\times} P_\ell', P^+ := E \underset{Z}{\times} E^* \setminus P'$$

and denote by $p_1' : P' \to E$ and $p_2' : P' \to E^*$ the canonical projections respectively. Set $P_i := \{(\eta, \xi) \in E_i \underset{Z}{\times} E_i^*; \langle \eta, \xi \rangle \ge 0\}$.

$$P := P_1 \underset{Z}{\times} \cdots \underset{Z}{\times} P_\ell, P^- := E \underset{Z}{\times} E^* \setminus P$$

and denote by $p_1'' : P \to E$ and $p_2'' : P \to E^*$ the canonical projections respectively. Let F and G be multi-conic objects on E and E^*, respectively. Then we set for short $\wedge_E$ (resp. $\vee_E^*$) the compostion of the Fourier-Sato transformations $\wedge_i$ on E_i (resp. the composition of the inverse Fourier-Sato transformations $\vee_i^*$) for each $i \in \{1, \dots, \ell\}$.

Following [8], we recall the notion of multi-microlocalization of sheaves. Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed submanifolds in X. We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi; p \in M_j, N \not\subseteq M_j \text{ and } M_j \not\subseteq N\}.$$

Let us consider the following conditions for χ.

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e., $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \emptyset$, we have

$$\left(\bigcap_{M_j \in NR_p(N)} T_p M_j \right) + T_p N = T_p X.$$

If χ satisfies the condition H2, then for any $p \in X$, there exist a neighborhood V of p in X, a system of local coordinates $(x_1, \dots, x_n)$ in V and a family of subsets $\{I_j\}_{j=1}^{\ell}$ of the set $\{1, 2, \dots, n\}$ for which the following conditions hold.

1. Either $I_k \subset I_j$, $I_j \subset I_k$ or $I_k \cap I_j = \emptyset$ holds ($k, j \in \{1, 2, \dots, \ell\}$).
2. A submanifold $M_j \in \chi$ with $p \in M_j (j = 1, 2, \dots, \ell)$ is defined by $\{x_i = 0; i \in I_j\}$ in V.

We set, for $N \in \chi$,

$$\iota_\chi(N) := \bigcap_{N \subsetneqq M_j} M_j.$$

Here $\iota_\chi(N) := X$ for convention if there exists no j with $N \subsetneqq M_j$. When there is no risk of confusion, we write for short $\iota(N)$. We also assume the condition

H3 $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \dots, \ell\}$.

In local coordinates let $I_1, \dots, I_\ell \subset \{1, \dots, \ell\}$ such that $M_i = \{x_k = 0; k \in I_i\}$. Note that the family χ satisfies the conditions H1, H2, and H3 if and only if $I_1, \dots, I_\ell$ satisfy the corresponding conditions

$$\text{(i) either } I_j \subsetneq I_k, I_k \subsetneq I_j \text{ or } I_j \cap I_k = \emptyset \text{ holds for any } j \neq k, \tag{1.1}$$
$$\text{(ii)} \left(\bigcup_{I_k \subsetneq I_j} I_k \right) \subsetneq I_j \text{ for any } j.$$

Hence, for any $j \in \{1, 2, \ldots, \ell\}$, the set

$$\hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneq I_j} I_k \right) \tag{1.2}$$

is not empty by the condition H3. By the conditions H1,H2 and H3, we have for $j \in \{1, \ldots, \ell\}$

$$I_j = \bigsqcup_{I_k \subset I_j} \hat{I}_k.$$

In particular $\bigcup_{1 \leq j \leq \ell} I_j = \hat{I}_1 \sqcup \cdots \sqcup \hat{I}_\ell$. Set for convenience $I_0 = \hat{I}_0 := \{1, \ldots, n\} \setminus \bigcup_{j=1}^{\ell} I_j$. Then, in local coordinates, we can write the coordinates $(x_1, \ldots, x_n)$ by

$$(x^{(0)}, x^{(1)}, \ldots, x^{(\ell)}),$$

where $x^{(j)}$ denotes the coordinates $(x_i)_{i \in \hat{I}_j}$ $(j = 0, \ldots, \ell)$.

Under the conditions H1-H3, we may obtain multi-normal deformation along χ(see [8, Section 1.2] for details). As usual we denote by $\tau_{M_1} : T_{M_1}X \to M_1$ the normal bundle. We first consider a normal deformation $\widetilde{X}_{M_1}$ along M_1 (see [12, Section 4.1]), i.e. an analytic manifold $\widetilde{X}_{M_1}$, a pair of mappings $p_{M_1} : \widetilde{X}_{M_1} \to X, t_1 : \widetilde{X}_{M_1} \to \mathbb{R}$, and an action of $\mathbb{R} \setminus \{0\}$ on $\widetilde{X}_{M_1}$ $(\widetilde{x}, r) \mapsto \widetilde{x} \cdot r$ such that $p_{M_1}^{-1}(X \setminus M_1) \simeq X \setminus M_1 \times \mathbb{R} \setminus \{0\}$, $t_1^{-1}(c) \simeq X$ for each $c \neq 0$ and $t_1^{-1}(0) \simeq T_{M_1}X$. We denote by Ω_{M_1} the open subset of $\widetilde{X}_{M_1}$ obtained as the inverse image of $\mathbb{R}^+$ by t_1, by $i_{\Omega_{M_1}}$ the embedding $\Omega_{M_1} \to \widetilde{X}_{M_1}$, $\tilde{p}_{M_1} = p_{M_1} \circ i_{\Omega_{M_1}}$ and $s_{M_1} : t_1^{-1}(0) \simeq T_{M_1}X \hookrightarrow \widetilde{X}_{M_1}$ the inclusion. We get the commutative diagram

$$\begin{tikzcd}
T_{M_1}X \arrow[r, "s_{M_1}"] \arrow[d, "\tau_{M_1}"'] & \widetilde{X}_{M_1} \arrow[r, "i_{\Omega_{M_1}}"] \arrow[d, "p_{M_1}"] & \Omega_{M_1} \arrow[dl, "\tilde{p}_{M_1}"] \\
M_1 \arrow[r, "i_{M_1}"] & X. &
\end{tikzcd}$$

Set $\widetilde{\Omega}_{M_1} = \{(x_1, t_1); t_1 \neq 0\}$ and $\widetilde{M_2} := \overline{(p_{M_1}|_{\widetilde{\Omega}_{M_1}})^{-1} M_2}$. Then $\widetilde{M_2}$ is a closed smooth submanifold of $\widetilde{X}_{M_1}$. Now we can define the normal deformation along M_1, M_2 as $\widetilde{X}_{M_1,M_2} := (\widetilde{\widetilde{X}_{M_1}})_{\widetilde{M_2}}$.

Then we can define recursively the normal deformation along χ as $\widetilde{X} := \widetilde{\left(\widetilde{X}_{M_1,\dots,M_{\ell-1}}\right)}_{\widetilde{M_\ell}}$. Set $S_\chi = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$ and $\Omega_\chi = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

$$\begin{tikzcd} S_\chi \arrow[r, "s"] \arrow[d, "\tau"'] & \widetilde{X} \arrow[r, "i_\Omega"] \arrow[d, "p"] & \Omega_\chi \arrow[dl, "\tilde{p}"] \\ M \arrow[r, "i_M"] & X. & \end{tikzcd}$$

Let us consider the canonical map $T_{M_j}\iota(M_j) \to M_j \hookrightarrow X$ ($j = 1, \cdots, \ell$), we write for short

$$S_\chi \simeq \underset{X, 1 \le j \le \ell}{\times} T_{M_j}\iota(M_j) := T_{M_1}\iota(M_1) \underset{X}{\times} T_{M_2}\iota(M_2) \underset{X}{\times} \dots \underset{X}{\times} T_{M_\ell}\iota(M_\ell).$$

Definition 1.1.5. The multi-specialization along χ is the functor

$$\begin{array}{rcl} \nu^{sa}_\chi \colon D^b(k_{X_{sa}}) & \longrightarrow & D^b(k_{S_{\chi sa}}), \\ \Psi & & \Psi \\ F & \longmapsto & s^{-1}R\Gamma_{\Omega_\chi} p^{-1}F. \end{array}$$

We also define the functor $\nu_\chi = \rho^{-1}\nu^{sa}_\chi : D^b(k_{X_{sa}}) \to D^b(k_{S_\chi})$.

Let S^*_χ be the dual vector bundle of S_χ:

$$S^*_\chi := T^*_{M_1}\iota(M_1) \times_X \cdots \times_X T^*_{M_\ell}\iota(M_\ell).$$

Definition 1.1.6. The multi-microlocalization along χ is the functor

$$\begin{array}{rcl} \mu^{sa}_\chi \colon D^b(k_{X_{sa}}) & \longrightarrow & D^b(k_{S^*_{\chi sa}}), \\ \Psi & & \Psi \\ F & \longmapsto & \nu^{sa}_\chi(F)^{\wedge_{S_\chi}}. \end{array}$$

We set $\mu_\chi := \rho^{-1}\mu^{sa}_\chi : D^b(k_{X_{sa}}) \to D^b(k_{S^*_\chi})$.

Now we are going to restate a stalk formula for multi-microlocalization proved in [8, Theorem 2.11]. As the problem is local, we may assume that $X = \mathbb{R}^n$ and $q = 0$ with coordinates $(x_1, \dots, x_n)$, and that there exists a subset I_k ($k = 1, 2, \dots, \ell$) in $\{1, 2, \dots, n\}$ with the conditions ((1.1) (i),(ii)) such that each submanifold M_k is given by $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n; x_i = 0\ (i \in I_k)\}$. Recall that $\hat{I}_k$ was defined by (1.2) and that we set $M = \cap_k M_k$ and $n_k = \sharp \hat{I}_k$. Then locally we have

$$X = M \times (N_1 \times N_2 \times \cdots \times N_\ell) = M \times N, \tag{1.3}$$

where N_k is $\mathbb{R}^{n_k}$ with coordinates $x^{(k)} = (x_i)_{i\in\hat{I}_k}$. Set, for $k \in \{1,\dots,\ell\}$,

$$\begin{aligned}
J_{<k} &:= \{j \in \{1,\dots,\ell\}, I_j \subsetneq I_k\},\\
J_{>k} &:= \{j \in \{1,\dots,\ell\}, I_j \supsetneq I_k\}, \qquad (1.4)\\
J_{\not\parallel k} &:= \{j \in \{1,\dots,\ell\}, I_j \cap I_k = \emptyset\}.
\end{aligned}$$

Let $p = p_1 \times \cdots \times p_\ell = (q;\xi^{(1)},\dots,\xi^{(\ell)}) \in T^*_{M_1}\iota(M_1) \times_X \cdots \times_X T^*_{M_\ell}\iota(M_\ell)$ and consider the following conic subset in N

$$\gamma_k := \left\{ (x^{(1)}, x^{(2)}, \cdots, x^{(\ell)}) \in N \;\middle|\; \begin{array}{l} x^{(j)} = 0,\ (j \in J_{<k} \sqcup J_{\not\parallel k}),\\ x^{(j)} \in \mathbb{R}^{n_j},\ (j \in J_{>k}),\\ \langle x^{(j)}, \xi^{(k)}\rangle > 0,\ (j = k) \end{array} \right\}. \qquad (1.5)$$

Theorem 1.1.7. *Let $p = p_1 \times \cdots \times p_\ell = (q;\xi^{(1)},\dots,\xi^{(\ell)}) \in S^*_\chi$, and let $F \in D^b(k_{X_{sa}})$. Then we have*

$$H^k(\mu_\chi F)_p \simeq \varinjlim_{G,U} H^k_G(U;F).$$

Here U is an open subanalytic neighborhood of q in X and G is a closed subanalytic subset in the form $M \times \sum_{k=1}^{\ell} G_k$ with G_k being a closed subanalytic convex cone in N satisfying $G_k \setminus \{0\} \subset \gamma_k$, where γ_k is defined in (1.5).

In [8, Theorem 4.1], the authors proved the existence of Uchida's triangle ([28, Appendix A]) in the context of multi-microlocal analysis. In this paper we are going to show that it holds for sheaves on subanalytic sites as well.

Theorem 1.1.8. *Let $F \in D^b_{(\mathbb{R}^+)^\ell}(k_{E_{sa}})$. There exists a natural isomorphism*

$$\tau^! R\tau_{!!} F \simeq Rp_{1*} p_2^!(F^{\wedge_E}),$$

and the natural morphism $F \to \tau^! R\tau_{!!} F$ is embedded to the following distinguished triangle:

$$F \longrightarrow \tau^! R\tau_{!!} F \longrightarrow Rp_{1*} R\Gamma_{P^+}(p_2^!(F^{\wedge_E})) \xrightarrow{+1} .$$

Next we explain the notion of the μhom in multi-microlocalizations (see [6]). Let χ satisfy the conditions H1-H3. Let A be an $\ell \times \ell$ matrix by

$$a_{ji} = \begin{cases} 1, & (i \in I_j)\\ 0, & (i \notin I_j). \end{cases}$$

We call A an associated matrix with χ. $X^2 = X \times X$ and $\Delta_X \subset X^2$ denotes its diagonal, i.e., $\Delta_X = \{(x, y) \in X^2; x = y\}$.

Define the ambient space $\widehat{X}$ of the universal family as

$$(X^2)^\ell = X^2 \times \cdots \times X^2.$$

Define the universal family $\widehat{\chi} = \{\widehat{M}_1, \ldots, \widehat{M}_\ell\}$ of closed submanifolds in $\widehat{X}$ in the following manner: Each $\widehat{M}_j$ has a form

$$\widehat{M}_j = Z_{j1} \times Z_{j2} \times \cdots \times Z_{j\ell} \subset \widehat{X},$$

where $Z_{jk} \subset X^2$ is either X^2 or Δ_X determined by the rules:

$$Z_{jk} := \begin{cases} X^2, & (a_{jk} = 0), \\ \Delta_X, & (a_{jk} = 1). \end{cases}$$

For the universal family $\widehat{\chi}$, we may consider the multi-normal deformation $\widetilde{(X^2)^\ell}_{\widehat{\chi}}$ of $(X^2)^\ell$ along $\widehat{\chi}$ and the multi-microlocalization $\mu^{sa}_{\widehat{\chi}}$ (see [6] for details.). And we have

$$S^*_{\widehat{\chi}} = T^*X \times \cdots \times T^*X.$$

Definition 1.1.9 (Multi-microlocal Hom functors)**.** Let $F_1, \ldots, F_\ell$ and G be in $D^b(X_{sa})$. We define the functor

$$\mu hom^{sa}_{\widehat{\chi}}(F_1, \ldots, F_\ell; G) := \mu^{sa}_{\widehat{\chi}}(\mathrm{R}\mathscr{H}om(p_2^{-1}(F_1 \boxtimes \cdots \boxtimes F_\ell), p_1^! Ri_{\Delta *}G))$$

and we set the functor

$$\mu hom_{\widehat{\chi}}(F_1, \ldots, F_\ell; G) := \rho^{-1} \mu hom^{sa}_{\widehat{\chi}}(F_1, \ldots, F_\ell; G),$$

where

$$X^\ell \xleftarrow{p_1} \widehat{X} \xrightarrow{p_2} X^\ell \xleftarrow{i_\Delta} X$$

- $i_\Delta : X \hookrightarrow X^\ell$ is defined by $x \in X \to (x, x, \ldots, x) \in X^\ell$,
- $p_1 : (X^2)^\ell \to X^\ell$ is the canonical projection $(x_1, y_1) \times \cdots \times (x_\ell, y_\ell) \to (x_1, \ldots, x_\ell)$ and
- $p_2 : (X^2)^\ell \to X^\ell$ is the canonical projection $(x_1, y_1) \times \cdots \times (x_\ell, y_\ell) \to (y_1, \ldots, y_\ell)$.

1.2 A stalk formula for the multi-microlocal Hom functors

In this section we are going to prove a stalk formula for the multi-microlocal Hom functors. As the question being local, we may assume X is a vector space $\mathbb{R}^n$. Let γ be a closed convex cone with vertex at 0. One denotes by X_γ the set X endowed with the induced γ-topology (see [12, page.161]), and by $\phi_\gamma : X \to X_\gamma$ the natural continuous map from X (with the usual topology) to X_γ. Set $Z(\gamma) := \{(x,y) \in X \times X; y - x \in \gamma\}$.

Lemma 1.2.1. *Let $F \in D^b_{\mathbb{R}\text{-}c}(k_X)$ and V be a relatively compact open subanalytic set. For a closed convex cone γ in X, we have*

$$Rp_{1!!}(p_2^{-1}(R\rho_*F)_V)_{Z(\gamma)} \simeq R\rho_*(\phi_{\tilde{\gamma}}^{-1}R\phi_{\gamma *}(F_V)),$$

where p_1 and p_2 the first and second projection from $X \times X$ to X.

Proof. This follows immediately from [20, Proposition 2.2.1], [20, Proposition 1.3.3], [20, Proposition 1.3.2] and [12, Proposition 3.5.4]. In fact,

$$\begin{aligned} Rp_{1!!}(p_2^{-1}(R\rho_*F)_V)_{Z(\gamma)} \simeq & Rp_{1!!}(R\rho_*(p_2^{-1}(F_V)_{Z(\gamma)}) & (1.6)\\ \simeq & R\rho_*Rp_{1!}(p_2^{-1}(F_V))_{Z(\gamma)} & (1.7)\\ \simeq & R\rho_*(\phi_\gamma^{-1}R\phi_{\gamma*}(F_V)). \end{aligned}$$

The isomorphism (2.4.2) follows since F is constructible. The isomorphism (1.7) follows since p_1 is proper on $\operatorname{supp}(p_2^{-1}(F_V))_{Z(\gamma)}$. □

Theorem 1.2.2. *Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in \overbrace{T^*X \times_X \cdots \times_X T^*X}^{\ell}$. $G_1, \dots, G_\ell \in Ob(D^b_{\mathbb{R}\text{-}c}(k_X))$ and $F \in Ob(D^b(k_{X_{sa}}))$ we have*

$$\begin{aligned} & H^j\left(\mu hom_{\widetilde{\chi}}(R\rho_*G_1, R\rho_*G_2, \dots, R\rho_*G_\ell; F)\right)_p \\ \simeq & \varinjlim_{V,\tilde{\gamma}} H^j(R\Gamma(i_\Delta^{-1}V; \mathrm{R}\mathscr{H}om(i_\Delta^{-1}R\rho_*\phi_{\tilde{\gamma}}^{-1}R\phi_{\tilde{\gamma}*}((\boxtimes_{i=1}^{\ell} G_i)_V), F))) \\ \simeq & \varinjlim_{U,\tilde{\gamma}} H^j\left(R\Gamma\left(U; \mathrm{R}\mathscr{H}om(R\rho_*(i_\Delta^{-1}\phi_{\tilde{\gamma}}^{-1}R\phi_{\tilde{\gamma}*}((G_1 \boxtimes \cdots \boxtimes G_\ell)_{U\times\cdots\times U})), F)\right)\right), \end{aligned}$$

where $\tilde{\gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_\ell \subset X^\ell$, each $\tilde{\gamma}_k$ is a closed conic subanalytic proper convex subset of $\gamma_k \cup \{0\}$,

$$\gamma_k := \left\{ (y^{(1)}, \cdots, y^{(\ell)}) \in X^\ell \;\middle|\; \begin{array}{l} y^{(j)} = 0, (j \in J_{<k} \sqcup J_{\nparallel k}), \\ y^{(j)} \in \mathbb{R}^n, (j \in J_{>k}), \\ \langle y^{(j)}, \xi^{(k)} \rangle < 0, (j = k) \end{array} \right\}$$

and U ranges through the family of open subanalytic neighborhoods of q. V runs through a family of open subanalytic neighborhoods of $i_\Delta(q)$ in X^ℓ.

Proof. As the question being local, we may assume X is a vector space $\mathbb{R}^n$ and $q = 0$ with coordinates $(x_1, \dots, x_n)$. Clearly (1.3) becomes $X^{2\ell} = X^\ell \times X^\ell$. Set

$$\delta_k := \left\{ (y^{(1)}, \cdots, y^{(\ell)}) \in X^\ell \;\middle|\; \begin{array}{l} y^{(j)} = 0, (j \in J_{<k} \sqcup J_{\not\sharp k}), \\ y^{(j)} \in \mathbb{R}^n, (j \in J_{>k}), \\ \langle y^{(j)}, \xi^{(k)} \rangle > 0, (j = k) \end{array} \right\}.$$

We propose the following identification φ.

$$\begin{array}{ccc} X^\ell \times X^\ell & \xleftrightarrow{\quad\quad\varphi\quad\quad} & X^\ell \times X^\ell \\ \uplus & & \uplus \\ (x_1, \dots, x_\ell; y_1, \dots, y_\ell) & \longleftrightarrow & (x_1, \dots, x_\ell; x_1 - y_1, \dots, x_\ell - y_\ell) \end{array} \tag{1.8}$$

We have the following claim.

Claim 1.2.3. Let G be a closed subanalytic subset in the form $X^\ell \times \sum_{k=1}^{\ell} G_k$ with G_k being a closed subanalytic proper convex cone in X^ℓ satisfying $G_k \setminus \{0\} \subset \delta_k$. Then for each such G there exists a closed conic subanalytic proper convex subset $\tilde{\gamma}_k \setminus \{0\} \subset -\delta_k = \gamma_k$ such that

$$G \subset \varphi^{-1}(Z(\tilde{\gamma})),$$

where $\tilde{\gamma} = \tilde{\gamma}_1 + \tilde{\gamma}_2 + \cdots + \tilde{\gamma}_\ell$ and

$$Z(\tilde{\gamma}) := \left\{ (x_1, x_2, \dots, x_\ell; y_1, y_2, \dots, y_\ell) \in X^\ell \times X^\ell \mid (y_1 - x_1, \dots, y_\ell - x_\ell) \in \tilde{\gamma} \right\}.$$

Moreover for such $\tilde{\gamma}$ we may find a closed subanalytic subset $G' = X^\ell \times \sum_{k=1}^{\ell} G'_k$ with G'_k being a closed subanalytic proper convex cone in X^ℓ satisfying $G'_k \setminus \{0\} \subset \delta_k$ so that $\varphi^{-1}(Z(\tilde{\gamma})) \subset G'$.

In fact, let G be such a conic set. Then there exists $G_k \setminus \{0\} \subset \delta_k$. We may find $\tilde{\gamma}_k$ such that $G_k \subset -\tilde{\gamma}_k$ and $-\tilde{\gamma}_k \setminus \{0\} \subset -\delta_k$. By the identification (1.8), $\varphi^{-1}(Z(\tilde{\gamma})) = \left\{ (x; y) \in X^\ell \times X^\ell \mid -y \in \tilde{\gamma} \right\}$. If $(x, y) \in G$ then $y \in \sum_{k=1}^{\ell} G_k$. And $y \in -\tilde{\gamma}$ thus $(x, y) \in \varphi^{-1}(Z(\tilde{\gamma}))$. Finally take a closed subanalytic proper convex cone G'_k in X^ℓ so that $-\tilde{\gamma}_k \subset G'_k$ and $G'_k \setminus \{0\} \subset \delta_k$ for each k. It is easy to see $\varphi^{-1}(Z(\tilde{\gamma})) \subset G' = X^\ell \times \sum_{k=1}^{\ell} G'_k$. Therefore by Theorem 1.1.7 and Lemma 1.2.1 we obtain the chain of isomorphisms (put $\overset{\ell}{\underset{i=1}{\boxtimes}} G_i := G_1 \boxtimes \cdots \boxtimes G_\ell$)

$$
\begin{aligned}
& H^k \mu hom_{\widetilde{X}}(R\rho_* G_1, R\rho_* G_2, \ldots, R\rho_* G_\ell; F)_p \\
\simeq\ & H^k \mu_{\widetilde{X}}(\mathrm{R}\mathscr{H}\!om(p_2^{-1}(R\rho_* \overset{\ell}{\underset{i=1}{\boxtimes}} G_i), p_1^! Ri_{\Delta *}F))_p \\
\simeq\ & \varinjlim_{G,U,V} H^k(R\Gamma_G(U \times V; \mathrm{R}\mathscr{H}\!om(p_2^{-1}(R\rho_* \overset{\ell}{\underset{i=1}{\boxtimes}} G_i), p_1^! Ri_{\Delta *}F))) && (1.9) \\
\simeq\ & \varinjlim_{\tilde{\gamma},U,V} H^k(R\Gamma_{Z(\tilde{\gamma})}(U \times V; \mathrm{R}\mathscr{H}\!om(p_2^{-1}(R\rho_* \overset{\ell}{\underset{i=1}{\boxtimes}} G_i), p_1^! Ri_{\Delta *}F)) && (1.10) \\
\simeq\ & \varinjlim_{\tilde{\gamma},U,V} H^k(R\Gamma(U \times X; \mathrm{R}\mathscr{H}\!om((p_2^{-1}(R\rho_* \overset{\ell}{\underset{i=1}{\boxtimes}} G_i)_V)_{Z(\tilde{\gamma})}, p_1^! Ri_{\Delta *}F))) \\
\simeq\ & \varinjlim_{\tilde{\gamma},U,V} H^k(R\Gamma(U; \mathrm{R}\mathscr{H}\!om(Rp_{1!!}((p_2^{-1}(R\rho_* \overset{\ell}{\underset{i=1}{\boxtimes}} G_i)_V)_{Z(\tilde{\gamma})}), Ri_{\Delta *}F))) \\
\simeq\ & \varinjlim_{\tilde{\gamma},U,V} H^k(R\Gamma(i_\Delta^{-1}U; \mathrm{R}\mathscr{H}\!om(i_\Delta^{-1}Rp_{1!!}(p_2^{-1}(R\rho_* \overset{\ell}{\underset{i=1}{\boxtimes}} G_i)_V)_{Z(\tilde{\gamma})}, F))) \\
\simeq\ & \varinjlim_{\tilde{\gamma},U} H^k(R\Gamma(i_\Delta^{-1}U; \mathrm{R}\mathscr{H}\!om(i_\Delta^{-1}R\rho_*\phi_{\tilde{\gamma}}^{-1}R\phi_{\tilde{\gamma}*}((\overset{\ell}{\underset{i=1}{\boxtimes}} G_i)_U), F))). && (1.11)
\end{aligned}
$$

G runs through a family of closed subanalytic subsets in the form $X^\ell \times \sum_{k=1}^{\ell} G_k$ with G_k being a closed subanalytic proper convex cone in X^ℓ satisfying $G_k \setminus \{0\} \subset \delta_k$, U and V runs through a family of open subanalytic neighborhoods of $i_\Delta(q)$ in X^ℓ. Notice that U is different from the one in the statement of theorem. The first isomorphism follows from [20, Proposition 1.3.3] and [22, Proposition 1.3.1]. (1.9) follows from Theorem 1.1.7. (1.10) is a consequence of the claim 1.2.3. Finally, Lemma 1.2.1 implies the isomorphism (1.11). We have used adjunctions many times throughout the chain of isomorphisms. □

Lemma 1.2.4. *Let γ be a closed proper convex cone (with vertex at 0) in a vector space V, G be a closed convex subset. If $\gamma^a + G$ is closed then*

$$\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_G \simeq \mathbb{C}_{\gamma^a + G}.$$

Here we denote by "a" the antipodal map $x \mapsto -x$.

Proof. Let us first prove $\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a} \xrightarrow{\sim} \mathbb{C}_{G+\gamma^a}$. By adjunction, there exists a morphism $\phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a} \to \mathbb{C}_{G+\gamma^a}$. Then taking a stalk of the cohomology proves the assertion. In fact, by [12, Theorem 3.5.3] for any $x \in X$

$$\varinjlim_{x \in U} H^k(U; \phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a}) \simeq \varinjlim_{x \in U} H^k(U+\gamma; \mathbb{C}_{G+\gamma^a})$$

and since $(x+\gamma)\cap G+\gamma^a$ is convex when $(x+\gamma)\cap G+\gamma^a \neq \emptyset$ we have $R\Gamma((x+\gamma)\cap G+\gamma^a; \mathbb{C}_{G+\gamma^a}) \simeq \mathbb{C}$. Thus it is enough to prove if $x \notin G+\gamma^a$ then $(x+\gamma)\cap G+\gamma^a = \emptyset$. By contradiction, if we may find $g_1, g_2 \in \gamma$ and $g_3 \in G$ so that $x + g_1 = g_3 - g_2$ then $x = g_3 - g_1 - g_2$. This implies $x \in G + \gamma^a$ which is a contradiction.

Now we prove the general assertion. We have a morphism $\mathbb{C}_{G+\gamma^a} \to \mathbb{C}_G$. Therefore by the first assertion we get a morphism

$$\mathbb{C}_{G+\gamma^a} \simeq \phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_{G+\gamma^a} \to \phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_G.$$

Then taking a stalk of the cohomology proves the assertion. In fact, by [12, Theorem 3.5.3]

$$\varinjlim_{x \in U} H^k(U; \phi_\gamma^{-1} R\phi_{\gamma *}\mathbb{C}_G) \simeq \varinjlim_{x \in U} H^k(U+\gamma; \mathbb{C}_G)$$

and since $(x+\gamma)\cap G$ is convex when $(x+\gamma)\cap G \neq \emptyset$ we have $R\Gamma((x+\gamma)\cap G; \mathbb{C}_G) \simeq \mathbb{C}$. Thus it is enough to prove if $x \notin G+\gamma^a$ then $(x+\gamma)\cap G = \emptyset$. By contradiction, if we may find $g_1 \in \gamma$ and $g_2 \in G$ so that $x+g_1 = g_2$ then $x = g_2 - g_1$. This implies $x \in G+\gamma^a$ which is a contradiction. □

Corollary 1.2.5 (Normal crossing type)**.** *Assume X is a vector space. Let $\widehat{\chi} = \{\widehat{D}_1, \dots, \widehat{D}_\ell\}$ where*

$$\widehat{D}_j = Z_{j1} \times Z_{j2} \times \cdots \times Z_{j\ell} \subset \widehat{X},$$
$$Z_{jk} := \begin{cases} \Delta_X, & (j=k), \\ X^2, & (o.w.). \end{cases}$$

Let $M_1, \dots, M_\ell$ be ℓ closed subanalytic convex subsets of X. Then for $p = (q; \xi_1, \dots, \xi_\ell) \in T^*X \times_X \cdots \times_X T^*X$ with $M_k \subset \{x \in X; \langle x-q, \xi_k\rangle \geq 0\}$ $(k = 1, \dots, \ell)$, we have

$$\begin{aligned} &H^j(\mu hom_{\widehat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}, \dots, \mathbb{C}_{M_\ell}; F))_p \\ \simeq &\varinjlim_{U, \widetilde{\delta_1}, \cdots, \widetilde{\delta_\ell}} H^j_{(M_1+\widetilde{\delta_1})\cap(M_2+\widetilde{\delta_2})\cap\cdots\cap(M_\ell+\widetilde{\delta_\ell})\cap U}(U; F), \end{aligned}$$

where each $\widetilde{\delta_k}$ is a closed subanalytic proper convex cone of $\delta_k \cup \{0\}$, $\delta_k = \{x \in X; \langle x, \xi_k\rangle > 0\}$ and U is an open subanalytic neighborhood of q in X.

Proof. By Theorem 1.2.2 we have

$$
\begin{aligned}
& H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1}, \dots, \mathbb{C}_{M_\ell}; F))_p \\
\simeq & \varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j(\bigcap_{j=1}^{\ell} U_j; \mathrm{R}\mathscr{H}om(R\rho_* i_\Delta^{-1}(\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}(\mathbb{C}_{M_1\cap U_1\times M_2\cap U_2\times\cdots\times M_\ell\cap U_\ell})), F)),
\end{aligned}
$$

where $\tilde{\gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_\ell \subset X^\ell$, each $\tilde{\gamma}_k$ is a closed conic subanalytic proper convex subset of $\gamma_k \cup \{0\}$,

$$
\gamma_k := \left\{ (x^{(1)}, \cdots, x^{(\ell)}) \in X^\ell \;\middle|\; \begin{array}{l} \langle x^{(j)}, \xi_k \rangle < 0, (j = k), \\ x^{(j)} = 0, (\text{o.w.}) \end{array} \right\}
$$

and each U_j ranges through the family of open subanalytic neighborhoods of q in X.

Let $\tilde{\gamma}$ be a cone as above. We may find open neighborhoods $U_1, U_2, \dots, U_\ell$ of q such that:

$$
M_1 \cap U_1 \times M_2 \cap U_2 \times \cdots \times M_\ell \cap U_\ell = M_1 \times M_2 \times \cdots \times M_\ell \cap (U_1 \times \cdots \times U_\ell + \tilde{\gamma}).
$$

In fact, for each j we may take an open neighborhood U_j of q which is an intersection of $\tilde{\gamma}_j$-open set in M_j and $\{x \in X; \langle x - q, \xi_j \rangle > -\epsilon\}$ $(\epsilon > 0)$.

Thus since for such $U_1, U_2, \dots, U_\ell$ the direct and inverse image of $\phi_{\tilde{\gamma}}$ commute with $(\cdot)_{U_1\times\cdots\times U_\ell+\tilde{\gamma}}$ we have:

$$
(\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}\mathbb{C}_{M_1\cap U_1\times\cdots\times M_\ell\cap U_\ell})|_{U_1\times\cdots\times U_\ell} \simeq (\phi_{\tilde{\gamma}}^{-1} R\phi_{\tilde{\gamma}*}\mathbb{C}_{M_1\times\cdots\times M_\ell})|_{U_1\times\cdots\times U_\ell}. \tag{1.12}
$$

Therefore by Lemma 1.2.4 we have,

$$
\begin{aligned}
&H^j(\mu hom_{\widehat{\chi}}(\mathbb{C}_{M_1},\dots,\mathbb{C}_{M_\ell};F))_p \\
\simeq &\varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j(\bigcap_{j=1}^{\ell} U_j;\mathrm{R}\mathscr{H}om(R\rho_* i_\Delta^{-1}(\phi_{\tilde{\gamma}}^{-1}R\phi_{\tilde{\gamma}*}(\mathbb{C}_{M_1\cap U_1\times M_2\cap U_2\times\cdots\times M_\ell\cap U_\ell})),F)) \\
\simeq &\varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j(\bigcap_{j=1}^{\ell} U_j;\mathrm{R}\mathscr{H}om(R\rho_* i_\Delta^{-1}((\phi_{\tilde{\gamma}}^{-1}R\phi_{\tilde{\gamma}*}(\mathbb{C}_{M_1\times M_2\times\cdots\times M_\ell}))_{U_1\times\cdots\times U_\ell}),F)) \\
\simeq &\varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j(\bigcap_{j=1}^{\ell} U_j;\mathrm{R}\mathscr{H}om(R\rho_* i_\Delta^{-1}(\mathbb{C}_{M_1\times M_2\times\cdots\times M_\ell+\tilde{\gamma}^a\cap U_1\times\cdots\times U_\ell}),F)) \\
\simeq &\varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j(\bigcap_{j=1}^{\ell} U_j;\mathrm{R}\mathscr{H}om(R\rho_*(i_\Delta^{-1}(\mathbb{C}_{\prod_{j=1}^{\ell}(M_j+q_j(\tilde{\gamma}_j^a))\cap\prod_{j=1}^{\ell}U_j})),F)) \\
\simeq &\varinjlim_{U_1,\dots,U_\ell,\tilde{\gamma}} H^j(\bigcap_{j=1}^{\ell} U_j;\mathrm{R}\mathscr{H}om(R\rho_*\mathbb{C}_{(M_1+q_1(\tilde{\gamma_1}^a))\cap\cdots\cap(M_\ell+q_\ell(\tilde{\gamma_\ell}^a))\cap\bigcap_{j=1}^{\ell}U_j},F)) \\
\simeq &\varinjlim_{U,\tilde{\gamma}} H^j_{(M_1+q_1(\tilde{\gamma_1}^a))\cap(M_2+q_2(\tilde{\gamma_2}^a))\cap\cdots\cap(M_\ell+q_\ell(\tilde{\gamma_\ell}^a))\cap U}(U;F),
\end{aligned}
$$

where q_k is k-th projection and U ranges through the family of open subanalytic neighborhoods of q in X. The second isomorphism follows from (1.12) and the third one follows from Lemma 1.2.4. The fourth isomorphism follows from the fact that $M_1\times M_2\times\cdots\times M_\ell+\tilde{\gamma}=(M_1+q_1(\tilde{\gamma_1}^a))\times(M_2+q_2(\tilde{\gamma_2}^a))\times\cdots\times(M_\ell+q_\ell(\tilde{\gamma_\ell}^a)$. The fifth one follows since $i_\Delta^{-1}\mathbb{C}_Z\simeq\mathbb{C}_{i_\Delta^{-1}(Z)}$ for a locally closed subset Z. □

Example 1.2.6. Let $X=\mathbb{C}^2$ and $\widehat{\chi}=\{\Delta\times\mathbb{C}^2,\mathbb{C}^2\times\Delta\}$. Consider closed subsets $M_1=\{y_1=0,x_1\geq 0\}=\mathbb{R}_{\geq 0}\times\mathbb{C}$ and $M_2=\{y_2=0,x_2\geq 0\}=\mathbb{C}\times\mathbb{R}_{\geq 0}$.

By Corollary 1.2.2 for any subanalytic sheaf $F\in D^b(\mathbb{C}_{X_{sa}})$ and $p=(q;\zeta_1,\zeta_2)\in T^*X\times_X T^*X=T^*\mathbb{C}^2\times_{\mathbb{C}^2}T^*\mathbb{C}^2=\overbrace{\underbrace{\mathbb{C}^2}_{\text{base space}}\times\underbrace{\mathbb{C}^2\times\mathbb{C}^2}_{\text{fiber space}}}^{\text{total space}}$ with $M_k\subset\{z\in X;\mathrm{Re}\langle z-q,\zeta_k\rangle\geq 0\}(k=1,2)$,

$$
\begin{aligned}
&H^j(\mu hom_{\widehat{\chi}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};F))_p \\
\simeq &\varinjlim_{U,\widetilde{\delta_1},\widetilde{\delta_2}} H^j_{(M_1+\widetilde{\delta_1})\cap(M_2+\widetilde{\delta_2})\cap U}(U;F),
\end{aligned}
$$

where each $\widetilde{\delta_k}$ is a closed subanalytic proper convex cone of $\delta_k\cup\{0\}$, $\delta_k=\{z\in\mathbb{C}^2;\mathrm{Re}\langle z,\zeta_k\rangle>0\}$ and U ranges through the family of open subanalytic neighborhoods of q. Here $\langle(z_1,z_2),(z_3,z_4)\rangle:=z_1z_3+z_2z_4$.

Let $p = ((1,1); (\sqrt{-1},0),(0,\sqrt{-1}))$, we have

$$\begin{aligned}&H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};F))_p\\ &\simeq \varinjlim_{U} H^j_{H\times H\cap U}(U;F),\end{aligned}$$

where U ranges through the family of open subanalytic neighborhoods of $(1,1)$ and $H := \{x + iy; y \leq 0\}$. If $F = R\rho_*\mathscr{O}_X$ then by [10, Theorem 2.2.2] or the useful criterion [12, (2.9.14)] for $j \neq 2$ we have

$$\begin{aligned}&H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};R\rho_*\mathscr{O}_X))_p\\ &\simeq \varinjlim_{U} H^j_{H\times H\cap U}(U;R\rho_*\mathscr{O}_X) = 0.\end{aligned}$$

When $p = ((1,0); (\sqrt{-1},0),(0,\sqrt{-1}))$ we have

$$\begin{aligned}&H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};F))_p\\ &\simeq \varinjlim_{\lambda,U} H^j_{G_\lambda\times H\cap U}(U,F),\end{aligned}$$

where $G_\lambda = \lambda + \mathbb{R}_{\geq 0} \subset \mathbb{C}$ and λ runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda \subset \{y_1 < 0\} \cup \{0\}$ and U ranges through the family of open subanalytic neighborhoods of $(1,0)$. If $F = R\rho_*\mathscr{O}_X$ then by the useful criterion [12, (2.9.14)] for $j \neq 2$ we have

$$H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};R\rho_*\mathscr{O}_X))_p = 0.$$

Finally consider $p = ((0,0); (\sqrt{-1},0),(0,\sqrt{-1}))$. We obtain

$$\begin{aligned}&H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};F))_p\\ &\simeq \varinjlim_{\lambda_1,\lambda_2,U} H^j_{G_{\lambda_1}\times G_{\lambda_2}\cap U}(U,F),\end{aligned}$$

where each $G_{\lambda_k} = \lambda_k + \mathbb{R}_{\geq 0} \subset \mathbb{C}$ and λ_k runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda_k \subset \{y_k < 0\} \cup \{0\}$ and U ranges through the family of open subanalytic neighborhoods of $(0,0)$. If $F = R\rho_*\mathscr{O}_X$ then by the useful criterion [12, (2.9.14)] for $j \neq 2$ we have

$$H^j(\mu hom_{\widetilde{X}}(\mathbb{C}_{M_1},\mathbb{C}_{M_2};R\rho_*\mathscr{O}_X))_p = 0.$$

Summing up computations above, $H^2(\mu hom_{\widetilde{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_* \mathscr{O}_X))_p$ is isomorphic to

$$\begin{cases} \varinjlim_U H^j_{H\times H\cap U_1}(U; F) & \text{if } p = ((1,1); (\sqrt{-1}, 0), (0, \sqrt{-1})), \\ \varinjlim_{\lambda, V} H^j_{G_\lambda \times H \cap V}(V, F) & \text{if } p = ((1,0); (\sqrt{-1}, 0), (0, \sqrt{-1})), \\ \varinjlim_{\lambda_1, \lambda_2, W} H^j_{G_{\lambda_1}\times G_{\lambda_2}\cap W}(W, F) & \text{if } p = ((0,0); (\sqrt{-1}, 0), (0, \sqrt{-1})). \end{cases}$$

Here, U_1 ranges through the family of open subanalytic neighborhoods of $(1,1)$ and $H := \{x + iy; y \leq 0\}$. $G_\lambda = \lambda + \mathbb{R}_{\geq 0} \subset \mathbb{C}$ and λ runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda \subset \{y_1 < 0\} \cup \{0\}$ and V ranges through the family of open subanalytic neighborhoods of $(1,0)$. Each $G_{\lambda_k} = \lambda_k + \mathbb{R}_{\geq 0} \subset \mathbb{C}$ and λ_k runs through the family of closed conic subanalytic proper convex subsets in $\mathbb{C}$ with its vertex being 0 such that $\lambda_k \subset \{y_k < 0\} \cup \{0\}$ and W ranges through the family of open subanalytic neighborhoods of $(0,0)$. If $j \neq 2$, $H^j(\mu hom_{\widetilde{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_* \mathscr{O}_X))_p \simeq 0$.

1.3 Multi-microlocal Sato triangle

In this section we construct multi-microlocal Sato's triangle. Suppose $\chi = \{M_1, M_2, \ldots, M_\ell\}$ satisfies the conditions H1, H2 and H3.

Proposition 1.3.1. *Let $i : M \to X$ be the canonical embedding and $\pi : S^*_\chi \to M$ be the canonical projection where $M = \bigcap_{j=1}^{\ell} M_j$. For $F \in D^b(k_{X_{sa}})$*

$$\mu^{sa}_\chi(F)|_M \simeq R\pi_* \mu^{sa}_\chi(F) \simeq R\Gamma_M(F)|_M \simeq i^! F.$$

Proof. It is enough to prove the second isomorphism. By Theorem 1.1.8, for $H = \nu^{sa}_\chi(F)$ we have

$$\tau^! R\tau_{!!} H \simeq Rp_{1*} p_2^! H^{\wedge_{S_\chi}}.$$

By the base change formula([20, Proposition 2.2.9]) on the Cartesian square:

$$\begin{array}{ccc} S_\chi \times_M S^*_\chi & \xrightarrow{p_2} & S^*_\chi \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ S_\chi & \xrightarrow[\tau]{} & M \end{array}$$

we have $Rp_{1*} p_2^! H^{\wedge_{S_\chi}} \simeq \tau^! R\pi_* H^{\wedge_{S_\chi}}$. Thus

$$\tau^! R\tau_{!!} H \simeq \tau^! R\pi_* H^{\wedge_{S_\chi}}.$$

Applying $i^!$ both sides:

$$R\tau_{!!}H \simeq R\pi_* H^{\wedge s_\chi}.$$

Since $R\tau_{!!}\nu_\chi^{sa}(F) \simeq R\Gamma_M(F)$ by [8, Proposition 2.6], we get

$$R\Gamma_M(F) \simeq R\pi_* \mu_\chi^{sa}(F).$$

□

Next we extend the result in [28, Corollary A.2] to subanalytic sheaf theory.

Proposition 1.3.2. *Let $\tau : E \to Z$ be a vector bundle and E^* be its dual. Set $P = \{(x, y) \in E \times_Z E^*; \langle x, y\rangle \geq 0\}$. For a conic object $G \in D^b_{\mathbb{R}^+}(k_{E_{sa}})$, there is a natural isomorphism*

$$\tau^{-1}R\tau_* G \otimes \omega_{E/Z} \simeq Rp_{1!!}p_2^{-1}G^\wedge$$

and a commutative diagram

$$\begin{tikzcd}
G & Rp_{1!!}(p_2^{-1}G^\wedge)_P \otimes \omega_{E/Z} \arrow[l] \\
\tau^{-1}R\tau_* G \arrow[u] & Rp_{1!!}p_2^{-1}G^\wedge \otimes \omega_{E/Z}, \arrow[l] \arrow[u]
\end{tikzcd}$$

where every horizontal arrow is an isomorphism.

Proof. We have a commutative diagram

$$\begin{tikzcd}
G & Rp_{1!!}(p_2^!G^\wedge)_P \arrow[l, "\sim"'] & Rp_{1!!}p_2^!G^\wedge \arrow[l] \\
\tau^{-1}R\tau_* G \arrow[u] & \tau^{-1}R\tau_* Rp_{1!!}(p_2^!G^\wedge)_P \arrow[l, "\sim"'] \arrow[u] & \tau^{-1}R\tau_* Rp_{1!!}p_2^!G^\wedge, \arrow[l, "\alpha"'] \arrow[u, "\beta"']
\end{tikzcd}$$

where each horizontal arrow of the left square is an isomorphism by [22, Theorem 3.2.6]. To show α is an isomorphism, it is enough to prove

$$R\tau_* Rp_{1!!}H \to R\tau_* Rp_{1!!}H_P$$

is an isomorphism for a conic object $H \in D^b_{\mathbb{R}^+}(k_{E\times_M E^*{}_{sa}})$ on $E \times_M E^*{}_{sa}$. We make the use of the following Cartesian diagram:

$$\begin{array}{ccccccc}
E \times_Z E^* & \xrightarrow{\tau'} & Z \times E^* & \xhookrightarrow{i'} & E \times_Z E^* & \xrightarrow{p_2} & E^* \\
{\scriptstyle p_1}\downarrow & & {\scriptstyle p_1'}\downarrow & & {\scriptstyle p_1}\downarrow & & \\
E & \xrightarrow[\tau]{} & Z & \xhookrightarrow[i]{} & E. & &
\end{array}$$

We have the commutative diagram,

$$\begin{array}{ccccc}
R\tau_* Rp_{1!!}H_P & \xrightarrow{\sim} & i^{-1}Rp_{1!!}H_P & \xrightarrow{\sim} & Rp'_{1!!}i'^{-1}H_P \\
\uparrow & & \uparrow & & \uparrow \\
R\tau_* Rp_{1!!}H & \xrightarrow{\sim} & i^{-1}Rp_{1!!}H & \xrightarrow{\sim} & Rp'_{1!!}i'^{-1}H,
\end{array}$$

where horizontal isomorphisms follows from $R\tau_* \simeq i^{-1}$ ([22, Lemma 3.1.5]) and the base change formula ([20, Proposition 2.2.8]). Since

$$i'(Z \times_Z E^*) \subset P$$

third vertical arrow is an isomorphism. Thus α is an isomorphism. We shall prove β is an isomorphism. Since $p_2 = p_2 \circ i' \circ \tau'$ and by the base change formula([20, Proposition 2.2.9]), we have

$$\begin{aligned}
Rp_{1!!}p_2^{-1}G^\wedge \otimes \omega_{E/Z} &\simeq Rp_{1!!}\tau'^{-1}i'^{-1}p_2^{-1}G^\wedge \otimes \omega_{E/Z} \\
&\simeq \tau^{-1}i^{-1}Rp_{1!!}p_2^{-1}G^\wedge \otimes \omega_{E/Z} \\
&\simeq \tau^{-1}R\tau_* Rp_{1!!}p_2^! G^\wedge.
\end{aligned}$$

□

Proposition 1.3.3. *Let G be a multi-conic object on E^*_{sa}. We have*

$$G^{\vee^*_E} \simeq Rp''_{1!!}p_2''^{-1}G \otimes \omega_{E/Z} \simeq Rp_{1!!}(p_2^! G)_P.$$

(For the notation see Subsection 1.1.2.)

Proof. By induction assume $\ell = 2$. We have the following diagram:

$$\begin{tikzcd}
& P \arrow[r, "p''_{2,2}"] \arrow[d, "p''_{1,1}"'] \arrow[ll, bend right, "p''_1"'] \arrow[rr, bend left, "p''_2"] \arrow[dr, phantom, "\square"] & P_1 \underset{Z}{\times} E_2^* \arrow[r, "p''_{1,2}"'] \arrow[d, "p''_{1,1}"] & E^* \\
E & E_1 \underset{Z}{\times} P_2 \arrow[l, "p''_{2,1}"'] \arrow[r, "p''_{2,2}"'] & E_1 \underset{Z}{\times} E_2^*. &
\end{tikzcd}$$

Then by Proposition 1.1.2 we have

$$\begin{aligned}
G^{\vee_1^*\vee_2^*} &\simeq Rp_{2,1!!}(p_{2,2}^{-1}Rp_{1,1!!}(p_{1,2}^{-1}G)_{P_1})_{P_2} \otimes \omega_{E/Z} \\
&\simeq Rp''_{2,1!!}p''^{-1}_{2,2}Rp''_{1,1!!}p''^{-1}_{1,2}G \otimes \omega_{E/Z} \\
&\simeq Rp''_{2,1!!}Rp''_{1,1!!}p''^{-1}_{2,2}p''^{-1}_{1,2}G \otimes \omega_{E/Z} \\
&\simeq Rp''_{1!!}p''^{-1}_{2}G \otimes \omega_{E/Z}.
\end{aligned}$$

Assume $\ell > 2$. Set $E' := \underset{Z,2\leq i\leq \ell}{\times} E_i$, $E'^* := \underset{Z,2\leq i\leq \ell}{\times} E_i'^*$, $P_{E'} := \underset{Z,2\leq i\leq \ell}{\times} P_i$. We have the following commutative diagram:

$$\begin{tikzcd}
& P \arrow[r, "p''_{E',2}"] \arrow[d, "p''_{1,1}"'] \arrow[ll, bend right, "p''_1"'] \arrow[rr, bend left, "p''_2"] \arrow[dr, phantom, "\square"] & P_1 \underset{Z}{\times} E'^* \arrow[r, "p''_{1,2}"'] \arrow[d, "p''_{1,1}"] & E^* \\
E & E_1 \underset{Z}{\times} P_{E'} \arrow[l, "p''_{E',1}"'] \arrow[r, "p''_{E',2}"'] & E_1 \underset{Z}{\times} E'^*. &
\end{tikzcd}$$

By induction hypothesis, we have

$$(G^{\vee_1^*})^{\vee_{E'}^*} \simeq Rp''_{1!!}p''^{-1}_{2}G \otimes \omega_{E/Z}.$$

Therefore, the induction proceeds. $\square$

We need to extend the result [22, Proposition 3.2.9]:

Lemma 1.3.4. *Let E_1, E_2 be vector bundles over Z, a biconic object $F \in D^b_{\mathbb{R}^+\times\mathbb{R}^+}(k_{E_1\times_Z E_{2sa}})$ on $E_1 \times_Z E_{2sa}$ and $p : E_1 \times_Z E_2 \to E_1$ then $Rp_{!!}F \in D^b_{\mathbb{R}^+}(k_{E_{1sa}})$. Here the first $\mathbb{R}^+$-action on $E_1 \times_Z E_2$ is either $((z;\xi_1,\xi_2),t_1) \mapsto (z;t_1\xi_1,t_1\xi_2)$ or $((z;\xi_1,\xi_2),t_1) \mapsto (z;t_1\xi_1,\xi_2)$*

and the second one is $((z;\xi_1,\xi_2),t_2) \mapsto (z;\xi_1,t_2\xi_2)$*. Moreover,*

$$(Rp_{!!}F)^{\wedge_{E_1}} \simeq Rp'_{!!}F^{\wedge_{E_1}},$$

where $p' : E_1^* \times_Z E_2 \to E_1^*$*.*

Proof. The natural embedding $i : E_1 \to E_1 \times_Z E_2$ is a conic morphism with respect to the first action on $E_1 \times_Z E_2$. By [22, Proposition 2.5.13] we get $i^! F \in D^b_{\mathbb{R}^+}(k_{E_{1sa}})$. Since F is a conic object on $E_1 \times_Z E_{2sa}$ with respect to the second action, by [22, Lemma 3.1.5] we obtain $i^! F \simeq Rp_{!!}F$. Therefore $Rp_{!!}F \in D^b_{\mathbb{R}^+}(k_{E_{1sa}})$. Consider the diagram below where the squares are Cartesian:

$$\begin{tikzcd}
E_1^* \underset{Z}{\times} E_2 \arrow[r, "p'"] & E_1^* & \\
P' \underset{Z}{\times} E_2 \arrow[u, "q'_2"] \arrow[d, "q'_1"'] \arrow[r, "\tilde{p}"] \arrow[ur, phantom, "\square"] \arrow[dr, phantom, "\square"] & P' \arrow[u, "q_2"'] \arrow[d, "q_1"] \arrow[r, hook, "j"'] & E_1 \underset{Z}{\times} E_1^* \arrow[ul, "p_2"'] \arrow[dl, "p_1"] \\
E_1 \underset{Z}{\times} E_2 \arrow[r, "p"] & E_1, &
\end{tikzcd}$$

here $P' := \{(\eta,\xi) \in E_1 \underset{Z}{\times} E_1^*; \langle \eta,\xi\rangle \leq 0\}$. Notice that $Rq_{2!!}q_1^{-1}G \simeq G^{\wedge_{E_1}}$ for a conic object G on E_{1sa}. In fact,

$$\begin{aligned} Rp_{2!!}(p_1^{-1}G)_{P'} &\simeq Rp_{2!!}Rj_{!!}j^{-1}p_1^{-1}G \\ &\simeq Rq_{2!!}q_1^{-1}G. \end{aligned}$$

Therefore:

$$\begin{aligned} (Rp_{!!}F)^{\wedge_{E_1}} &\simeq Rq_{2!!}q_1^{-1}Rp_{!!}F \\ &\simeq Rq_{2!!}R\tilde{p}_{!!}q_1'^{-1}F \\ &\simeq Rp'_{!!}Rq'_{2!!}q_1'^{-1}F \\ &\simeq Rp'_{!!}(F^{\wedge_{E_1}}). \end{aligned}$$

□

Proposition 1.3.5. *Let F be a multi-conic object on E_{sa}. We have a natural isomorphism*

$$\tau^{-1}R\tau_* F \simeq Rp_{1!!}p_2^! F^{\wedge_E}$$

and the natural morphism $\tau^{-1}R\tau_*F \to F$ is embedded to the following distinguished triangle:

$$Rp_{1!!}(p_2^! F^{\wedge_E})_{P^-} \longrightarrow \tau^{-1}R\tau_*F \longrightarrow F \xrightarrow{+1} .$$

(For the notation see Subsection 1.1.2.)

Proof. If $\ell = 1$, the result follows by the Proposition 1.3.2. Assume $\ell > 1$, and $E' := \bigtimes_{Z,i=2}^{\ell} E_i$, $E'^* := \bigtimes_{Z,i=2}^{\ell} E_i^*$, and $P'_{E'} := \bigtimes_{Z,i=2}^{\ell} P'_i$. Moreover, let $\wedge_{E'}$(resp. $\vee_{E'}^*$) be the composition of $\wedge_i$(resp. $\vee_i^*$) for $i = 2, \dots, \ell$. Consider the following diagram:

$$\begin{tikzcd}
 & E \underset{Z}{\times} E^* \arrow[r, "p_{E',2}"] \arrow[rr, bend left, "p_2"] \arrow[dl, bend right, "p_1"'] \arrow[d, "p_{1,1}"'] \arrow[dr, phantom, "\square"] & E_1 \underset{Z}{\times} E^* \arrow[r, "p_{1,2}"'] \arrow[d, "p_{1,1}"] & E^* \\
E \arrow[r, "p_{E',1}"'] & E_1 \underset{Z}{\times} E' \underset{Z}{\times} E'^* \arrow[r, "p_{E',2}"'] & E_1 \underset{Z}{\times} E'^*. &
\end{tikzcd}$$

By the commutative diagram

$$\begin{tikzcd}
E \arrow[r, "\tau_1"] \arrow[d, "\tau_{E'}"'] \arrow[dr, phantom, "\square"] & E' \arrow[d, "\tau_{E'}"] \\
E_1 \arrow[r, "\tau_1"'] & Z
\end{tikzcd}$$

and [20, Proposition 2.4.8] and the dual base change formula we have

$$\begin{aligned}
\tau_1^{-1}R\tau_{1*}\tau_{E'}^{-1}R\tau_{E'*}F &\simeq \tau_1^{-1}\tau_{E'}^{-1}R\tau_{1*}R\tau_{E'*}F \\
&\simeq \tau^{-1}R\tau_*F.
\end{aligned} \tag{1.13}$$

Hence, by Proposition 1.3.2 and induction hypothesis, we obtain the following commutative diagram:

$$\begin{tikzcd}
F & F^{\wedge_{E'}\vee_{E'}^*} \arrow[l, "\sim"'] & F^{\wedge_{E'}\vee_{E'}^*\wedge_1\vee_1^*} \arrow[l, "\sim"'] \\
\tau_{E'}^{-1}R\tau_{E'*}F \arrow[u] & & Rp_{E',1!!}p_{E',2}^!(F^{\wedge_{E'}}) \arrow[ll, "\sim", "\beta_1"'] \arrow[u] \\
\tau_1^{-1}R\tau_{1*}\tau_{E'}^{-1}R\tau_{E'*}F \arrow[u] & & \tau_1^{-1}R\tau_{1*}Rp_{E',1!!}p_{E',2}^!(F^{\wedge_{E'}}) \arrow[ll, "\sim", "\beta_2"'] \arrow[u] \\
\tau^{-1}R\tau_*F \arrow[u, "\alpha", "\sim"'] & & Rp_{1,1!!}p_{1,2}^!((Rp_{E',1!!}p_{E',2}^!(F^{\wedge_{E'}}))^{\wedge_1}). \arrow[ll, "\sim", "\beta_3"'] \arrow[u, "\sim", "\gamma"']
\end{tikzcd}$$

The isomorphism α comes from (1.13). β_1 is an isomorphism by induction hypothesis. β_2 is the isomorphism $\tau_1^{-1}R\tau_{1*}(\beta_1)$. γ follows from Proposition 1.3.2. β_3 is a composition of isomorphisms.

By Lemma 1.3.6 below for a multi-conic object H on $E_1 \times_Z E'^*_{sa}$ we have $H^{\vee^*_{E'}\vee_1} \simeq H^{\vee_1\vee^*_{E'}}$. Therefore, we have

$$\begin{aligned} H^{\vee^*_{E'}\wedge_1} &\simeq (H^{\vee^*_{E'}\vee_1})^{id\times a} \otimes \omega_{E_1/Z}^{\otimes -1} \\ &\simeq (H^{\vee_1\vee^*_{E'}})^{id\times a} \otimes \omega_{E_1/Z}^{\otimes -1} \\ &\simeq H^{\wedge_1\vee^*_{E'}}. \end{aligned}$$

Thus, we get the following chain of isomorphisms by Proposition 1.3.3 and [8, Proposition 2.2].

$$\begin{aligned} F^{\wedge_{E'}\vee^*_{E'}\wedge_1\vee_1{}^*} &\simeq F^{\wedge_{E'}\wedge_1\vee^*_{E'}\vee_1{}^*} \\ &\simeq Rp''_{1!!}p''^{-1}_2 F^{\wedge_E} \otimes \omega_{E/Z}. \end{aligned}$$

Lastly by Lemma 1.3.4 and [22, Proposition 3.2.9] we have

$$\begin{aligned} & Rp_{1,1!!}p_{1,2}^{-1}\left(\left(Rp_{E',1!!}p_{E',2}^{-1}(F^{\wedge_{E'}})\right)^{\wedge_1}\right) \\ &\simeq Rp_{1,1!!}p_{1,2}^{-1}Rp'_{E',1!!}p'^{-1}_{E',2}(F^{\wedge_E}) \qquad (1.14) \\ &\simeq Rp_{1,1!!}Rp_{E',1!!}p'^{-1}_{1,2}p'^{-1}_{E',2}(F^{\wedge_E}) \qquad (1.15) \\ &\simeq Rp_{1,1!!}Rp_{E',1!!}p_2^{-1}(F^{\wedge_E}) \\ &\simeq Rp_{1!!}p_2^{-1}(F^{\wedge_E}), \end{aligned}$$

where morphisms on (1.14) are $p'_{E',2} : E' \times_Z E^* \to E^*$ and $p'_{E',1} : E_1^* \times_Z E' \times_Z E'^* \to E_1^* \times_Z E'$. On (1.15) we used the base change formula([20, Proposition 2.2.9]) on the Cartesian square as shown below.

$$\begin{tikzcd} E \underset{Z}{\times} E^* \arrow[r, "p'_{1,2}"] \arrow[d, "p_{E',1}"'] & E' \underset{Z}{\times} E^* \arrow[d, "p'_{E',1}"] \\ E_1 \underset{Z}{\times} E_1^* \underset{Z}{\times} E' \arrow[r, "p_{1,2}"'] & E_1^* \underset{Z}{\times} E' \end{tikzcd}$$

By the above isomorphisms,

$$
\begin{tikzcd}
 & \tau^{-1}R\tau_*F \arrow[r] & F & \\
 & Rp_{1,1!!}p^!_{1,2}(Rp_{E',1!!}p^!_{E',2}F^{\wedge_{E'}})^{\wedge_1} \arrow[r] \arrow[u, "\sim"'] & F^{\wedge_{E'}\vee^*_{E'}\wedge_1\vee_1{}^*} \arrow[u, "\sim"'] & \\
Rp_{1!!}(p^!_2(F^{\wedge_E}))_{P^-} \arrow[r] \arrow[d, equal] & Rp_{1!!}p^!_2F^{\wedge_E} \arrow[r] \arrow[u, "\sim"'] \arrow[d, equal] & Rp''_{1!!}p''^{-1}_2(F^{\wedge_E}) \otimes \omega_{E/Z} \arrow[r, "+1"] \arrow[u, "\sim"'] \arrow[d, "\sim"'] & {} \\
Rp_{1!!}(p^!_2(F^{\wedge_E}))_{P^-} \arrow[r] & Rp_{1!!}p^!_2F^{\wedge_E} \arrow[r] & Rp_{1!!}(p^!_2F^{\wedge_E})_P \arrow[r, "+1"] & {} .
\end{tikzcd}
$$

□

Lemma 1.3.6. *Let E_1 and E'^* be as in the proof of Proposition 1.3.5. For a multi-conic object H on $E_1 \times_Z E'^*_{sa}$ we have $H^{\vee^*_{E'}\vee_1} \simeq H^{\vee_1\vee^*_{E'}}$.*

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
 & P'_1 \underset{Z}{\times} P'_{E'} \arrow[r, "p'_{E',2}"] \arrow[dl, "p'_1"'] \arrow[d, "p'_{1,2}"] \arrow[rr, bend left, "p'_2"] & P'_1 \underset{Z}{\times} E'^* \arrow[r, "p'_{1,1}"] \arrow[d, "p'_{1,2}"] & E_1 \underset{Z}{\times} E'^* \\
E^*_1 \underset{Z}{\times} E' & E^*_1 \underset{Z}{\times} P'_{E'} \arrow[l, "p'_{E',1}"] \arrow[r, "p'_{E',2}"'] & E^*_1 \underset{Z}{\times} E'^*, &
\end{tikzcd}
$$

where the square is Cartesian. Then we have

$$
\begin{aligned}
H^{\vee^*_{E'}\vee_1} \simeq Rp'_{E',1*}p'^!_{E',2}Rp'_{1,2*}p'^!_{1,1}H &\simeq Rp'_{E',1*}Rp'_{1,2*}p'^!_{E',2}p'^!_{1,1}H \\
&\simeq Rp'_{1*}p'^!_2H.
\end{aligned}
$$

For the same reason, we obtain $H^{\vee_1\vee^*_{E'}} \simeq Rp'_{1*}p'^!_2H$. □

Proposition 1.3.7. *Let $F \in D^b(k_{X_{sa}})$. Let i and M be as in Proposition 1.3.1. Then:*

$$
R\pi_{!!}\mu^{sa}_{\chi}(F) \simeq R\Gamma_M(\mu^{sa}_{\chi}(F)) \simeq i^{-1}F \otimes \omega_{M/X}.
$$

Proof. Set $H := \nu^{sa}_{\chi}(F)$ and $E := S_{\chi}$, then by Proposition 1.3.5 we have

$$
\tau^{-1}R\tau_*H \otimes \omega_{E/M} \simeq Rp_{1!!}p^{-1}_2H^{\wedge_E}.
$$

By the base change formula([20, Proposition 1.4.6]) on the Cartesian square,

$$\begin{array}{ccc} E\times_M E^* & \xrightarrow{p_2} & E^* \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow[\tau]{} & M \end{array}$$

we have

$$\tau^{-1}R\pi_{!!}H^{\wedge_E} \simeq Rp_{1!!}p_2^{-1}H^{\wedge_E}.$$

Thus

$$\tau^{-1}R\pi_{!!}H^{\wedge_E} \simeq \tau^{-1}R\tau_*H\otimes\omega_{E/M}.$$

We apply i^{-1} to get

$$R\pi_{!!}H^{\wedge_E} \simeq R\tau_*H\otimes\omega_{M/X}.$$

Since $R\tau_*H \simeq F|_M$ ([8, Proposition 2.6]) we have,

$$R\pi_{!!}\nu^{sa}_{\chi}(F)^{\wedge_E} \simeq F|_M\otimes\omega_{M/X}.$$

□

Corollary 1.3.8 (multi-microlocal Sato's triangle)**.** *We get the distinguished triangle:*

$$F|_M\otimes\omega_{M/X} \longrightarrow R\Gamma_M(F)|_M \longrightarrow R\dot{\pi}_*\mu^{sa}_{\chi}(F) \xrightarrow{+1},$$

where $\dot{\pi}$ is the restriction of $\pi : S^*_{\chi} \to M$ to $S^*_{\chi}\setminus M$.

Proof. We finish the proof by applying the above results(Proposition 1.3.1, Proposition 1.3.7) to the distinguished triangle:

$$R\pi_{!!}\mu^{sa}_{\chi}(F) \longrightarrow R\pi_*\mu^{sa}_{\chi}(F) \longrightarrow R\dot{\pi}_*\mu^{sa}_{\chi}(F) \xrightarrow{+1}.$$

□

Now we construct Sato's triangle for multi-microlocal Hom functors. We replace X and χ with an ambient space $\hat{X}$ and the universal family $\hat{\chi}$.

Convention 1.3.9. Since if we let $i : T^*X\times_X\cdots\times_X T^*X \to S^*_{\hat{\chi}}$ be a canonical embedding we get by [6, Lemma 2.18],

$$Ri_*i^{-1}\mu hom^{sa}_{\hat{\chi}}(G_1,\ldots,G_\ell;F) \simeq \mu hom^{sa}_{\hat{\chi}}(G_1,\ldots,G_\ell;F),$$

$\mu hom^{sa}_{\widetilde{X}}(G_1,\ldots,G_\ell;F)$ may be seen as an object on $T^*X\times_X\cdots\times_X T^*X$. We take this convention throughout this paper.

Proposition 1.3.10. *Let $\pi : T^*X\times_X\cdots\times_X T^*X \to X$ be a canonical projection and $G_1,\ldots,G_\ell, F \in Ob(D^b(k_{X_{sa}}))$. Then*

$$\begin{aligned} R\pi_*\mu hom^{sa}_{\widetilde{X}}(G_1,\ldots,G_\ell;F) &\simeq \mathrm{R}\mathscr{H}om(G_1\boxtimes\cdots\boxtimes G_\ell, Ri_{\Delta *}F)\\ &\simeq Ri_{\Delta *}\,\mathrm{R}\mathscr{H}om(G_1\otimes\cdots\otimes G_\ell, F). \end{aligned}$$

In particular,

$$\mathrm{Hom}_{D^b(k_{X_{sa}})}(\overset{\ell}{\underset{i=1}{\otimes}} G_i, F) \simeq H^0(T^*X\times_X\cdots\times_X T^*X; \mu hom^{sa}_{\widetilde{X}}(G_1,\ldots,G_\ell;F)).$$

Proof. We have the following chain of isomorphisms:

$$\begin{aligned} &R\pi_*\mu^{sa}_{\widetilde{X}}\,\mathrm{R}\mathscr{H}om(q_2^{-1}(G_1\boxtimes\cdots\boxtimes G_\ell), q_1^! Ri_{\Delta *}F)\\ \simeq&\delta^!\,\mathrm{R}\mathscr{H}om(q_2^{-1}(G_1\boxtimes\cdots\boxtimes G_\ell), q_1^! Ri_{\Delta *}F)\\ \simeq&\,\mathrm{R}\mathscr{H}om(\delta^{-1}q_2^{-1}(G_1\boxtimes\cdots\boxtimes G_\ell), \delta^! q_1^! Ri_{\Delta *}F)\\ \simeq&\,\mathrm{R}\mathscr{H}om(G_1\boxtimes\cdots\boxtimes G_\ell, Ri_{\Delta *}F)\\ \simeq&Ri_{\Delta *}\,\mathrm{R}\mathscr{H}om(G_1\otimes\cdots\otimes G_\ell, F), \end{aligned}$$

where δ is a diagonal embedding $X^\ell \to (X^2)^\ell, (x_1,\ldots,x_\ell)\mapsto(x_1,x_1,x_2,x_2,\ldots,x_\ell,x_\ell)$. The first isomorphism follows from Proposition 1.3.1. The second isomorphism follows from Proposition 1.1.4. □

Proposition 1.3.11. *Let $G_1,\ldots,G_\ell \in D^b_{\mathbb{R}\text{-}c}(k_X)$ and $F \in D^b(k_{X_{sa}})$. We obtain the following isomorphisms:*

$$\begin{aligned} R\pi_{!!}\mu hom^{sa}_{\widetilde{X}}(G_1,G_2,\ldots G_\ell,F) &\simeq \mathrm{R}\mathscr{H}om(G_1\boxtimes\cdots\boxtimes G_\ell, k_{X^\ell})\otimes Ri_{\Delta *}F\\ &\simeq Ri_{\Delta *}(D'G_1\otimes\cdots\otimes D'G_\ell\otimes F), \end{aligned}$$

where we set $D'F := \mathrm{R}\mathscr{H}om(F,k_X)$ for $F\in D^b(k_{X_{sa}})$.

Proof. We have the isomorphisms by Proposition 1.3.7:

$$
\begin{aligned}
& R\pi_{!!}\mu hom^{sa}_{\widehat{\chi}}(G_1, G_2, \dots G_\ell, F) \\
\simeq & \delta^{-1}\left(\mathrm{R}\mathcal{H}om(p_2^{-1}(G_1 \boxtimes \cdots \boxtimes G_\ell), p_1^! Ri_{\Delta *}F)\right) \otimes \omega_{X^\ell/\widehat{X}} \\
\simeq & \delta^{-1}(\mathrm{R}\mathcal{H}om(G_1 \boxtimes \cdots \boxtimes G_\ell, k_{X^\ell}) \boxtimes Ri_{\Delta *}F) && (1.16) \\
\simeq & \mathrm{R}\mathcal{H}om(G_1 \boxtimes \cdots \boxtimes G_\ell, k_{X^\ell}) \otimes Ri_{\Delta *}F \\
\simeq & Ri_{\Delta *}(D'G_1 \otimes \cdots \otimes D'G_\ell \otimes F). && (1.17)
\end{aligned}
$$

Here $\delta : X^\ell \hookrightarrow (X^2)^\ell$ is a diagonal embedding. The isomorphism (1.16) follows from [22, Lemma 5.3.9]. (1.17) follows from [12, Proposition 3.4.4] and the projection formula. □

Corollary 1.3.12. *For $G_1, G_2, \dots, G_\ell \in D^b_{\mathbb{R}\text{-}c}(k_X)$ and $F \in D^b(k_{X_{sa}})$, we have the following distinguished triangle:*

$$
\overset{\ell}{\underset{i=1}{\otimes}} D'G_i \otimes F \longrightarrow \mathrm{R}\mathcal{H}om(\overset{\ell}{\underset{i=1}{\otimes}} G_i, F) \longrightarrow R\dot{\pi}_*\mu hom^{sa}_{\widehat{\chi}}(G_1, \dots, G_\ell; F) \xrightarrow{+1} ,
$$

where $\dot{\pi}$ is the restriction of $\pi : T^*X \times_X \cdots \times_X T^*X \to X$ to $T^*X \times_X \cdots \times_X T^*X \setminus X$.

Example 1.3.13. Let $X = \mathbb{C}^2$ and $\widehat{\chi} = \{\Delta \times \mathbb{C}^2, \mathbb{C}^2 \times \Delta\}$. Consider closed subsets $M_1 = \{y_1 = 0, x_1 \geq 0\} = \mathbb{R}_{\geq 0} \times \mathbb{C}$ and $M_2 = \{y_2 = 0, x_2 \geq 0\} = \mathbb{C} \times \mathbb{R}_{\geq 0}$.

Since $D'(\mathbb{C}_{M_1}) \simeq \mathbb{C}_{\{y_1=0, x_1>0\}}[-1]$, $D'(\mathbb{C}_{M_2}) \simeq \mathbb{C}_{\{y_2=0, x_2>0\}}[-1]$ we obtain

$$
(\mathrm{R}\mathcal{H}om(\mathbb{C}_{M_1 \cap M_2}, R\rho_*\mathcal{O}_X))_{(0,0)} \simeq \left(R\dot{\pi}_*\mu hom_{\widehat{\chi}}(\mathbb{C}_{M_1}, \mathbb{C}_{M_2}; R\rho_*\mathcal{O}_X)\right)_{(0,0)} .
$$

Chapter 2

Strong Regularity and Microsupport Estimates for Multi-Microlocalizations of Subanalytic Sheaves

2.0.1 Notations

We basically follow the notations of [12], [8] and [6]. Let X be a complex manifold, $X_{\mathbb{R}}$ the underlying real analytic manifold and $\overline{X}$ the complex conjugate manifold. We set $\mathcal{O}^t_X := \mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_{\mathbb{R}}})$, $\mathcal{O}^w_X := \mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_{\overline{X}}}(\rho_!\mathcal{O}_{\overline{X}}, C^{\infty,w}_{X_{\mathbb{R}}})$. Here $\mathcal{D}b^t_{X_{\mathbb{R}}}$ is a subanalytic sheaf of temperate distributions on $X_{\mathbb{R}}$ and $C^{\infty,w}_{X_{\mathbb{R}}}$ is a subanalytic sheaf of Whitney C^∞-functions.

Let C denote a small abelian category. [15] defined the functor $J : D^b(\mathrm{Ind}(C)) \to (D^b(C))^\wedge$ by setting for $F \in D^b(\mathrm{Ind}(C))$ and $G \in D^b(C)$

$$J(F)(G) = \mathrm{Hom}_{D^b(\mathrm{Ind}(C))}(G, F).$$

We use the category of compact support constructible sheaves $\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X)$ as C in this paper.

2.1 Strongly regular subanalytic sheaves

In this section, we develop the notion of strong regularities of subanalytic sheaves.

Definition 2.1.1 (strongly regular subanalytic sheaves)**.** Let V and Ω be subsets in T^*X. We say $F \in D^b(k_{X_{sa}})$ is strongly regular along V on Ω at $x \in X$ if there exist $F' \in D^b(k_{X_{sa}})$ isomorphic to F in a neighborhood of x, a small and filtrant category I and a functor $I \to D^{[a,b]}_{\mathbb{R}\text{-c}}(k_X)$, $i \mapsto F_i$ such that $J(F') \simeq$ "$\varinjlim_i$"$J(R\rho_* F_i)$ and $\mathrm{SS}(F_i) \cap \Omega \subset V$. If for any $x \in X$ there exists a neighborhood

U of x such that $F \in D^b(k_{X_{sa}})$ is strongly regular along V at $x \in X$ on $\pi^{-1}(U)$ we say $F \in D^b(k_{X_{sa}})$ is strongly regular along V.

Proposition 2.1.2. Let $F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{+1}$ be a distinguished triangle in $D^b(k_{X_{sa}})$. Suppose there exists a distinguished triangle of functors from a small filtrant inductive category I to $D^{[a,b]}_{\mathbb{R}\text{-c}}(k_X)$, $F \mapsto F_i$, $G \mapsto G_i$ and $H \mapsto H_i$,

$$F_i \xrightarrow{f_i} G_i \xrightarrow{g_i} H_i \xrightarrow{+1},$$

such that $J(F) \simeq$ "$\varinjlim_i$"$J(F_i)$, $f \simeq$ "$\varinjlim_i$"f_i, $J(G) \simeq$ "$\varinjlim_i$"$J(G_i)$ and $g \simeq$ "$\varinjlim_i$"g_i. Assume moreover $\mathrm{SS}(F_i) \subset V$ and $\mathrm{SS}(G_i) \subset V$ and the correspondence is functorial with respect to morphisms of triangles, that is for any $i \to j$, $j \to k$ and $j \to k$ the following associated diagram is commutative:

$$\begin{tikzcd}
F_i \arrow[r, "f_i"] \arrow[d] \arrow[dd, bend right] & G_i \arrow[r, "g_i"] \arrow[d] \arrow[dd, bend right] & H_i \arrow[r, "+1"] \arrow[d] \arrow[dd, bend right] & {} \\
F_j \arrow[r, "f_j"] \arrow[d] & G_j \arrow[r, "g_j"] \arrow[d] & H_j \arrow[r, "+1"'] \arrow[d] & {} \\
F_k \arrow[r, "f_k"] & G_k \arrow[r, "f_k"] & H_k \arrow[r, "+1"] & {} .
\end{tikzcd}$$

Then H is strongly regular along V.

Proof. By the triangular inequality, we have $\mathrm{SS}(H_i) \subset V$. Then it remains to prove $J(H) \simeq$ "$\varinjlim_{i\in I}$"$J(H_i)$. By [17, Lemma 2.3.1], this is equivalent to prove

$$H^k(H) \simeq \text{“}\varinjlim_{i\in I}\text{”} H^k(H_i),$$

for any k. Consider the following long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(F) \arrow[r] & H^k(G) \arrow[r] & H^k(H) \arrow[r] & \cdots \\
\cdots \arrow[r] & \text{“}\varinjlim_i\text{”} H^k(F_i) \arrow[r] \arrow[u, "\sim"] & \text{“}\varinjlim_i\text{”} H^k(G_i) \arrow[r] \arrow[u, "\sim"] & \text{“}\varinjlim_i\text{”} H^k(H_i) \arrow[r] \arrow[u, dotted] & \cdots .
\end{tikzcd}$$

By five lemma for each k the dotted arrow is an isomorphism. $\square$

By the following theorem, we understand the solutions to regular $\mathcal{D}$-modules are naturally strongly regular, while Martins [19] proved the regularity in the sense of [14] in the case of temperate holomorphic functions. Let X be a complex manifold and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-

module. We set for short:

$$\mathrm{Sol}^t(\mathcal{M}) := \mathrm{R}\mathscr{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathscr{O}^t_X),$$

$$\mathrm{Sol}^w(\mathcal{M}) := \mathrm{R}\mathscr{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathscr{O}^w_X).$$

Theorem 2.1.3. *If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module regular along an involutive vector subbundle V of T^*X in the sense of Kashiwara-Oshima [11], then* $\mathrm{Sol}^w(\mathcal{M})$ *and* $\mathrm{Sol}^t(\mathcal{M})$ *are strongly regular along V.*

Proof. We may assume $X = Z \times Y$, for some complex manifolds Z and Y, that f is the projection $X \to Y$ and that $V = X \times_Y T^*Y$. By [16, Lemma 3.6] we may assume $\mathcal{M} = \mathcal{D}_{X\to Y}$. Let us prove first for the case of Whitney holomorphic functions. By [22, Corollary A.4.8], we have

$$f^{-1}\mathscr{O}^w_Y \simeq \mathrm{R}\mathscr{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathscr{O}^w_X).$$

There exists a small filtrant inductive system $\{F_i\}_{i\in I} \in \mathrm{C}^{[a,b]}(\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_Y))$ so that $F := \mathscr{O}^w_Y$ is quasi-isomorphic to Q“$\varinjlim_i$”F_i. Then by [14, Theorem 3.1] and [17, Proposition 2.3.2] we obtain $J(f^{-1}F) \simeq$ “$\varinjlim_i$”$J(f^{-1}F_i)$ and one has

$$\mathrm{SS}(f^{-1}F_i) \subset X \times_Y T^*Y, \text{ for all } i \in I.$$

It follows that $f^{-1}F$ is strongly regular along V.

Next we prove the case of $\mathrm{Sol}^t(\mathcal{M})$, the proof is similar. In fact, we may use [22, Corollary A.3.7] in this case. We have

$$f^{-1}\mathscr{O}^t_Y \simeq \mathrm{R}\mathscr{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{D}_{X\to Y}, \mathscr{O}^t_X).$$

And the assertion follows. □

2.2 An estimate of supports and microsupports of multi-microlocalizations for strongly regular subanalytic regular sheaves

In this section, we study an estimation of supports of multi-microlocalizations and its application to $\mathcal{D}$-modules. Let χ be a family of submanifolds $\{M_1, \dots, M_\ell\}$ of a manifold X which satisfies the conditions H1, H2 and H3 of [8].

2.2.1 An estimate of supports

Theorem 2.2.1. *Let V be a subset in T^*X. Assume $F \in D^b(k_{X_{sa}})$ is strongly regular along V. Then*

$$\mu_\chi^{sa}(F)_{C_{\chi^*}(V)\cap S_\chi^*} \xleftarrow{\sim} \mu_\chi^{sa}(F).$$

$$\mathrm{supp}(\rho^{-1}\mu_\chi^{sa}F) \subset C_{\chi^*}(V) \cap S_\chi^*.$$

Proof. Set $Z := C_{\chi^*}(V) \cap S_\chi^*$. By the strong regularity, there exist F' isomorphic to F in a neighborhood U of x, a small and filtrant category I and a functor $I \to D^{[a,b]}_{\mathbb{R}\text{-c}}(k_X), i \mapsto F_i$ such that $J(F') \simeq$ "$\varinjlim_i$"$J(R\rho_* F_i)$ and $\mathrm{SS}(F_i) \cap \Omega \subset V$. By [14, Theorem 3.3],

$$\begin{aligned} J(\mu_\chi^{sa}(F')_Z) &\simeq \text{"}\varinjlim_i\text{"}J(R\rho_*\mu_\chi(F_i) \otimes k_Z) \\ &\simeq \text{"}\varinjlim_i\text{"}J(R\rho_*\mu_\chi(F_i)) \\ &\simeq J(\mu_\chi^{sa}(F')). \end{aligned}$$

The second isomorphism follows from [8, Theorem 3.6]. By [15, Corollary 15.4.4], J is conservative. Thus $\mu_\chi^{sa}(F')_Z \simeq \mu_\chi^{sa}(F')$. Therefore for any $x \in X$, there is a neighborhood U so that

$$\mu_\chi^{sa}(F)_{C_{\chi^*}(V)\cap S_\chi^*} \simeq \mu_\chi^{sa}(F)$$

on $\pi^{-1}(U)$.

Let $p \notin C_{\chi^*}(V) \cap S_\chi^*$. By the strong regularity along V, there exist a small filtrant system $\{F_i\}$ in $C^{[a,b]}(\mathrm{Mod}^c_{\mathbb{R}\text{-c}}(k_X))$ with $\mathrm{SS}(F_i) \subset V$ such that F is quasi-isomorphic to $\varinjlim_i \rho_* F_i$ on a neighborhood W of $\pi(p)$. And also we find a multi-conic open neighborhood U along χ of p such that $C_{\chi^*}(V) \cap U = \emptyset$ and $U \subset \pi^{-1}(W)$. We have:

$$H^k\mu_\chi^{sa}(F_W) \simeq \varinjlim_i \rho_* H^k\mu_\chi F_{iW},$$

for any $k \in \mathbb{Z}$. Then $\mu_\chi^{sa}(F)|_U = 0$ since $\mathrm{supp}(\mu_\chi F_i) \subset C_{\chi^*}(\mathrm{SS}(F_i)) \cap S_\chi^*$ by [8, Theorem 3.6] and $C_{\chi^*}(\mathrm{SS}(F_i)) \subset C_{\chi^*}(V)$. □

Proposition 2.2.2. *Let $G_1, \dots, G_\ell \in D^b_{\mathbb{R}\text{-c}}(k_X)$, $F \in D^b(k_{X_{sa}})$. Assume F is strongly regular along V. Set $Z := C_{\hat{\chi}^*}(\mathrm{SS}(G_1)^a \times \cdots \times \mathrm{SS}(G_\ell)^a \times i_{\Delta\pi}{}^t i_\Delta'^{-1}(V)) \cap S_{\hat{\chi}}^*$, here $i_{\Delta\pi}{}^t i_\Delta'^{-1}(V) = \{(x_1, \dots, x_n; \xi_1, \dots, \xi_n); x_1 = \cdots = x_n, \xi_1 + \cdots + \xi_n \in V\}$ in local coordinates. Then we have*

$$\mu hom_{\hat{\chi}}^{sa}(G_1, \dots, G_\ell; F)_Z \simeq \mu hom_{\hat{\chi}}^{sa}(G_1, \dots, G_\ell; F).$$

Proof. By [14, Theorem 3.3] and [12, Proposition 5.4.1, Proposition 5.4.2, Proposition 5.4.4], $\mathrm{R}\mathcal{H}om(p_2^{-1}(G_1 \boxtimes \cdots \boxtimes G_\ell), p_1^! Ri_{\Delta *}F))$ is strongly regular along $\mathrm{SS}(G_1)^a \times \cdots \times \mathrm{SS}(G_\ell)^a \times i_{\Delta\pi}{}^t i'^{-1}_\Delta(V)$. Thus by Theorem 2.2.1 the result follows. □

Lemma 2.2.3. *Let A be a subset in T^*X. We have:*

$$C_{\chi^*}(A) \cap S^*_\chi \subset \overline{A}^\chi \cap S^*_\chi.$$

Here for a subset D in T^*X we define $(x^{(0)}, 0, \ldots, 0; 0, \xi^{(1)}, \ldots, \xi^{(\ell)}) \in \overline{D}^\chi$ if and only if there exist sequences $\{(t_{1,n}, \ldots, t_{\ell,n})\} \subset (\mathbb{R}^+)^\ell$ and $\{(x_n^{(0)}, \ldots, x_n^{(\ell)}; \xi^{(0)}, \ldots, \xi^{(\ell)})\} \subset D$ such that $(t_{\hat{J}_0,n}x_n^{(0)}, t_{\hat{J}_1^c,k}\xi_k^{(1)}, \ldots, t_{\hat{J}_\ell^c,k}\xi_k^{(\ell)}) \to (x^{(0)}, \xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(\ell)})$ and $t_{j,n} \to +\infty$, $j = 1, \ldots, \ell$.

Proof. By [8, Lemma 1.4], $(x^{(0)}, \ldots, x^{(\ell)}; \xi^{(0)}, \ldots, \xi^{(\ell)}) \in C_{\chi^*}(A)$ if and only if there exist sequences

$$\{(x_k^{(0)}, \ldots, x_k^{(\ell)}; \xi_k^{(0)}, \ldots, \xi_k^{(\ell)})\} \subset A$$

and $(t_{1,k}, \ldots, t_{n,k}) \subset (\mathbb{R}^+)^\ell$ such that

$$(t_{\hat{J}_0,k}x_k^{(0)}, \ldots, t_{\hat{J}_\ell,k}x_k^{(\ell)}; t_{\hat{J}_0^c,k}\xi_k^{(0)}, \ldots, t_{\hat{J}_\ell^c,k}\xi_k^{(\ell)}) \to (x^{(0)}, \ldots, x^{(\ell)}; \xi^{(0)}, \ldots, \xi^{(\ell)}) \tag{2.1}$$

and $t_{1,n}, \ldots, t_{\ell,n} \to \infty$. This gives the result. □

Corollary 2.2.4. *Let $G_1, \ldots, G_\ell \in D^b_{\mathbb{R}\text{-c}}(k_X)$, $F \in D^b(k_{X_{sa}})$. Assume F is strongly regular along V. Suppose that $\mathrm{SS}(G_1) \times \cdots \times \mathrm{SS}(G_\ell) \cap i_{\Delta\pi}{}^t i'^{-1}_\Delta(V) \subset T^*_X X \times \cdots \times T^*_X X$. Then*

$$D'G_1 \otimes \cdots \otimes D'G_\ell \otimes F \xrightarrow{\sim} \mathrm{R}\mathcal{H}om(G_1 \otimes \cdots \otimes G_\ell, F).$$

Proof. By Lemma 2.2.3, by the assumption we have:

$$\begin{aligned} & C_{\hat{\chi}^*}(\mathrm{SS}(G_1)^a \times \cdots \times \mathrm{SS}(G_\ell)^a \times i_{\Delta\pi}{}^t i'^{-1}_\Delta(V)) \cap \dot{S}^*_{\hat{\chi}} \\ \subset \; & \overline{\mathrm{SS}(G_1) \times \cdots \times \mathrm{SS}(G_\ell) \cap i_{\Delta\pi}{}^t i'^{-1}_\Delta(V)}^\chi \cap \dot{S}^*_{\hat{\chi}} \\ = \; & \emptyset. \end{aligned}$$

Finally use the distinguished triangle in [23, Corollary 3.12] and Proposition 2.2.2. □

Corollary 2.2.5. *Assume $F \in D^b(k_{X_{sa}})$ is strongly regular along V. Let U and U' be open multi-conic subsets in S_χ with convex fibers, $U \subset U'$. If $C_{\chi^*}(V) \cap U^{\circ a} \subset U'^{\circ a}$, then*

$$R\Gamma(U'; \nu_\chi(F)) \to R\Gamma(U; \nu_\chi(F))$$

is an isomorphism.

Proof. Set $\gamma := U^{\circ a}, \gamma' := U'^{\circ a}$. By the following commutative diagram

$$\begin{array}{ccc} R\Gamma(U'; \nu_\chi(F)) & \longleftarrow & R\Gamma_{\gamma'}(S^*_\chi; \mu_\chi(F) \otimes or_{M/X}[d]) \\ \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle\alpha} \\ R\Gamma(U; \nu_\chi(F)) & \longleftarrow & R\Gamma_{\gamma}(S^*_\chi; \mu_\chi(F) \otimes or_{M/X}[d]) \end{array} \tag{2.2}$$

with horizontal isomorphisms by [22, Remark 3.2.7, Proposition 3.2.8] and [8, Proposition 2.2], where α and β are morphisms induced by restrictions. Therefore by Proposition 2.2.1, α is an isomorphism. □

2.2.2 Microsupport estimate of multi-microlocalization

By making use of the techniques developed in [14], we may obtain the microsupport estimate of multi-microlocalization. Recall [14, Definition 4.4]: let $\Lambda_i, i \in I$ be a family of closed conic subsets of T^*X, indexed by the objects of a small and filtrant category I. $p \in T^*X$ does not belong to $\lim_i \Lambda_i$ if there exists an open neighborhood U of p and a cofinal subset J of I such that $\Lambda_j \cap U = \emptyset$ for every $j \in J$.

Theorem 2.2.6. *Let $F \in D^b(k_{X_{sa}})$ be strongly regular along V. We have:*

$$\mathrm{SS}(\mu^{sa}_\chi(F)) \subset \mathrm{C}_{\chi^*}(V).$$

In particular,

$$\mathrm{SS}(\rho^{-1}\mu^{sa}_\chi(F)) \subset \mathrm{C}_{\chi^*}(V).$$

Proof. Since F is strongly regular along V, for any $x \in X$, we find F' isomorphic to F in a neighborhood U of x so that we have $J(F') \simeq$ "$\varinjlim_i$"$J(R\rho_* F_i)$ with $\mathrm{SS}(F_i) \cap \pi^{-1}(U) \subset V$. By [14, Theorem 3.3] we have

$$J(\mu^{sa}_\chi(F')) \simeq \text{“}\varinjlim_i\text{”} J(R\rho_*\mu_\chi(F_i)).$$

By [14, (4.1)] and [8, Theorem 3.6.] on $\pi^{-1}U$ we have

$$\begin{aligned} \mathrm{SS}(\mu^{sa}_\chi F') &\subset \lim_i \mathrm{SS}(\mu_\chi(F_i)) \\ &\subset \lim_i \mathrm{C}_{\chi^*}(\mathrm{SS}(F_i)) \subset \mathrm{C}_{\chi^*}(V). \end{aligned}$$

By [19, Proposition 3.13 (ii)], the second assertion follows. □

Corollary 2.2.7. *Let $F \in \mathrm{D}^b(k_{X_{sa}})$ be strongly regular along V. We have:*

$$\mathrm{SS}(\mu_M^{sa}(F)) \subset \mathrm{C}_{T^*_M X}(V).$$

Theorem 2.2.8. *Let W be a subset of T^*Y. Let $f : Y \to X$ be a morphism of analytic manifolds. Assume $F \in D^b(k_{X_{sa}})$ is strongly regular along V in T^*X. If f is non-characteristic for V on W then*

$$\rho^{-1} f^{-1} F \otimes \omega_{Y/X} \to \rho^{-1} f^! F$$

is an isomorphism on W.

Proof. Identify Y with the graph of f in $X \times Y$ and denote by q the projection $T^*_Y(X \times Y) \to Y$. Then by [22, Theorem 5.1.2] we get:

$$\begin{cases} f^{-1} F \otimes \omega_{Y/X} \simeq Rq_{!!}\mu_Y(F \boxtimes \omega_Y), \\ f^! F \simeq Rq_* \mu_Y(F \boxtimes \omega_Y). \end{cases} \tag{2.3}$$

We have the following distinguished triangle:

$$\rho^{-1} f^{-1} F \otimes \omega_{Y/X} \longrightarrow \rho^{-1} f^! F \longrightarrow R\dot{q}_* \rho^{-1} \mu_Y^{sa}(F) \xrightarrow{+1} .$$

And by the assumption we have:

$$\dot{q}_\pi {}^t\dot{q}'^{-1}(\mathrm{C}_{T^*_Y X}(V)) \cap W = \emptyset.$$

Therefore the result follows from Corollary 2.2.7 and [12, Proposition 5.5.4]. □

Corollary 2.2.9. *Let $G_1, \dots, G_\ell \in D^b_{\mathbb{R}\text{-c}}(X)$. Assume $F \in D^b(k_{X_{sa}})$ is strongly regular along V. Then:*

$$\mathrm{SS}(\mu hom_{\hat{\chi}}(G_1, \dots, G_\ell; F)) \subset \mathrm{C}_{\hat{\chi}^*}(\mathrm{SS}(G_1 \boxtimes \cdots \boxtimes G_\ell)^a \times i_{\Delta\pi} {}^t i'^{-1}_\Delta(V)).$$

Moreover,

$$\overset{\ell}{\underset{i=1}{\otimes}} D' G_i \otimes \rho^{-1} F \to \rho^{-1}\, \mathrm{R}\mathscr{H}om(\overset{\ell}{\underset{i=1}{\otimes}} G_i, F)$$

is an isomorphism on $T^*X \setminus R\dot{\pi}_\pi {}^t\dot{\pi}'^{-1}\mathrm{C}_{\hat{\chi}^*}(\mathrm{SS}(G_1 \boxtimes \cdots \boxtimes G_\ell)^a \times i_{\Delta\pi} {}^t i'^{-1}_\Delta(V))$.

Proof. Consider the distinguished triangle

$$\overset{\ell}{\underset{i=1}{\otimes}} D' G_i \otimes \rho^{-1} F \longrightarrow \rho^{-1}\, \mathrm{R}\mathscr{H}om(\overset{\ell}{\underset{i=1}{\otimes}} G_i, F) \longrightarrow H \xrightarrow{+1} ,$$

where $H := R\dot{\pi}_*\mu hom_{\hat{\chi}}(G_1, \dots, G_\ell; F)$. □

2.3 Multi-microlocalizations and inverse images

In this section, our goal is division theorems for regular $\mathcal{D}$-modules. For this purpose we study a theorem of multi-microlocalizations and inverse images which is a generalization of [12, Theorem 6.7.1] in the context of the theory of multi-microlocalizations. Let $f : Y \to X$ be a morphism of manifolds, let $\chi^M = \{M_1, \dots, M_\ell\}$ and $\chi^N = \{N_1, \dots, N_\ell\}$ be two families of closed submanifolds of X and Y respectively which satisfy H1, H2 and H3 with $f(N_j) \subset M_j$, $j = 1, \dots, \ell$. By [6, Proposition 2.5] for $F \in D^b(k_{X_{sa}})$ there exists a commutative diagram of canonical morphisms:

$$\begin{tikzcd} R^t f'_{\chi!}(\omega_{N/M} \otimes f_{\chi\pi}^{-1}\mu_{\chi^M} F) \arrow[r] \arrow[d, "\beta"] & \mu_{\chi^N}(\omega_{Y/X} \otimes f^{-1}F) \arrow[d, "\alpha"] \\ R^t f'_{\chi*} f^!_{\chi\pi}\mu_{\chi^M} F & \mu_{\chi^N}(f^! F). \arrow[l] \end{tikzcd} \tag{2.4}$$

Let us consider $\chi = \{M_1, \dots, M_\ell\}$ is of normal crossing type. $M := \bigcap_{M_i \in \chi} M_i$ ($N := \bigcap_{N_i \in \chi} N_i$). Let $\pi : S^*_\chi \to T^*_M X$ be an identification in an obvious manner:

$$\begin{array}{rccc} \pi: & S^*_\chi & \longrightarrow & T^*_M X \\ & \Cup & & \Cup \\ & (x, \xi_1), (x, \xi_2), \dots, (x, \xi_\ell) & \longmapsto & (x, \xi_1, \dots, \xi_\ell). \end{array} \tag{2.5}$$

We get the following diagram if χ^M and χ^N are of normal crossing type:

$$\begin{tikzcd} S^*_{\chi^N} \arrow[d, "\pi", "\sim"'] & N \times_M S^*_{\chi^M} \arrow[l, "{}^t f'_\chi"'] \arrow[r, "f_{\chi\pi}"] \arrow[d, "\sim"'] & S^*_{\chi^M} \arrow[d, "\pi", "\sim"'] \\ T^*_N Y & N \times_M T^*_M X \arrow[l, "{}^t f'_N"'] \arrow[r, "f_{N\pi}"] & T^*_M X. \end{tikzcd}$$

2.3.1 Classical sheaf case

We first study multi-microlocalizations and inverse images in the case of classical sheaves.

Lemma 2.3.1. Let V be a closed conic subset of T^*X. If

$$V \cap \dot{T}^*_M X = \emptyset,$$

then

$$C_{\chi^{M*}}(V) \cap \dot{S}^*_{\chi^M} = \emptyset.$$

Proof. Assume that there exists a point

$$(x^{(0)}; \xi^{(1)}, \ldots, \xi^{(\ell)}) \in \dot{S}^*_{\chi^M} \cap C_{\chi^{M*}}(V).$$

Then there exist sequences

$$\{(x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(\ell)}; \xi_n^{(1)}, \ldots, \xi_n^{(\ell)})\}_{n=1}^{\infty} \subset V,$$

$$\{(c_n^{(1)}, \ldots, c_n^{(\ell)})\}_{n=1}^{\infty} \subset (\mathbb{R}^+)^{\ell},$$

such that:

$$\begin{cases} \lim\limits_{n\to\infty} c_n^{(j)} = \infty, & (j = 1, \ldots, \ell), \\ \lim\limits_{n\to\infty} (x_n^{(0)}, x_n^{(1)} c_n^{(\hat{J}_1)}, \ldots, x_n^{(\ell)} c_n^{(\hat{J}_\ell)}; \xi_n^{(0)} c_n, \xi_n^{(1)} c_n^{(\hat{J}_1)}, \ldots, \xi_n^{(\ell)} c_n^{(\hat{J}_\ell)}) & \\ \quad = (x^{(0)}, 0, \ldots, 0; 0, \xi^{(1)}, \ldots, \xi^{(\ell)}), & \end{cases}$$

where

$$c_n := \prod_{j=1}^{\ell} c_n^{(j)}, \quad \hat{J}_j := \{1, \ldots, \ell\} \setminus J_j, \quad c_n^{(J)} := \prod_{j\in J} c_n^{(j)} \quad \text{for any } J \subset \{1, \ldots, \ell\}.$$

In particular, we have

$$\lim_{n\to\infty} (x_n^{(1)}, \ldots, x_n^{(\ell)}, \xi_n^{(0)} c_n) = (0, \ldots, 0, 0).$$

Since $(\xi^{(1)}, \ldots, \xi^{(\ell)}) \neq 0$, we may assume that

$$t_n := |(\xi_n^{(0)}, \xi_n^{(1)}, \ldots, \xi_n^{(\ell)})| > 0.$$

We consider the sequence

$$\{(x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(\ell)}; t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \ldots, \xi_n^{(\ell)}))\}_{n=1}^{\infty} \subset V.$$

By extracting a subsequence, we may assume that there exists $\zeta_0 \neq 0$ such that

$$\lim_{n\to\infty} t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \ldots, \xi_n^{(\ell)}) = \zeta_0.$$

Choose $1 \leq k \leq \ell$ with
$$\lim_{n\to\infty} \xi_n^{(k)} c_n^{(\hat{J}_k)} = \xi^{(k)} \neq 0.$$
Since $\lim_{n\to\infty} c_n^{(j)} = \infty$, we have
$$\lim_{n\to\infty} \frac{\xi_n^{(0)}}{\xi_n^{(k)}} = \lim_{n\to\infty} \frac{\xi_n^{(0)} c_n}{\xi_n^{(k)} c_n^{(\hat{J}_k)} c_n^{(J_k)}} = 0.$$

Therefore, we see that $\zeta_0 = (0, \zeta_0^{(1)}, \zeta_0^{(2)}, \cdots, \zeta_0^{(\ell)}) \neq 0$. Hence
$$\lim_{n\to\infty} (x_n^{(0)}, x_n^{(1)}, \ldots, x_n^{(\ell)}; t_n^{-1}(\xi_n^{(0)}, \xi_n^{(1)}, \ldots, \xi_n^{(\ell)})) = (x^{(0)}, 0, \ldots, 0; \zeta_0) \in \dot{T}_M^* X \cap V,$$
which contradicts the non-characteristic condition. □

Theorem 2.3.2. *Let any family of submanifolds be of normal crossing type. Let $F \in D^b(X)$ and V be an open subset of $T_N^* Y$. Assume:*

(i) f is non-characteristic for F on V,

(ii) $S^*_{\chi^N} \cap C_{\chi^{N*}}(f^\sharp_\infty(\mathrm{SS}(F))) \cap \pi^{-1}(V) = \emptyset$,

(iii) $f_{N\pi} : {}^t f_N'^{-1}(V) \to T_M^* X$ is non-characteristic for $C_{T_M^*X}(\mathrm{SS}(F))$,

(iv) $f_{\chi\pi} : {}^t f_\chi'^{-1}(\pi^{-1}(V)) \to S^*_{\chi^M}$ is non-characteristic for $C_{\chi^{M*}}(\mathrm{SS}(F))$,

(v) ${}^t f'^{-1}(V) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset Y \times_X T_M^* X$,

(vi) ${}^t f_\chi'$ is proper on ${}^t f_\chi'^{-1}(\pi^{-1}(V)) \cap f_{\chi\pi}^{-1}(C_{\chi^{M*}}(\mathrm{SS}(F)) \cap S^*_{\chi^M})$.

Then
$$R^t f'_{\chi!}(\omega_{N/M} \otimes f_{\chi\pi}^{-1} \mu_{\chi^M}(F))|_{\pi^{-1}(V)} \to \mu_{\chi^N}(\omega_{Y/X} \otimes f^{-1}F)|_{\pi^{-1}(V)}$$
and
$$\mu_{\chi^N}(f^! F)|_{\pi^{-1}(V)} \to R^t f'_N f^!_{N\pi} \mu_{\chi^M}(F)|_{\pi^{-1}(V)}$$
are isomorphisms.

Proof. It is enough to prove in the case that f is a closed embedding. By [8, Theorem 3.6], [12, Corollary 6.4.4] and the assumption (iv), $\omega_{N/M} \otimes f_{\chi\pi}^{-1} \mu_{\chi_M} F \to f^!_{\chi\pi} \mu_{\chi^M} F$ is an isomorphism on ${}^t f_\chi'^{-1}(\pi^{-1}(V))$. By the assumption (vi), ${}^t f'_\chi$ is proper on ${}^t f_\chi'^{-1}(\pi^{-1}(V)) \cap \mathrm{supp}(f_{\chi\pi}^{-1} \mu_{\chi_M} F)$. Thus the vertical arrow β in (2.4) is an isomorphism on $\pi^{-1}(V)$. By the assumption (ii), the vertical arrow α in (2.4) is an isomorphism on $\pi^{-1}(V)$. In fact there is a following distinguished triangle:
$$\omega_{Y/X} \otimes f^{-1} F \longrightarrow f^! F \longrightarrow G \xrightarrow{+1},$$

where $G := R\dot{q}_*\mu_Y F$ and q is the projection $T^*_Y(X \times Y) \to Y$. Applying μ_{χ^N} we obtain the following distinguished triangle:

$$\mu_{\chi^N}(\omega_{Y/X} \otimes f^{-1}F) \longrightarrow \mu_{\chi^N}(f^!F) \longrightarrow \mu_{\chi^N}(G) \xrightarrow{+1} .$$

Then by [8, Theorem 3.6], [12, Proposition 5.5.4] and the assumption (ii), $\mathrm{supp}(\mu_{\chi^N}(G)) \cap \pi^{-1}(V) \subset S^*_{\chi^N} \cap C_{\chi^{N*}}(\dot{q}_\pi {}^t\dot{q}'^{-1} C_{T^*_Y X}(\mathrm{SS}(F))) \cap \pi^{-1}(V) = S^*_{\chi^N} \cap C_{\chi^{N*}}(f^\sharp_\infty(\mathrm{SS}(F))) \cap \pi^{-1}(V) = \emptyset$.

Therefore, it suffices to prove one of the horizontal arrows in (2.4). Consider following chain of isomorphisms:

$$\begin{aligned} T_\chi f^{-1}\nu_{\chi^M}(F) &\simeq T_\chi f^{-1} s_X^{-1} R\Gamma_{\Omega_X} p^{-1}F \\ &\simeq s_Y^{-1}\tilde{f}'^{-1} R\Gamma_{\Omega_X} p^{-1}F \\ &\simeq s_Y^{-1}\tilde{f}'^{-1} Rj_{X*}\tilde{p}_X^{-1}F \end{aligned}$$

and

$$\begin{aligned} \nu_{\chi^N}(f^{-1}F) &\simeq s_Y^{-1} Rj_{Y*}\tilde{f}^{-1}\tilde{p}_X^{-1}F \\ &\simeq s_Y^{-1}(\tilde{f}'^! Rj_{X*}\tilde{p}_X^{-1}F \otimes \omega^{\otimes -1}_{\tilde{Y}/\tilde{X}}). \end{aligned}$$

Consider a distinguished triangle:

$$\omega_{\tilde{Y}/\tilde{X}} \otimes \tilde{f}'^{-1} Rj_{X*}\tilde{p}_X^{-1}F \longrightarrow \tilde{f}'^! Rj_{X*}\tilde{p}_X^{-1}F \longrightarrow H \xrightarrow{+1} .$$

Then H is supported in S_{χ^N}. And it is enough to prove $\pi^{-1}(V) \cap \mathrm{SS}((s_Y^{-1}H)^\wedge) = \emptyset$. Recall the identification [8, Theorem 3.3]. Thus by the identification (i.e., We perform the proof in the space $T^*S_{\chi^N}$.), it is enough to prove $\pi^{-1}(V) \cap \mathrm{SS}(H) = \emptyset$. By [12, Corollary 6.4.4] it suffices to show:

Claim 2.3.3. for any $q_0 \in \pi^{-1}(V)$, there exists $q \in S_\chi \times_{\tilde{Y}} T^*\tilde{Y}$ such that $q_0 = {}^t s'_Y(q) \in \pi^{-1}(V)$, and $\tilde{f}'$ is non-characteristic for $Rj_{X*}\tilde{p}_X^{-1}F$ at q.

We choose local coordinates systems $(x) = (x^{(0)}, x^{(1)}, \dots, x^{(\ell)})$ on X, $(y) = (y^{(0)}, y^{(1)}, \dots, y^{(\ell)})$ on Y such that $M_k = \{x^i = 0\,(i \in I_k)\}$, $N_k = \{y^i = 0\,(i \in I_k)\}$. We denote by

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}, t_1, \dots, t_\ell; \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}, \tau_1, \dots, \tau_\ell),$$

the coordinates on $\tilde{X}$, by

$$(y^{(0)}, y^{(1)}, \dots, y^{(\ell)}, t_1, \dots, t_\ell; \eta^{(0)}, \eta^{(1)}, \dots, \eta^{(\ell)}, \tau_1, \dots, \tau_\ell),$$

the coordinates on $\tilde{Y}$. We write:

$$f(y) = (g_0(y), g_1(y), \dots, g_\ell(y)),$$

$$\tilde{f}(y, t_1, \dots, t_\ell) = (\tilde{g}_0(y, t_1, \dots, t_\ell), \tilde{g}_1(y, t_1, \dots, t_\ell), \tilde{g}_\ell(y, t_1, \dots, t_\ell), t_1, \dots, t_\ell).$$

Hence:

$$t_{J_k}\tilde{g}_k(y^{(0)}, y^{(1)}, \dots, y^{(\ell)}, t_1, \dots, t_\ell) = g_k(y^{(0)}, t_{J_1}y^{(1)}, \dots, t_{J_\ell}y^{(\ell)}),$$

$$\tilde{g}_0(y^{(0)}, y^{(1)}, \dots, y^{(\ell)}, t_1, \dots, t_\ell) = g_0(y^{(0)}, t_{J_1}y^{(1)}, \dots, t_{J_\ell}y^{(\ell)}).$$

Note that:

$$\begin{aligned} &{}^t f'(y^{(0)}, ty^{(1)}, \dots, ty^{(\ell)}) \cdot (t\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}) \\ = &\begin{pmatrix} \tilde{g}_{0,y^{(0)}}(y,t) \cdot t\xi^{(0)} + \tilde{g}_{1,y^{(0)}}(y,t) \cdot t\xi^{(1)} + \dots + \tilde{g}_{\ell,y^{(0)}}(y,t) \cdot t\xi^{(\ell)} \\ \tilde{g}_{0,y^{(1)}}(y,t) \cdot \xi^{(0)} + \tilde{g}_{1,y^{(1)}}(y,t) \cdot \xi^{(1)} + \dots + \tilde{g}_{\ell,y^{(1)}}(y,t) \cdot \xi^{(\ell)} \\ \dots \\ \tilde{g}_{0,y^{(\ell)}}(y,t) \cdot \xi^{(0)} + \tilde{g}_{1,y^{(\ell)}}(y,t) \cdot \xi^{(1)} + \dots + \tilde{g}_{\ell,y^{(\ell)}}(y,t) \cdot \xi^{(\ell)} \end{pmatrix}. \end{aligned} \tag{2.6}$$

We set $q = (y_0, 0, 0, 0, 0; 0, \eta_1, \dots, \eta_\ell, \tau_1, \dots, \tau_\ell)$, $x_0 = g_0(y_0, 0, 0)$, $q_0 = (y_0; \eta_1, \dots, \eta_\ell) \in \pi^{-1}(V)$. Let us assume $\tilde{f}'$ is characteristic for $Rj_{X*}\tilde{p}_X^{-1}F$ at q. Applying [12, Proposition 6.2.4 (iii) (b)] ($A = T^*_{\tilde{Y}}\tilde{Y}$, $B = \mathrm{SS}(Rj_{X*}\tilde{p}_X^{-1}F)$) we find sequences:

$$\{(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell)\} \subset \tilde{Y}$$

and

$$\{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}, t_n^1, \dots, t_n^\ell; \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}, \tau_n^1, \dots, \tau_n^\ell)\} \subset \mathrm{SS}(Rj_{X*}\tilde{p}_X^{-1}F),$$

such that:

$$\begin{cases} (y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell) \xrightarrow{n} (y_0, 0, 0), \\ (x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}, t_n^1, \dots, t_n^\ell) \xrightarrow{n} (x_0, 0, 0), \end{cases} \tag{2.7}$$

$$\begin{cases} \tau_n^1 + \sum_{k=0}^{\ell} \tilde{g}_{k,t_1}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell) \cdot \xi_n^{(k)} \xrightarrow{n} \tau_1, \\ \dots \\ \tau_n^\ell + \sum_{k=0}^{\ell} \tilde{g}_{k,t_\ell}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell) \cdot \xi_n^{(k)} \xrightarrow{n} \tau_\ell, \end{cases} \tag{2.8}$$

$$\begin{cases} \sum_{k=0}^{\ell} \tilde{g}_{k,y^0}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell) \cdot \xi_n^{(k)} \xrightarrow{n} 0, \\ \sum_{k=0}^{\ell} \tilde{g}_{k,y^1}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell) \cdot \xi_n^{(k)} \xrightarrow{n} \eta_1, \\ \cdots \\ \sum_{k=0}^{\ell} \tilde{g}_{k,y^\ell}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell) \cdot \xi_n^{(k)} \xrightarrow{n} \eta_\ell, \end{cases} \tag{2.9}$$

$$|\xi_n^{(0)}| + \cdots + |\xi_n^{(\ell)}| + |\tau_n^1| + \cdots + |\tau_n^\ell| \xrightarrow{n} \infty, \tag{2.10}$$

$$|(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}, t_n^1, \dots, t_n^\ell) - \tilde{f}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, \hat{t}_n^1, \dots, \hat{t}_n^\ell)| \cdot (\sum_{k=0}^{\ell} |\xi_n^{(k)}| + \sum_{k=1}^{\ell} |\tau_n^k|) \xrightarrow{n} 0. \tag{2.11}$$

By (2.8), (2.10) we obtain:

$$|\xi_n^{(0)}| + |\xi_n^{(1)}| + \cdots + |\xi_n^{(\ell)}| \xrightarrow{n} \infty. \tag{2.12}$$

In fact, assume by contradiction that $|\xi_n^{(0)}|+|\xi_n^{(1)}|+\cdots+|\xi_n^{(\ell)}|$ is bounded. By (2.10), $|\tau_n^1|+\cdots+|\tau_n^\ell| \to +\infty$. We may assume $|\tau_n^k| \to +\infty$ for some $1 \le k \le \ell$. Since $|\xi_n^{(0)}| + |\xi_n^{(1)}| + \cdots + |\xi_n^{(\ell)}|$ is bounded, $|\tau_n^k| \to +\infty$ contradicts to (2.8).

Moreover (2.11) implies in particular:

$$|t_n^j - \hat{t}_n^j| \cdot (|\xi_n^{(0)}| + |\xi_n^{(1)}| + \cdots + |\xi_n^{(\ell)}| + |\tau_n^1| + \cdots + |\tau_n^\ell|) \xrightarrow{n} 0. \tag{2.13}$$

Then by Lemma 2.3.4 below, we may assume $t_n^j > 0$ and

$$(x_n^{(0)}, -t_n x_n^{(1)}, \dots, -t_n x_n^{(\ell)}; t_n \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}) \in \mathrm{SS}(F)$$

(keep in mind of the identification [8, Theorem 3.3]), where $t_n = \prod_{k=1}^{\ell} t_{k,n}$. By (2.6), (2.9) and (2.13) we have:

$$^t f'(y_n^{(0)}, -t_n y_n^{(1)}, \dots, -t_n y_n^{(\ell)}) \cdot (t_n \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}) \xrightarrow{n} (0, \eta_1, \dots, \eta_\ell). \tag{2.14}$$

Since f is non-characteristic for F at $\pi(q_0)$ (hypothesis (i)), this implies:

$$\{|t_n\xi_n^{(0)}| + |\xi_n^{(1)}| + \cdots + |\xi_n^{(\ell)}|\} \text{ is bounded.} \tag{2.15}$$

Thus by (2.12) and (2.15), $|\xi_n^{(0)}| \xrightarrow{n} \infty$. Since the sequence $\{(t_n\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})\}$ is bounded, we may assume after extracting a subsequence so that $\xi_n^{(1)} \xrightarrow{n} \xi_1^{(1)}$, …, $\xi_n^{(\ell)} \xrightarrow{n} \xi_1^{(\ell)}$, $t_n\xi_n^{(0)} \xrightarrow{n} \xi_1^{(0)}$. Then by (2.14)

$$(y^{(0)}, 0, 0; \xi_1^{(0)}, \xi_1^{(1)}, \dots, \xi_1^{(\ell)}) \in {}^t f'^{-1}(V) \cap f_\pi^{-1}(\mathrm{SS}(F)).$$

By the hypothesis (v), we get $\xi_1^{(0)} = 0$. Thus $(t_n\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}) \xrightarrow{n} (0, \xi_1^{(1)}, \dots, \xi_1^{(\ell)})$. We may assume that $\xi_n^{(0)}/|\xi_n^{(0)}|$ has a limit $\xi_2^{(0)}$. We obtain a sequence

$$\{(x_n^{(0)}, -t_n x_n^{(1)}, \dots, -t_n x_n^{(\ell)}; t_n\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})\}$$

in $\mathrm{SS}(F)$ such that:

$$\begin{aligned}
(x_n^{(0)}, -t_n x_n^{(1)}, \dots, -t_n x_n^{(\ell)}) &\xrightarrow{n} (x_0, 0, \dots, 0),\\
(t_n\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}) &\xrightarrow{n} (0, \xi_1^{(1)}, \dots, \xi_1^{(\ell)}),\\
\frac{1}{t_n|\xi_n^{(0)}|}(t_n\xi_n^{(0)}, t_n x_n^{(1)}, \dots, t_n x_n^{(\ell)}) &\xrightarrow{n} (\xi_2^{(0)}, 0, \dots, 0).
\end{aligned}$$

Then $(x_0, 0, \dots, 0; \xi_2^{(0)}, \xi_1^{(1)}, \dots, \xi_1^{(\ell)}) \in C_{T_M^*X}(\mathrm{SS}(F))$, and hypothesis (iii) implies $g_{0,y^0}(y^0, 0, 0) \cdot \xi_2^{(0)} = \tilde{g}_{y^0}(y^0, 0, 0) \cdot \xi_2^{(0)} \neq 0$. By (2.9) we have (recall that $|\xi_n^0| \xrightarrow{n} \infty$):

$$\sum_{k=0}^{\ell} \tilde{g}_{k,y^0}(y_n^{(0)}, y_n^{(1)}, \dots, y_n^{(\ell)}, t_n^1, \dots, t_n^\ell) \cdot \frac{\xi_n^{(k)}}{|\xi_n^{(0)}|} \xrightarrow{n} 0.$$

We get a contradiction, since $|\xi_n^{(1)}|$, $|\xi_n^{(2)}|$, … and $|\xi_n^{(\ell)}|$ are bounded. □

Lemma 2.3.4. Suppose there exists a sequence

$$(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}, t_n^1, \dots, t_n^\ell; \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}, \tau_n^1, \dots, \tau_n^\ell)$$

in $\mathrm{SS}(Rj_{X*}\tilde{p}_X^{-1}F)$ which satisfies the conditions (2.8)–(2.11). Then we find sequences

$$\{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})\} \subset \mathrm{SS}(F)$$

and $\{(t_n^1, \dots, t_n^\ell)\} \subset (\mathbb{R}^+)^\ell$ so that they satisfy the conditions (2.8)–(2.11). Also since $\mathrm{SS}(F)$ is conic, we have

$$(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; t_n\xi_n^{(0)}, t_n\xi_n^{(1)}, \dots, t_n\xi_n^{(\ell)}) \in \mathrm{SS}(F),$$

where $t_n = \prod_{k=1}^{\ell} t_{k,n}$.

Proof. By [12, Theorem 6.3.1] we have

$$\mathrm{SS}(Rj_{\Omega_X *}\tilde{p}_X^{-1}F) \subseteq \mathrm{SS}(\tilde{p}_X^{-1}F)\hat{+}N^*(\Omega_X).$$

By [12, Proposition 5.4.5] we have

$$\mathrm{SS}(\tilde{p}_X^{-1}F) = \{(x_0, \frac{x^{(0)}}{t_{\hat{j}_1}}, \dots, \frac{x^{(\ell)}}{t_{\hat{j}_\ell}}; t_1, t_2, \dots, t_\ell; \xi_0, t_{\hat{j}_1}\xi^{(1)}, \dots, t_{\hat{j}_\ell}\xi^{(\ell)}; \tau_1, \tau_2, \dots, \tau_\ell);$$
$$t_j > 0, (x^{(0)}, x^{(1)}, \dots, x^{(\ell)}; \xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)}) \in \mathrm{SS}(F)\}.$$

Here we do not explicitly calculate τ's because we do not use them. By [12, Remark 6.2.8] and the fact that $N^*(\Omega_X) \subset \{(x; t; 0; \tau)\}$, for each $n \in \mathbb{N}$, we get sequences

$$\{(x_{n,m}^{(0)}, x_{n,m}^{(1)}, \dots, x_{n,m}^{(\ell)}; \xi_{n,m}^{(0)}, \xi_{n,m}^{(1)}, \dots, \xi_{n,m}^{(\ell)})\}_{m=1}^{\infty} \subset \mathrm{SS}(F)$$

$$\{(t_{1,n,m}, t_{2,n,m}, \dots, t_{\ell,n,m})\} \subset (\mathbb{R}^+)^\ell$$

such that

$$(x_{n,m}^{(0)}, \frac{x_{n,m}^{(1)}}{t_{\hat{j}_1,n,m}}, \dots, \frac{x_{n,m}^{(\ell)}}{t_{\hat{j}_\ell,n,m}}; t_{1,n,m}, \dots, t_{\ell,n,m}) \xrightarrow{m} (x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}, t_{1,n}, t_{2,n}, \dots, t_{\ell,n}),$$

$$(\xi_{n,m}^{(0)}, t_{\hat{j}_1,n,m}\xi_{n,m}^{(1)}, \dots, t_{\hat{j}_\ell,n,m}\xi_{n,m}^{(\ell)}) \xrightarrow{m} (\xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)}).$$

Then extracting a subsequence we find sequences

$$\{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; \xi_n^{(0)}, \xi_n^{(1)}, \dots, \xi_n^{(\ell)})\}_{n=1}^{\infty} \subset \mathrm{SS}(F),$$

and $\{(t_{1,n}, t_{2,n}, \dots, t_{\ell,n})\} \subset (\mathbb{R}^+)^\ell$ so that they satisfy desired conditions. Since $\mathrm{SS}(F)$ is conic, we have

$$\{(x_n^{(0)}, x_n^{(1)}, \dots, x_n^{(\ell)}; t_n\xi_n^{(0)}, t_n\xi_n^{(1)}, \dots, t_n\xi_n^{(\ell)})\} \subset \mathrm{SS}(F).$$

□

Remark 2.3.5. By [12, Remark 6.2.6], if we replace the condition (i) of Theorem 2.3.2 with the following condition (1) then we do not need (ii).

(1) f is non-characteristic for F.

2.3.2 Strongly regular subanalytic sheaf case

We study multi-microlocalizations and inverse images in the case of strongly regular subanalytic sheaves.

Theorem 2.3.6. *Let any family of submanifolds be of normal crossing type. Let $F \in D^b(k_{X_{sa}})$ be a strongly regular along a closed conic subset W and V be an open subset of $T^*_N Y$. Assume:*

(i) f is non-characteristic for W,

(ii) $f_{N\pi} : {}^t f_N'^{-1}(V) \to T^*_M X$ is non-characteristic for $C_{T^*_M X}(W)$,

(iii) $f_{\chi\pi} : {}^t f_\chi'^{-1}(\pi^{-1}(V)) \to S^*_{\chi^M}$ is non-characteristic for $C_{\chi^{M*}}(W)$,

(iv) ${}^t f'^{-1}(V) \cap f_\pi^{-1}(W) \subset Y \times_X T^*_M X$,

(v) ${}^t f_\chi'$ is proper on ${}^t f_\chi'^{-1}(\pi^{-1}(V)) \cap f_{\chi\pi}^{-1}(C_{\chi^{M*}}(W) \cap S^*_{\chi^M})$.

Then

$$R^t f'_{\chi!}(\omega_{N/M} \otimes f_{\chi\pi}^{-1} \mu_{\chi^M}(F))|_{\pi^{-1}(V)} \to \mu_{\chi^N}(\omega_{Y/X} \otimes f^{-1}F)|_{\pi^{-1}(V)}$$

and

$$\mu_{\chi^N}(f^! F)|_{\pi^{-1}(V)} \to R^t f'_\chi f^!_{\chi\pi} \mu_{\chi^M}(F)|_{\pi^{-1}(V)}$$

are isomorphisms.

Proof. We may assume f is a closed embedding. By Theorem 2.2.6, [12, Corollary 6.4.4] and the assumption (iii), $\omega_{N/M} \otimes f_{\chi\pi}^{-1} \mu_{\chi^M} F \to f^!_{\chi\pi} \mu_{\chi^M} F$ is an isomorphism on ${}^t f_\chi'^{-1}(\pi^{-1}(V))$.

By the assumption (i), the vertical arrow α in (2.4) is an isomorphism. In fact, by the assumption (i), f is non-characteristic for SS(F). By [22, Proposition 5.3.8], $\omega_{Y/X} \otimes f^{-1}F \to f^! F$ is an isomorphism. Applying μ_{χ^N} we have an isomorphism $\mu_{\chi^N}(\omega_{Y/X} \otimes f^{-1}F) \to \mu_{\chi^N}(f^! F)$.

Thus it is enough to show the first morphism in the theorem is an isomorphism. In particular, it is enough to show

$$R^t f'_{\chi!}(\omega_{N/M} \otimes f_{\chi\pi}^{-1} \mu_{\chi X}(F))|_{\pi^{-1}(V)} \to \mu_{\chi Y}(\omega_{Y/X} \otimes f^{-1}F)|_{\pi^{-1}(V)} \simeq \mu_{\chi Y}(f^! F)|_{\pi^{-1}(V)}$$

is an isomorphism. We have

$$\begin{aligned} T_\chi f^{-1} \nu^{sa}_{\chi X}(F) &\simeq T_\chi f^{-1} s_X^{-1} R\Gamma_{\Omega_X} p_X^{-1} F && (2.16)\\ &\simeq s_Y^{-1} \tilde{f}'^{-1} R\Gamma_{\Omega_X} p_X^{-1} F && (2.17) \end{aligned}$$

and

$$\begin{aligned} \nu^{sa}_{XY}(f^! F) &\simeq s_Y^{-1} R\Gamma_{\Omega_Y} p_Y^{-1} f^! F & (2.18)\\ &\simeq s_Y^{-1} \tilde{f}'^! R\Gamma_{\Omega_Y} p_X^{-1} F. & (2.19)\end{aligned}$$

Since there is a natural morphism $\omega_{\tilde{Y}/\tilde{X}} \otimes \tilde{f}'^{-1}G \to \tilde{f}'^! G$ for G we obtain the following distinguished triangle:

$$\omega_{\tilde{Y}_N/\tilde{X}_M} \otimes \rho^{-1}\tilde{f}'^{-1} R\Gamma_{\Omega_X} p_X^{-1} F \longrightarrow \rho^{-1}\tilde{f}'^! R\Gamma_{\Omega_X} p_X^{-1} F \longrightarrow H \xrightarrow{+1} .$$

Then H is supported in S_χ. And it is enough to prove $\pi^{-1}(V) \cap \mathrm{SS}((s_Y^{-1}H)^\wedge) = \emptyset$. Thus by the identification [8, Theorem 3.3], it is enough to prove $\pi^{-1}(V) \cap \mathrm{SS}(H) = \emptyset$. By Theorem 2.2.8 it is enough to prove:

Claim 2.3.7. For any $q_0 \in V$ which has $q \in S_\chi \times_{\tilde{Y}} T^*\tilde{Y}$ such that $q_0 = {}^t s'_Y(q) \in V$, we have $\tilde{f}'$ is non-characteristic for $\mathrm{SS}(R\Gamma_{\Omega_X} p_X^{-1} F)$ at q.

Using Lemma 2.3.8, the rest of the proof goes along the same lines in Theorem 2.3.2 replacing $\mathrm{SS}(F)$ with W. In fact, if we assume $\tilde{f}'$ is characteristic for $\mathrm{SS}(R\Gamma_{\Omega_X} p_X^{-1} F)$ at q, we get a contradiction. □

Lemma 2.3.8. *Let Ω_X be an open subset in X. $F \in D^b(k_{X_{sa}})$ be strongly regular along V. Then:*

$$\mathrm{SS}(R\Gamma_{\Omega_X} p_X^{-1} F) \subset {}^t\tilde{p}'_X \tilde{p}_{X\pi}^{-1} V \hat{+} N^*(\Omega_X).$$

In particular,

$$\mathrm{SS}(\rho^{-1} R\Gamma_{\Omega_X} p_X^{-1} F) \subset {}^t\tilde{p}'_X \tilde{p}_{X\pi}^{-1} V \hat{+} N^*(\Omega_X).$$

Proof. By the strong regularity we have $J(F) \simeq$ “$\varinjlim_i$”$J(F_i)$ and $\mathrm{SS}(F_i) \subset V$. By [14, Theorem 3.3], $J(R\Gamma_{\Omega_X} p_X^{-1} F) \simeq$ “$\varinjlim_i$”$J(R\Gamma_{\Omega_X} p_X^{-1} F_i)$. By [14, (4.1)] we have

$$\begin{aligned} \mathrm{SS}(R\Gamma_{\Omega_X} p_X^{-1} F) &\subset \varliminf_i \mathrm{SS}(R\Gamma_{\Omega_X} p_X^{-1} F_i) & (2.20)\\ &= \varliminf_i \mathrm{SS}(Rj_{X*} \tilde{p}_X^{-1} F_i) & (2.21)\\ &\subset \varliminf_i N^*(\Omega_X) \hat{+}\, \mathrm{SS}(\tilde{p}_X^{-1} F_i) & (2.22)\\ &\subset N^*(\Omega_X) \hat{+}\, {}^t\tilde{p}'_X \tilde{p}_{X\pi}^{-1} V. & (2.23)\end{aligned}$$

Here (2.22) follows from [12, Theorem 6.3.1]. □

2.4 Applications to $\mathcal{D}$-modules

2.4.1 Solution of $\mathcal{D}$-modules in complex domains with growth conditions.

Recall the following triangle in [23, Theorem 1.8]:

Theorem 2.4.1. *Let $F \in D^b(k_{X_{sa}})$ be an object of the derived category of subanalytic sheaves. We have the following distinguished triangle:*

$$\nu_\chi^{sa}(F) \to \tau^{-1}R\Gamma_M(F) \otimes or_{M/X}[n] \to Rp_{1*}R\Gamma_{P^+}p_2^{-1}\mu_\chi^{sa}(F) \otimes or_{M/X}[n] \xrightarrow{+1} .$$

Here $n = \operatorname{Codim} M$.

Corollary 2.4.2. *Let $F \in D^b(k_{X_{sa}})$. Assume F is strongly regular along V. If $\dot{S}^*_\chi \cap C_{\chi^*}(V) = \emptyset$, then*

$$\nu_\chi^{sa}(F) \xrightarrow{\sim} \tau^{-1}R\Gamma_M(F) \otimes or_{M/X}[n].$$

Proof. By Theorem 2.4.1 and Theorem 2.2.1, the result follows. $\square$

Let $\chi := \{Y_1, Y_2, \ldots, Y_\ell\}$ satisfy the conditions H1, H2 and H3 of [7] and assume each Y_i and $Y := \bigcap Y_i$ are complex submanifolds of X. Let $f : Y \hookrightarrow X$ be the canonical embedding. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We define the inverse image of $\mathcal{M}$ by

$$Df^*\mathcal{M} := \mathcal{O}_Y \overset{L}{\underset{f^{-1}\mathcal{O}_X}{\otimes}} f^{-1}\mathcal{M}.$$

Assume that Y is non-characteristic for $\mathcal{M}$; that is, $T^*_YX \cap Char\mathcal{M} \subset T^*_XX$. Then, it is known that $Df^*\mathcal{M}$ is identified with $Df^*\mathcal{M} := H^0Df^*\mathcal{M}$, and $Df^*\mathcal{M}$ is a coherent $\mathcal{D}_Y$-module.

Theorem 2.4.3. *Assume that Y is non-characteristic for an involutive subbundle V in T^*X and $\mathcal{M}$ is regular coherent $\mathcal{D}_X$-module along V. Then*

$$\begin{aligned}\mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \nu_\chi^{sa}(\mathcal{O}_X^\lambda)) &\simeq \tau^{-1}\,\mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_Y}(\rho_!Df^*\mathcal{M}, \mathcal{O}_Y^\lambda)\\ &\simeq \tau^{-1}f^{-1}\,\mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathcal{O}_X^\lambda),\end{aligned}$$

where $\lambda = w$ or t.

Proof. By the assumption, Y is non-characteristic for $\mathcal{M}$. Recall $\mathrm{SS}(\mathrm{Sol}^\lambda(\mathcal{M})) = Char(\mathcal{M})$ by [21, Corollary 2.2.3]. By [22, Proposition 5.3.8]

$$\tau^{-1}R\Gamma_Y\,\mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathcal{O}_X^\lambda) \otimes \omega_{Y/X}^{\otimes -1} \simeq \tau^{-1}f^{-1}\,\mathrm{R}\mathcal{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathcal{O}_X^\lambda). \tag{2.24}$$

By [21, Theorem 3.1.1] we have:

$$\tau^{-1} f^{-1} \mathrm{R}\mathscr{H}om_{\rho_! \mathcal{D}_X}(\rho_! \mathcal{M}, \mathscr{O}^{\lambda}_X) \simeq \tau^{-1} \mathrm{R}\mathscr{H}om_{\rho_! \mathcal{D}_Y}(\rho_! Df^* \mathcal{M}, \mathscr{O}^{\lambda}_Y). \tag{2.25}$$

Since $\mathrm{Sol}^{\lambda}(\mathcal{M})$ is strongly regular along V by Theorem 2.1.3, it is enough to check $\dot{S}^*_{\chi} \cap C_{\chi^*}(V) = \emptyset$ by Corollary 2.4.2. By Lemma 2.3.1, we have

$$\dot{S}^*_{\chi} \cap C_{\chi*}(V) = \emptyset.$$

Hence we obtain the desired result. □

Corollary 2.4.4. *Assume that Y is non-characteristic for an involutive subbundle V in T^*X and $\mathcal{M}$ is regular coherent $\mathcal{D}_X$-module along V. Then*

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_{\chi}(\mathscr{O}^{\lambda}_X)) &\simeq \tau^{-1} \mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(Df^* \mathcal{M}, \mathscr{O}_Y) & (2.26)\\ &\simeq \tau^{-1} f^{-1} \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathscr{O}_X), & (2.27) \end{aligned}$$

where $\lambda = w$ or t.

Remark 2.4.5. If $\chi = \{M\}$ is a single submanifold then we do not need $\mathcal{D}$-modules $\mathcal{M}$ to be regular to get the assertion of the Corollary thanks to [22, Proposition 5.3.4.]. If we take $\lambda = w$ and χ as a normal crossing divisor, Corollary 2.4.4 asserts that the initial value problem for Majima's strongly asymptotically developable functions is uniquely solvable for a non-characteristic regular $\mathcal{D}$-module $\mathcal{M}$.

2.4.2 The division theorems with growth conditions for regular $\mathcal{D}$-modules

We state division theorems with coefficients of temperate and Whitney multi-microfunctions for regular $\mathcal{D}$-modules. Let $f : Y \hookrightarrow X$ be a closed embedding of complex manifolds. Let $\chi^N = \{N_1, \ldots, N_\ell\}$ be a family of closed submanifolds of Y. Let $\chi^M = \{M_1, \ldots, M_\ell\}$ be a family of closed submanifolds of X. We also assume the conditions below:

1. $f(N_j) \subset M_j$, $j = 1, \ldots, \ell$,
2. $M_j \subseteq M_{j'}$ iff $N_j \subseteq N_{j'}$.

Assume that these families are of normal crossing type. Recall the following diagram:

$$\begin{tikzcd} S^*_{\chi^N} \arrow[d, "\pi", "\sim"'] & N \times_M S^*_{\chi^M} \arrow[l, "{}^t f'_\chi"'] \arrow[r, "f_{\chi\pi}"] \arrow[d, "\sim"'] & S^*_{\chi^M} \arrow[d, "\pi", "\sim"'] \\ T^*_N Y & N \times_M T^*_M X \arrow[l, "{}^t f'_N"'] \arrow[r, "f_{N\pi}"] & T^*_M X, \end{tikzcd}$$

where π is defined in (2.5).

Theorem 2.4.6. *Assume that these families are of normal crossing type. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module regular along an involutive subbundle W. Let V be an open subset of $T^*_N Y$. Assume:*

(i) f *is non-characteristic for* W,

(ii) $f_{N\pi} : {}^t f_N'^{-1}(V) \to T^*_M X$ *is non-characteristic for* $C_{T^*_M X}(W)$,

(iii) $f_{\chi\pi} : {}^t f_\chi'^{-1}(\pi^{-1}(V)) \to S^*_{\chi^M}$ *is non-characteristic for* $C_{\chi^{M*}}(W)$,

(iv) ${}^t f'^{-1}(V) \cap f_\pi^{-1}(W) \subset Y \times_X T^*_M X$,

(v) ${}^t f'_\chi$ *is proper on* ${}^t f_\chi'^{-1}(\pi^{-1}(V)) \cap f_{\chi\pi}^{-1}(C_{\chi^{M*}}(W) \cap S^*_{\chi^M})$.

Then we have:

$$\mathrm{R}^t f'_{\chi *}(f_{\chi\pi}^{-1}\,\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{\chi^M}(\mathscr{O}_X^\lambda)) \otimes or_{S_{\chi^{M*}}})[d] \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mu_{\chi^N}(\mathscr{O}_Y^\lambda) \otimes or_{S_{\chi^{N*}}})$$

on $\pi^{-1}(V)$. *Here* $d := d_X^{\mathbb{C}} - d_Y^{\mathbb{C}} - (d_M - d_N)$, $\lambda = \emptyset, t$ *or* w.

Proof. Let $F := \mathrm{R}\mathscr{H}om_{\rho_!\mathcal{D}_X}(\rho_!\mathcal{M}, \mathscr{O}_X^\lambda)$. By Theorem 2.1.3, F is strongly regular along W. By Theorem 2.3.6 we have

$$R^t f'_{\chi *}(\omega_{N/M} \otimes f_\pi^{-1}\mu_{\chi^M}(F)) \xrightarrow{\sim} \mu_{\chi^N}(\omega_{Y/X} \otimes f^{-1}F)$$

on $\pi^{-1}(V)$. Finally by the assumption (i) and Cauchy-Kowaleskaya-Kashiwara theorem with growth conditions [21, Theorem 3.1.1] we obtain

$$\begin{aligned} \mu_{\chi^N}(f^{-1}F) &\simeq \mu_{\chi^N}(\mathrm{R}\mathscr{H}om_{\rho_!\mathcal{D}_Y}(\rho_! Df^*\mathcal{M}, \mathscr{O}_Y^\lambda)) && (2.28) \\ &\simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mu_{\chi^N}(\mathscr{O}_Y^\lambda)). && (2.29) \end{aligned}$$

□

Remark 2.4.7. If we suppose $\lambda = \emptyset$, χ is a single submanifold and $\mathcal{M}$ is elliptic, one recovers [9, Theorem 1]. If we suppose $\lambda = t$, χ is a single submanifold and $\mathcal{M}$ is elliptic, one recovers [2, Theorem 1].

Definition 2.4.8. We say χ is a simultaneously linearizable family compatible with complexification if there exists a local coordinate transformation such that we can have the situation described in [7, Section 4.3].

Corollary 2.4.9. *Let χ^N be a simultaneously linearizable family compatible with complexification. Assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module regular along an involutive subbundle V and $M = N$. Let W be a open set in $T^*_N Y$. Let Y be non-characteristic for V and assume:*

(i) ${}^t f'^{-1}(W) \cap f_\pi^{-1}(V) \subset Y \times_X T^*_M X$.

For $j <$ $\mathrm{Codim}_{\mathbb{C}} M$, we have

$$R^j \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{\chi^M}(\mathscr{O}^\lambda_X)) \simeq 0$$

on $\pi^{-1}(W)$. Here $\mathrm{Codim}_{\mathbb{C}} M$ is the complex codimension of the maximal complex linear subspace contained in M under the local coordinate system described in Definition 2.4.8.

Proof. Since $N = M$, $f_{N\pi} : N \times_M T^*_M X \to T^*_M X$ and $f_{\chi\pi} : N \times_M S^*_{\chi^M} \to S^*_{\chi^M}$ are isomorphisms.

Note that $N \times_M T^*_M X \subset Y \times_X T^*_Y X$. By the assumption (i), $f_\pi^{-1}(W) \cap N \times_M T^*_M X \subset f_\pi^{-1}(W) \cap Y \times_X T^*_Y X \subset T^*_X X$. By Lemma 2.3.1, we have $C_{\chi^{M*}}(W) \cap \dot{S}^*_{\chi^M} = \emptyset$. Since $\mathrm{supp}(\mu_{\chi^M}(F)) \subset C_{\chi^{M*}}(W) \cap S^*_{\chi^M}$, ${}^t f'_\chi$ is proper on ${}^t f'^{-1}_\chi(\pi^{-1}(V)) \cap \mathrm{supp}(f^{-1}_{\chi\pi}\mu_{\chi^M} F)$.

By Theorem 2.4.6 we have,

$$\mathrm{R}^t f'_* \,\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{\chi^M}(\mathscr{O}^\lambda_X))[d] \simeq \mathrm{R}\mathscr{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mu_{\chi^N}(\mathscr{O}^\lambda_Y)),$$

on $\pi^{-1}(W)$. It is well-known that $H^k(Df^*\mathcal{M}) = 0$ for $k \neq 0$ and $Df^*\mathcal{M}$ is a coherent $\mathcal{D}_Y$-module. From [7, Section 4], we have $H^k(\mu_{\chi^N}(\mathscr{O}^\lambda_Y)) = 0$ for $k < \mathrm{Codim}_{\mathbb{C}} N$. Thus we obtain that $\mathrm{R}^j \mathscr{H}om(Df^*\mathcal{M}, \mu_{\chi^N}(\mathscr{O}^\lambda_Y)) \simeq 0$ for $j < \mathrm{Codim}_{\mathbb{C}} N$. Therefore for any $p \in W$ and for $j < d + \mathrm{Codim}_{\mathbb{C}} N$

$$(\mathrm{R}^{jt} f'_{\chi *} \,\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{\chi^M}(\mathscr{O}^\lambda_X)))_{{}^t f'_\chi(p)} \simeq 0.$$

By the finitenes of ${}^t f'_\chi$ we obtain

$$\mathrm{R}^j \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_{\chi^M}(\mathscr{O}^\lambda_X)) \simeq 0,$$

for $j < d + \mathrm{Codim}_{\mathbb{C}} N$. □

Corollary 2.4.10 (Bochner's tube theorem)**.** *Let χ^M be a simultaneously linearizable family compatible with complexification. Assume $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module regular along an involutive subbundle V. Let Y be non-characteristic for V. Let U be an open multi-conic subset of S_{χ^M}. Let $\tilde{U}$ be a convex hull of U in each fiber. We set $W := S^*_{\chi^M} \setminus U^{\circ a}$ and assume:*

(i) ${}^t f'^{-1}(\pi(W)) \cap f_\pi^{-1}(V) \subset Y \times_X T^*_M X$.

We have:

$$\Gamma(\tilde{U}; \mathrm{R}^0\, \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_{\chi^M}(\mathscr{O}_X^\lambda))) \simeq \Gamma(U; \mathrm{R}^0\, \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}; \nu_{\chi^M}(\mathscr{O}_X^\lambda))).$$

Proof. By the following triangle obtained from [23, Theorem 1.8] we get the result.

$$\mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathscr{O}_X^\lambda)) \longrightarrow \tau^{-1}\, \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \rho^{-1}\mathrm{R}\Gamma_M(\mathscr{O}_X^\lambda)) \otimes \omega_{M/X}^{\otimes -1}$$

$$\longrightarrow Rp^+_{1*}(p_2^+)^{-1}\, \mathrm{R}\mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathscr{O}_X^\lambda)) \otimes \omega_{M/X}^{\otimes -1} \xrightarrow{+1}$$

In fact, by Corollary 2.4.9 we obtain the following exact sequence:

$$0 \to \Gamma(U; \mathrm{R}^0\, \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathscr{O}_X^\lambda))) \longrightarrow \Gamma(M; R^d\, \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \rho^{-1}\mathrm{R}\Gamma_M(\mathscr{O}_X^\lambda)) \otimes or_{M/X})$$

$$\longrightarrow \Gamma(S^*_\chi \setminus U^{\circ a}; \mathrm{R}^d\, \mathscr{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathscr{O}_X^\lambda)) \otimes or_{M/X}).$$

Similarly, we have an exact sequence as above with U replaced with $\tilde{U}$. Since $U^{\circ a} = \tilde{U}^{\circ a}$, by the five lemma the assertion follows. □

Remark 2.4.11. This is a generalization of the work [28, 1, 24].

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