Doctoral Thesis

Lower gradient estimates for viscosity solutions to first-order Hamilton–Jacobi equations

（1階ハミルトン・ヤコビ方程式の粘性解に対する
下からの勾配評価）

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Abstract

This dissertation investigates quantitative lower bounds on the spatial gradient of viscosity solutions to first-order Hamilton–Jacobi equations with convex and Lipschitz continuous Hamiltonians that may also depend on the solution itself.

We establish gradient estimates by employing two distinct approaches. The first approach is PDE-based. By combining inf-convolution regularization with a local comparison principle, we derive an explicit lower bound showing that the gradient magnitude cannot decay faster than exponentially in time. This result improves upon the estimate in [45] by providing a larger numerical bound and extending the validity to a broader domain.

The second approach is dynamical, relying on Herglotz' variational principle for contact Hamiltonian systems, as developed in [19]. By analyzing the evolution of initial gradients along characteristic curves of the contact flow, we obtain a sharper lower bound than that provided by the PDE approach.

In the special case where the Hamiltonian is positively homogeneous and independent of the solution, the dynamical analysis reveals an additional geometric property: the gradient on a given level set depends solely on the initial gradients located near the same level. This insight explains why modified level-set methods in numerical surface evolution algorithms are able to preserve sharp fronts. Several explicit examples, including the transport and eikonal equations, demonstrate the optimality of the proposed bounds.

This dissertation builds upon the author's prior work in [32, 35]. The paper [32] is joint work with the author's supervisor, Professor Nao Hamamuki of Hokkaido University. Both papers are included in this thesis with permission from their respective publishers.

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Contents

Chapter 1

Introduction

1.1 Equation and purpose

In this thesis, we consider the first-order Hamilton–Jacobi equation

$$u_t(x,t) + H(x,t,u(x,t),D_x u(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.1}$$

with the initial condition

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n. \tag{1.2}$$

Here, $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ is the unknown function, and we denote its partial derivatives by $u_t = \partial_t u$ and $D_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$. The Hamiltonian $H : \mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be continuous, and the initial datum $u_0 : \mathbb{R}^n \to \mathbb{R}$ is assumed to be Lipschitz continuous. Throughout this thesis, we work under these regularity assumptions on H and u_0. The Lipschitz constant of u_0 over $\mathbb{R}^n$ is denoted by $\|Du_0\|_{L^\infty(\mathbb{R}^n)}$.

There is abundant literature on upper bounds for the gradient of viscosity solutions. Such bounds have been applied in various contexts, including the existence and regularity of solutions to nonlinear elliptic and parabolic equations, as well as in singular perturbation problems (see [7]). Upper bound estimates can be derived using techniques such as the weak Bernstein method for viscosity solutions [8, 10] or the Ishii–Lions method [39]. For more details, see [46] and references therein.

In contrast, much less is known about lower bounds for the gradient. One reason is that the aforementioned techniques do not apply to deriving such estimates. Furthermore, as discussed in Example 6.10, lower bounds for the gradient generally fail to hold when the Hamiltonian $H = H(x,t,u,p)$ is not convex in the variable p. In such cases, the gradient may vanish instantly, even if the initial gradient is strictly positive. Therefore, it becomes essential to understand how to exploit the convex structure of the Hamiltonian H in order to derive valid lower bounds.

When the Hamiltonian takes the form $H = H(x,t,p)$, lower bounds for the gradient of viscosity solutions have already been studied in [45] (also [32]). These results have been applied to establish the uniqueness of solutions to nonlocal equations, such as those arising in dislocation dynamics [11, 12, 14]. Moreover, it has been shown that fattening of level-sets does not occur, and that level sets retain local Lipschitz regularity [45]. However, to the best of the author's knowledge, no results are currently available concerning lower bounds for the gradient of viscosity solutions to the general form (1.1)–(1.2).

The aim of this thesis is to derive quantitative lower bounds for the spatial gradient of viscosity solutions to (1.1)–(1.2), using two distinct approaches. The first is a PDE-based approach, developed in [45], and the second is a dynamical approach, based on the method introduced in [32]. We compare the resulting estimates and show that, in several aspects, the dynamical approach yields sharper bounds than the PDE-based approach. We also study the special case in which (1.1) corresponds to a level-set equation, which arises in surface evolution problems. In this context, we derive improved gradient estimates that highlight the geometric features of the solution. The gradient estimates established in this thesis not only guarantee Lipschitz regularity of level-sets (see [45, Theorem 5.4]), but also serve as a foundation for reinitialization algorithms in numerical computations (see Remark 4.16).

1.2 Assumptions and known results

We impose the following assumptions on the Hamiltonian H:

(H1) There exist constants $C_1 \geqq 0$ and $\beta \in \{0,1\}$ such that

$$|H(x,t,u,p) - H(y,t,u,p)| \leqq C_1(\beta + |p|)|x-y|$$

for all $(t,u,p) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$ and $x, y \in \mathbb{R}^n$.

(H2) There exist constants $A_2, B_2 \geqq 0$ such that

$$|H(x,t,u,p) - H(x,t,u,q)| \leqq (A_2|x| + B_2)|p-q|$$

for all $(x,t,u) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}$ and $p, q \in \mathbb{R}^n$.

(H3) There exists a constant $K_3 \geqq 0$ such that

$$|H(x,t,u,p) - H(x,t,v,p)| \leqq K_3|u-v|$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$ and $u, v \in \mathbb{R}$.

(H4) For $(x,t,u) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}$, the mapping $p \mapsto H(x,t,u,p)$ is convex in $\mathbb{R}^n$.

Here, $|\cdot|$ denotes the standard Euclidean norm. The assumptions (H1), (H2), and (H4) are consistent with those used in [32, 45]. To ensure the existence and uniqueness of viscosity solutions to (1.1)–(1.2), we further impose the following standard structural conditions (see [1, 37]):

(H5) There exists a constant $\gamma_0 \in \mathbb{R}$ such that

$$H(x,t,u,p) - H(x,t,v,p) \geqq \gamma_0(u-v)$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$ and $u, v \in \mathbb{R}$ with $v \leqq u$.

(H6) The function H is uniformly continuous in $\mathbb{R}^n \times [0,T] \times \mathbb{R} \times \overline{B_R(0)}$ for every $R > 0$.

Here, $B_R(x)$ denotes the open ball of radius R centered at x, and $\overline{B_R(0)}$ is its closure. These conditions are not directly used in the derivation of gradient estimates, but as discussed in Section 4.1, they are necessary when applying Herglotz' variational principle. Moreover, under these assumptions, the unique viscosity solution u is Lipschitz continuous in $\mathbb{R}^n \times [0,T)$.

We also impose the following condition on the initial data u_0:

(U) For a given $x_0 \in \mathbb{R}^n$ and $r > 0$, there exists a constant $\theta > 0$ such that

$$|p| \geqq \theta$$

for all $x \in B_r(x_0)$ and $p \in D^-u_0(x)$.

Here, $D^-u_0(x)$ denotes the subgradients of u_0 at x (see Definition 2.6). If $u_0 \in C^1(\mathbb{R}^n)$, then (U) implies $|Du_0(x)| \geqq \theta$ for all $x \in B_r(x_0)$. For example, the function $u_0(x) := \max\{1-|x|, 0\}$ is not differentiable at $x = 0$, but (U) holds in the viscosity sense since $D^-u_0(0) = \emptyset$ (see (6.9)).

As mentioned earlier, lower bounds for the gradient of viscosity solutions to

$$u_t(x,t) + H(x,t,D_xu(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{1.3}$$

with the initial condition (1.2) have been studied in [45] (also [32]). The result is summarized as follows.

Theorem 1.1 ([45, Theorem 4.2 (i)]). *Assume that the Hamiltonian $H = H(x,t,p)$ satisfies (H1), (H2), (H4), and suppose that the initial data u_0 satisfies (U). Let $u \in C(\mathbb{R}^n \times [0,T))$ be a viscosity solution of* (1.3)*–*(1.2)*. Then, for any $t_0 \in (0,T]$ satisfying*

$$\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 T} t_0 > 0,$$

we have

$$|p| \geqq e^{-\frac{5}{4}C_1 t}\sqrt{\theta^2 - 2\beta C_1 e^{\frac{5}{2}C_1 T} t_0} \tag{1.4}$$

for all $(x,t) \in \mathcal{D}(x_0,r) \cap (\mathbb{R}^n \times (0,t_0))$ and $p \in D_x^- u(x,t)$. Here, β and C_1 are constants from (H1) and

$$\mathcal{D}(x_0,r) := \{(x,t) \in B_r(x_0) \times (0,T) \mid e^{(A_2+B_2+A_2|x_0|)t}(1+|x-x_0|) < r+1\}, \tag{1.5}$$

where A_2 and B_2 are constants from (H2).

The domain $\mathcal{D}(x_0, r)$ is referred to as the *domain of dependence* in [45, Remark 6.1]. In that work, Olivier Ley establishes gradient estimates by employing the notion of Barron–Jensen solutions (see Definition 2.2), which are known to be equivalent to viscosity solutions when the Hamiltonian H is convex. His approach relies on a delicate analysis of inf-convolution approximation.

By contrast, in [32], Nao Hamamuki and the author obtain gradient estimates by employing the results of [2] and by analyzing how initial gradients propagate along solutions to associated Hamiltonian systems (see also [21, Chapter 6]). Their method requires the construction of an appropriate approximation scheme together with precise error estimates. Furthermore, they showed that the lower bounds derived via this dynamical approach are sharper and valid on a larger domain than those obtained in [45]. The principal result of [32] is stated as Corollary 1.5 below.

1.3 PDE-based approach and main result 1

This thesis presents two main results concerning lower bounds for the gradient of viscosity solutions to (1.1)–(1.2), each derived using a different approach.

The first main result is obtained via a PDE-based method, inspired by the approach of [45]. To derive the lower bound, we employ a key lemma (see Corollary 3.5), which states that if u is a Barron–Jensen solution of (1.1)–(1.2), then the inf-convolution u_ε defined in (3.12) is a viscosity subsolution of (1.1) with an explicit error term:

$$u_t(x,t) + H(x,t,u(x,t),D_x u(x,t)) = \frac{\beta C_1}{2} e^{\gamma t}\varepsilon^2,$$

where γ is a suitable constant. To apply this result, we first establish the equivalence between Barron–Jensen solutions and viscosity solutions under the convexity of the Hamiltonian H (see Theorem 2.4). Next, we apply the local comparison principle (see Theorem 3.6) to the subsolution u_ε and the supersolution u, yielding

$$u_\varepsilon(x,t) - u(x,t) \leqq \sup_{y\in\overline{B_r(x_0)}} e^{-K_3 t}(u_\varepsilon - u)(y,0) + e^{-K_3 t}\int_0^t e^{K_3 s}\cdot \frac{\beta C_1}{2} e^{\gamma s}\varepsilon^2\, ds \tag{1.6}$$

for all $(x,t) \in \overline{\mathcal{E}(x_0,r)}$, where

$$\mathcal{E}(x_0,r) := \{(x,t) \in B_r(x_0)\times(0,T) \mid R(x,t) + |x-x_0| < r\}, \tag{1.7}$$

$$R(x,t) := \begin{cases} \left(\dfrac{B_2}{A_2} + |x|\right)(e^{A_2 t} - 1) & \text{if } A_2 > 0, \\ B_2 t & \text{if } A_2 = 0, \end{cases} \tag{1.8}$$

where A_2 and B_2 are constants from (H2). Under the assumption (U), the first term on the right-hand side of (1.6) is estimated from above using the viscosity decreasing principle (see Lemma 3.11), resulting in

$$u_\varepsilon(x,t) - u(x,t) \leqq -\frac{\theta^2}{4} e^{-K_3 t}\varepsilon^2 + \frac{\beta C_1}{2} e^{\gamma t} t\varepsilon^2.$$

On the other hand, from the definition of subgradients and a Taylor expansion argument, there exist a modulus of continuity $\omega \in C([0,\infty))$ with $\lim_{r\to+0}\omega(r) = 0$ and a constant $M > 0$ such that

$$u_\varepsilon(x,t) - u(x,t) \geqq -\frac{|p|^2}{4} e^{\gamma t}\varepsilon^2 - M\varepsilon^2\omega(M\varepsilon^2)$$

for all $(x,t) \in \mathcal{E}(x_0,r)$ and $p \in D_x^- u(x,t)$. By combining these two estimates and taking the limit $\varepsilon \to +0$, we obtain a lower bound of $|p|$. This leads to our first main theorem.

Theorem 1.2 (Gradient estimates I)**.** *Assume that the Hamiltonian H satisfies (H1)–(H4), and suppose that the initial data u_0 satisfies (U). Let $u \in C(\mathbb{R}^n \times [0,T))$ be a viscosity solution of* (1.1)–(1.2)*. Then, for any $t_0 \in (0,T]$ satisfying*

$$\theta^2 e^{-((\frac{\beta}{2}+2)C_1 + 2K_3)t_0} - 2\beta C_1 t_0 > 0,$$

we have

$$|p| \geqq \sqrt{\theta^2 e^{-((\frac{\beta}{2}+2)C_1+2K_3)t} - 2\beta C_1 t} \tag{1.9}$$

for all $(x,t) \in \mathcal{E}(x_0,r)\cap(\mathbb{R}^n\times(0,t_0))$ and $p \in D_x^- u(x,t)$. Here, β, C_1, and K_3 are constants from (H1) and (H3), and $\mathcal{E}(x_0,r)$ is defined in (1.7)*.*

The proof of Theorem 1.2 is presented in Chapter 3. Although analogous arguments can also be used to derive upper gradient bounds, as in [45, Theorem 4.1], we do not investigate such estimates here. We emphasize that, by Theorem A.2, the domain $\mathcal{E}(x_0, r)$ defined in (1.7) is strictly larger than the domain $\mathcal{D}(x_0, r)$ defined in (1.5). Moreover, even in the case $K_3 = 0$, the lower bound in (1.9) is sharper than the one given in (1.4). This improvement is obtained by refining the method of [45] (see also Remark 3.12). For a geometric illustration of the domains $\mathcal{D}(x_0, r)$ and $\mathcal{E}(x_0, r)$ appearing in these estimates, we refer the reader to Appendix A.

1.4 Dynamical approach and main result 2

Our second main result is derived using a dynamical approach, developed in [32]. To estimate the gradient of solutions, we employ Herglotz' variational principle [19, 20, 22], and consider contact Hamiltonian systems (also known as the Lie equation):

$$\begin{cases} \xi'(s) = D_pH(\xi(s), s, u_\xi(s), \eta(s)), \\ \eta'(s) = -D_xH(\xi(s), s, u_\xi(s), \eta(s)) - D_uH(\xi(s), s, u_\xi(s), \eta(s))\eta(s), \\ u'_\xi(s) = \langle \eta(s), \xi'(s)\rangle - H(\xi(s), s, u_\xi(s), \eta(s)), \end{cases} \tag{1.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. To apply the variational principle, we impose additional conditions on the Hamiltonian H:

(H7) $H \in C^2(\mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n)$.

$(\mathrm{H4})_{\mathrm{st}}$ $D_{pp}H(x, t, u, p)$ is positive definite for all $(x, t, u, p) \in \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n$.

Additional conditions, denoted by (A1) and (A2) in Section 4.1, are also required; however these may be omitted when the uniqueness of the solution is ensured (see Remark 4.1). The assumptions (H7) and $(\mathrm{H4})_{\mathrm{st}}$ can be removed by approximating H with H_δ satisfying (H7) and $(\mathrm{H4})_{\mathrm{st}}$, and analyzing the solution u_δ of the approximated equation

$$(u_\delta)_t(x, t) + H_\delta(x, t, u_\delta(x, t), D_xu_\delta(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, T). \tag{1.11}$$

By applying the results of Section 4.2 to u_δ, we derive the gradient estimates for the original solution (1.1). See Section 2.2 and Section 4.3 for details.

Let us illustrate the outline of the idea for deriving gradient estimates. Let (x, t) be a differentiable point of u, and consider the solution (ξ, η, u_ξ) of (1.10) with the terminal conditions

$$\xi(t) = x, \quad \eta(t) = D_xu(x, t), \quad u_\xi(t) = u(x, t).$$

Then, by Proposition 4.6, the function u is differentiable at $(\xi(s), s)$ and satisfies $\eta(s) = D_xu(\xi(s), s)$ for all $s \in (0, t)$. Moreover, we have

$$\eta(0) \in D^-u_0(\xi(0)). \tag{1.12}$$

By integrating the second equation in (1.10) and applying Gronwall's lemma, we obtain the estimate

$$|D_xu(x, t)| \geqq |\eta(0)|e^{-(C_1+K_3)t} - \frac{\beta C_1}{C_1 + K_3}(1 - e^{-(C_1+K_3)t})$$

provided $(C_1, K_3) \neq (0, 0)$ (see Proposition 4.9). Similarly, from the first equation in (1.10), we obtain the estimate

$$|x - \xi(0)| \leqq R(x, t),$$

where $R(x, t)$ is defined in (1.8) (see Proposition 4.11). Combining these inequalities with (1.12) yields the desired lower bound. This is our second main theorem.

Theorem 1.3 (Gradient estimates II)**.** *Assume that the Hamiltonian H satisfies (H1)–(H6). Let $u \in C(\mathbb{R}^n \times [0, T))$ be the viscosity solution of* (1.1)–(1.2)*. Let $(x, t) \in \mathbb{R}^n \times (0, T)$ and $p \in D_x^-u(x, t)$. Then, if $(C_1, K_3) \neq (0, 0)$, we have*

$$\begin{aligned} &\underline{I}(x, t; u_0)e^{-(C_1+K_3)t} - \frac{\beta C_1}{C_1 + K_3}(1 - e^{-(C_1+K_3)t}) \\ &\leqq |p| \leqq \overline{S}(x, t; u_0)e^{(C_1+K_3)t} + \frac{\beta C_1}{C_1 + K_3}(e^{(C_1+K_3)t} - 1), \end{aligned}$$

and if $(C_1, K_3) = (0,0)$, *we have*

$$\underline{I}(x,t;u_0) \leqq |p| \leqq \overline{S}(x,t;u_0).$$

Here, β, C_1, *and* K_3 *are constants from (H1) and (H3), and*

$$\overline{S}(x,t;u_0) := \lim_{\alpha\to+0} \sup \left\{ |p| \;\middle|\; p \in D^- u_0(y),\ y \in \overline{B_{R(x,t)+\alpha}(x)} \right\}, \tag{1.13}$$

$$\underline{I}(x,t;u_0) := \lim_{\alpha\to+0} \inf \left\{ |p| \;\middle|\; p \in D^- u_0(y),\ y \in \overline{B_{R(x,t)+\alpha}(x)} \right\}, \tag{1.14}$$

where $R(x,t)$ *is defined in* (1.8).

Under the additional assumption (U), the following theorem follows immediately from Theorem 1.3.

Theorem 1.4 (Gradient estimates II under (U))**.** *Assume that the Hamiltonian* H *satisfies (H1)–(H6), and suppose that the initial data* u_0 *satisfies (U). Let* $u \in C(\mathbb{R}^n \times [0,T))$ *be the viscosity solution of* (1.1)–(1.2). *Let* $(x,t) \in \mathcal{E}(x_0,r)$ *and* $p \in D_x^- u(x,t)$, *where* $\mathcal{E}(x_0,r)$ *is defined in* (1.7). *Then, if* $(C_1, K_3) \neq (0,0)$, *we have*

$$|p| \geqq \theta e^{-(C_1+K_3)t} - \frac{\beta C_1}{C_1+K_3}(1 - e^{-(C_1+K_3)t}), \tag{1.15}$$

and if $(C_1, K_3) = (0,0)$, *we have*

$$|p| \geqq \theta.$$

The proof of Theorem 1.3 is presented in Chapter 4. In Section 4.4, we compare Theorems 1.2 and 1.4. Although both estimates are valid in the same domain $\mathcal{E}(x_0,r)$, the resulting bounds differ in general. They coincide when $(C_1, K_3) = (0,0)$; however, in general, the lower bound in (1.15) is strictly greater than that in (1.9) (see Theorem 4.14).

Furthermore, when $K_3 = 0$, Theorem 1.3 reduces exactly to Corollary 1.5, which was established in [32].

Corollary 1.5 ([32, Theorems 1.2, 6.2])**.** *Assume that the Hamiltonian* $H = H(x,t,p)$ *satisfies (H1), (H2), (H4), and (H6). Let* $u \in C(\mathbb{R}^n \times [0,T))$ *be the viscosity solution of* (1.3)–(1.2). *Let* $(x,t) \in \mathbb{R}^n \times (0,T)$ *and* $p \in D_x^- u(x,t)$. *Then, we have*

$$\underline{I}(x,t;u_0)e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq \overline{S}(x,t;u_0)e^{C_1 t} + \beta(e^{C_1 t} - 1). \tag{1.16}$$

Here, $\overline{S}(x,t;u_0)$ *and* $\underline{I}(x,t;u_0)$ *are defined in* (1.13) *and* (1.14), *respectively.*

Moreover, suppose that the initial data u_0 *satisfies (U). Then, we have*

$$|p| \geqq \theta e^{-C_1 t} - \beta(1 - e^{-C_1 t})$$

for all $(x,t) \in \mathcal{E}(x_0,r)$ *and* $p \in D_x^- u(x,t)$, *where* $\mathcal{E}(x_0,r)$ *is defined in* (1.7).

Remark 1.6. In [32, Theorem 1.2], the condition $p \in D^- u_0(y)$ in the definitions (1.13) and (1.14) actually corresponds to $p \in D^-_{\mathrm{pr}} u_0(y)$, where $D^-_{\mathrm{pr}} u_0(y)$ denotes the set of proximal subgradients of u_0 at y. Nevertheless, the resulting quantities are the same: replacing $p \in D^- u_0(y)$ and $D^-_{\mathrm{pr}} u_0(y)$ does not change the values of (1.13) and (1.14). See Appendix B for more details.

1.5 Surface evolution problem and main result 3

In Chapter 5, we study the Hamiltonian $H = H(x,t,p)$ that is positively homogeneous of degree 1 with respect to p. More precisely, we assume the following:

(H8) $H(x,t,\mu p) = \mu H(x,t,p)$ for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$ and $\mu \geqq 0$.

The Hamiltonian satisfying (H8) naturally arises in the level-set formulation of surface evolution problems. Under this structure, improved gradient estimates become available.

To explain the results, let us briefly recall the level-set method for surface evolution problems. The reader is referred to [30] for the details. For a given initial hypersurface $\Gamma(0) \subset \mathbb{R}^n$, we consider its evolution $\{\Gamma(t)\}_{t\in[0,T)}$. The motion of the interface is captured by representing each $\Gamma(t)$ as the zero level-set of an auxiliary function $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$:

$$\Gamma(t) = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}. \tag{1.17}$$

Suppose that the evolution law of $\Gamma(t)$ is given by the normal velocity:

$$V = g(x, t, \mathbf{n}) \quad \text{on } \Gamma(t), \tag{1.18}$$

where g is a given function, $\mathbf{n} = \mathbf{n}(x, t) \in \mathbb{R}^n$ is the unit normal vector to $\Gamma(t)$ at x, directed from the set $\{x \in \mathbb{R}^n \mid u(x, t) > 0\}$ to $\{x \in \mathbb{R}^n \mid u(x, t) < 0\}$, and $V = V(x, t) \in \mathbb{R}$ is the normal velocity of $\Gamma(t)$ at x in the direction of $\mathbf{n}$. If u is smooth near (x, t) and $D_x u(x, t) \neq 0$, then

$$\mathbf{n} = -\frac{D_x u(x, t)}{|D_x u(x, t)|}, \quad V = \frac{u_t(x, t)}{|D_x u(x, t)|}.$$

Substituting these into (1.18), we obtain the level-set equation (1.3) with the Hamiltonian

$$H(x, t, p) = -|p| g\left(x, t, -\frac{p}{|p|}\right).$$

This Hamiltonian satisfies (H8) for $p \neq 0$. If it admits a continuous extension to $p = 0$ and satisfies the assumptions (H1), (H2), (H4), and (H6), then our results are applicable.

In practice, the level-set method proceeds as follows:

(i) Choose an initial function u_0 such that $\Gamma(0) = \{x \in \mathbb{R}^n \mid u_0(x) = 0\}$;

(ii) Solve (1.3)–(1.2);

(iii) Obtain the desired surfaces $\Gamma(t)$ given by (1.17) with the solution u.

The resulting evolution $\Gamma(t)$ is called a level-set solution. An important feature of this method is that the level-set solutions are unique: the set $\{x \in \mathbb{R}^n \mid u(x, t) = 0\}$ depends only on the initial level-set $\{x \in \mathbb{R}^n \mid u_0(x) = 0\}$, not on the values of u_0 elsewhere. This naturally leads to the question: Do the gradients of u on the level-set $\{x \in \mathbb{R}^n \mid u(x, t) = 0\}$ depend only on the initial gradients of u_0 on $\{x \in \mathbb{R}^n \mid u_0(x) = 0\}$?

We give a positive answer to this question. In fact, we show that the sub- and superdifferentials $D^{\pm}_x u(x, t)$ at (x, t) where $u(x, t) = 0$ depend only on the values of $D^- u_0(y)$ at nearby points y such that $u_0(y) \approx 0$. (The result actually holds for any level-set $\{x \in \mathbb{R}^n \mid u(x, t) = \gamma\}$ with $\gamma \in \mathbb{R}$.) The key observation is that the solution u is constant along the trajectory $(\xi(s), s)$ for the solution ξ of (1.10). To rigorously justify this, we construct suitable approximations and perform error estimates. This yields our third main theorem.

Theorem 1.7 (Gradient estimates III for a homogeneous Hamiltonian)**.** *Assume that the Hamiltonian $H = H(x, t, p)$ satisfies (H1), (H2), (H4), (H6), (H8), and that*

$$H \text{ is locally Lipschitz continuous in } \mathbb{R}^n \times [0, T] \times \mathbb{R}^n. \tag{1.19}$$

Let $u \in C(\mathbb{R}^n \times [0, T))$ be the viscosity solution of (1.3)–(1.2)*. Let $(x, t) \in \mathbb{R}^n \times (0, T)$ and $p \in D^-_x u(x, t)$, and set $\gamma := u(x, t)$. Then,*

$$\underline{I}(x, t; u_0, \gamma) e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq \overline{S}(x, t; u_0, \gamma) e^{C_1 t} + \beta(e^{C_1 t} - 1).$$

Here, β and C_1 are constants from (H1), and

$$\overline{S}(x, t; u_0, \gamma) := \lim_{\alpha \to +0} \sup \left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_{R(x,t)+\alpha}(x)},\, |u_0(y) - \gamma| \leqq \alpha \right\},$$
$$\underline{I}(x, t; u_0, \gamma) := \lim_{\alpha \to +0} \inf \left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_{R(x,t)+\alpha}(x)},\, |u_0(y) - \gamma| \leqq \alpha \right\},$$

where $R(x, t)$ is defined in (1.8)*.*

Remark 1.8. In [32, Theorem 1.4], the condition $p \in D^- u_0(y)$ in the definitions $\overline{S}(x, t; u_0, \gamma)$ and $\underline{I}(x, t; u_0, \gamma)$ in fact corresponds to $p \in D^-_{\mathrm{pr}} u_0(y)$. Unlike in Remark 1.6, however, the equivalence between these two definitions has not been established. Thus, while the result of [32, Theorem 1.4] provides a stronger formulation, we adopt the present notation in this dissertation for the sake of simplicity. See also Appendix B.

We also show that the techniques used to prove Theorem 1.7 can be extended to m-homogeneous Hamiltonians with $m > 1$. For these extensions, see Theorems 5.4 and 5.6.

One of the typical surface evolution equations is the transport equation $V = \langle \mathbf{X}(x,t), \mathbf{n} \rangle$. Here, $\mathbf{X} : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a given vector field. The corresponding level-set equation is a linear one of the form

$$u_t(x,t) + \langle \mathbf{X}(x,t), D_x u(x,t) \rangle = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Moreover, the eikonal equation $V = \nu(x,t)$ is also a surface evolution equation, where $\nu : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is called a driving force. The level-set equation is given by

$$u_t(x,t) - \nu(x,t)|D_x u(x,t)| = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

In Chapter 6, we present several examples, including the above, and show cases where the lower and upper bounds are attained, confirming the optimality of our estimates. A representative case is when $H(x,t,u,p) = \lambda u + H_0(x,t,p)$, with $\lambda \in \mathbb{R}$ and $H_0 \in C(\mathbb{R}^n \times [0,T] \times \mathbb{R}^n)$ satisfying (H1), (H2), (H4), and (H6). This structure corresponds to a modified level-set equation as studied in [18, 31] (see also Remark 6.4). In such cases, a gradient estimate can be derived explicitly (see Proposition 6.1).

1.6 Literature overview

As mentioned in Section 1.2, lower bounds for the gradient of viscosity solutions to (1.3)–(1.2) have been studied in [32, 45]. Lower bound estimates for other types of equations have also been considered in the literature. In [15], such bounds are obtained for viscosity solutions of second-order geometric equations and applied to prove short-time uniqueness of solutions to geometric equations with nonlocal terms. The arguments rely on geometric properties of the equations and the continuous dependence of solutions. For related results concerning first-order nonlocal equations, see [4, 5, 11, 12, 14]; for second-order nonlocal equations, see [27, 28]; and for results specifically addressing continuous dependence, see [23, 41, 42, 43, 48].

In addition, lower bounds for the spatial Lipschitz norm $\|D_x u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ are investigated in [29, 33]. In [29], optimal-in-time lower bounds are derived for linear second-order uniformly parabolic equations and Hamilton–Jacobi equations with power-type Hamiltonians, assuming the Hölder continuity of the initial data. The proofs are based on explicit representation formulas and logarithmic Sobolev inequalities. More generally, [33] addresses fully nonlinear parabolic equations, constructing viscosity sub- and supersolutions using scaled solutions of modified heat equations, and applying the comparison principle.

For surface evolution problems, modified level-set equations were proposed in [31, 34] to preserve the initial gradient structure on the zero level-sets. Recently, the existence of smooth solutions near the zero level-set, as well as the existence and uniqueness of global viscosity solutions to these modified equations, was shown in [18]. For a comprehensive introduction to the level-set method, see [30].

The notion of a Barron–Jensen solution, introduced in [17], extends the concept of viscosity solutions by requiring the PDE to hold only for test functions that touch the solution from below. It has been shown in [17, 45] that Barron–Jensen solutions coincide with viscosity solutions when the Hamiltonian H is convex. This concept has also been employed for discontinuous viscosity solutions in [9], and extended to quasiconvex Hamiltonians in the context of L^∞-optimal control in [16]. Recently, [40] used Barron–Jensen solutions to prove the existence and uniqueness of extended real-valued solutions with lower semicontinuous initial data (see also [38]).

Two types of variational principles for contact Hamiltonian systems (also called Lie equations) are known in the literature. One is the implicit variational principle, introduced in [50], which connects Weak KAM theory and Aubry–Mather theory to contact Hamiltonian dynamics. See also [51, 52, 53] for developments in this direction. The other is Herglotz' variational principle, used in this thesis, and developed in [19, 20]. This principle provides explicit representations for solutions to contact Hamilton–Jacobi equations and is related to nonholonomic mechanics [47]. Representation formulas using this principle are further studied in [36], and applied to the propagation of singularities in [22]. For classical results on Hamiltonian systems, see [21, Chapter 6].

The remainder of this thesis is structured as follows. In Chapter 2, we collect definitions and known results that will be used throughout the proofs. In Chapter 3, we analyze the properties of the inf-convolution, establish a local comparison principle, and prove our first main result (Theorem 1.2). In Chapter 4, we study Herglotz' variational principle, analyze the properties of solutions to contact Hamiltonian systems, and prove our second main result (Theorem 1.3). We also compare the PDE-based and dynamical approaches. In Chapter 5, we investigate the case where the Hamiltonian is positively homogeneous, which arises in surface evolution problems, and prove the third main result (Theorem 1.7). In Chapter 6, we provide several examples of solutions to (1.1)–(1.2), demonstrating the sharpness of our gradient estimates. In Appendix A, we illustrate and compare the domains $\mathcal{D}(x_0, r)$ and $\mathcal{E}(x_0, r)$, which appear in the main theorems. In Appendix B, we clarify the relationship between the results obtained in this thesis and those presented in [32], emphasizing the role of proximal subgradients and demonstrating the equivalence between the corresponding gradient estimates.

Chapter 2

Preliminaries

In this chapter, we provide definitions, notations, and known results that will be used throughout the thesis. For the fundamental theory of viscosity solutions, see [1, 6, 24, 44, 49].

2.1 Viscosity solution and Barron–Jensen solution

We begin by defining the spaces of semicontinuous functions:

$$\mathrm{USC}(\mathbb{R}^n \times [0,T)) := \{u \mid u \text{ is upper semicontinuous in } \mathbb{R}^n \times [0,T)\},$$
$$\mathrm{LSC}(\mathbb{R}^n \times [0,T)) := \{u \mid u \text{ is lower semicontinuous in } \mathbb{R}^n \times [0,T)\}.$$

Definition 2.1 (Viscosity solution [24]). We say that a function $u \in \mathrm{USC}(\mathbb{R}^n \times [0,T))$ is a *viscosity subsolution* (resp., $u \in \mathrm{LSC}(\mathbb{R}^n \times [0,T))$ is a *viscosity supersolution*) of (1.1)–(1.2) if

- $u(x,0) \leqq u_0(x)$ (resp., $u(x,0) \geqq u_0(x)$) for all $x \in \mathbb{R}^n$, and
- for any $\phi \in C^1(\mathbb{R}^n \times (0,T))$ such that $u - \phi$ attains a local maximum (resp., minimum) at $(x,t) \in \mathbb{R}^n \times (0,T)$, we have

$$\phi_t(x,t) + H(x,t,u(x,t),D_x\phi(x,t)) \leqq 0 \quad (\text{resp., } \geqq 0).$$

We say that a function $u \in C(\mathbb{R}^n \times [0,T))$ is a *viscosity solution* if it is both a viscosity subsolution and a viscosity supersolution.

Definition 2.2 (Barron–Jensen solution [17]). We say that a function $u \in \mathrm{LSC}(\mathbb{R}^n \times [0,T))$ is a *Barron–Jensen solution* of (1.1)–(1.2) if

- the initial condtion (1.2) holds, and
- for any $\phi \in C^1(\mathbb{R}^n \times (0,T))$ such that $u - \phi$ attains a local minimum at $(x,t) \in \mathbb{R}^n \times (0,T)$, we have

$$\phi_t(x,t) + H(x,t,u(x,t),D_x\phi(x,t)) = 0.$$

Remark 2.3. By definition, every Barron–Jensen solution is also a viscosity supersolution.

When the Hamiltonian $H = H(x,t,u,p)$ is convex in p, the notions of viscosity and Barron–Jensen solutions coincide (see [17, Theorem 2.3] and [45, Theorem 3.1]). In [17], H is assumed to be Lipschitz continuous in t; in [45], H is independent of u. We extend the equivalence to our setting under assumption (H3).

Theorem 2.4. *Assume that the Hamiltonian H satisfies (H1), (H3), and (H4). Then, a function $u \in C(\mathbb{R}^n \times [0,T))$ is a viscosity solution of* (1.1)–(1.2) *if and only if it is a Barron–Jensen solution of* (1.1)–(1.2).

The proof of Theorem 2.4 relies on the properties of sup- and inf-convolutions. It is known that the sup-convolution (resp., inf-convolution) of a viscosity subsolution (resp., supersolution) remains a viscosity subsolution (resp., supersolution) of an approximate equation. In particular, Lemma 3.2 (see also [45, Lemma 3.2]) shows that the inf-convolution of a Barron–Jensen solution is a viscosity subsolution of the approximate equation. This property is essential for deriving lower gradient bounds (see Section 3.3).

Remark 2.5. If the Hamiltonian H is not convex in p, then a viscosity solution of (1.1)–(1.2) may not be a Barron–Jensen solution. For counterexamples, see [3, Remark 2.6].

We now introduce some notation and recall properties of generalized gradients of semi-concave functions. For details, see [21, Section 3]. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is *semi-concave* if there exists a constant $C \geqq 0$ such that $f - \frac{C}{2}|\cdot|^2$ is concave.

Definition 2.6. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$.

(1) We define the *subgradients* $D^- f(x)$ of f at x as

$$D^- f(x) := \left\{ D\phi(x) \;\middle|\; \phi \in C^1(\mathbb{R}^n),\ f - \phi \text{ attains a local minimum at } x \right\}.$$

The *supergradients* $D^+ f(x)$ of f at x is defined by replacing "a local minimum" by "a local maximum" in the above definition. Moreover, for a function $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ and point $(z, t) \in \mathbb{R}^n \times (0, T)$, we define

$$D_x^\pm u(z, t) = \{ p \in \mathbb{R}^n \mid (p, \tau) \in D^\pm u(z, t) \}.$$

(2) We define the *reachable gradients* $D^* f(x)$ of f at x as

$$D^* f(x) := \left\{ p \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \text{there exists } \{x_k\}_{k=1}^\infty \subset \mathbb{R}^n \setminus \{x\} \text{ such that} \\ f \text{ is differentiable at } x_k \text{ and} \\ x_k \to x,\ Df(x_k) \to p \text{ as } k \to \infty \end{array} \right\}.$$

Proposition 2.7 ([6, Proposition II.4.7(b)], [21, Proposition 3.3.4, Theorem 3.3.6])**.** *Let f be semi-concave on a nonempty convex set $K \subset \mathbb{R}^n$, and let x be an interior point of K.*

(1) *If f is not differentiable at x, then $D^- f(x) = \emptyset$.*

(2) *$D^* f(x) \neq \emptyset$. If f is differentiable at x, then $D^* f(x) = \{Df(x)\}$.*

2.2 Approximation of the Hamiltonian

When the Hamiltonian H satisfies only (H1)–(H6), we construct an approximation that satisfies both smoothness (H7) and strictly convexity $(\mathrm{H4})_{\mathrm{st}}$. This construction follows [32, Section 2.3].

We begin by extending H in time:

$$H(x, t, u, p) := \begin{cases} H(x, 0, u, p) & \text{if } t < 0, \\ H(x, T, u, p) & \text{if } t > T. \end{cases}$$

For $\delta \in (0, 1]$, we define the approximated Hamiltonian $H_\delta : \mathbb{R}^{2n+2} \to \mathbb{R}$ by

$$H_\delta(x, t, u, p) := (H * \rho_\delta)(x, t, u, p) + k_\delta(p) \quad ((x, t, u, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n), \tag{2.1}$$

where $k_\delta(p) := \delta\sqrt{|p|^2 + 1}$ and

$$(H * \rho_\delta)(x, t, u, p) = (H * \rho_\delta)(z) := \int_{B_\delta(0)} H(z - w)\rho_\delta(w)\, dw.$$

Here, $\rho_\delta : \mathbb{R}^{2n+2} \to \mathbb{R}$ is the standard Friedrichs mollifier such that $\rho_\delta \geqq 0$ in $\mathbb{R}^{2n+2}$ and

$$\mathrm{supp}(\rho_\delta) \subset \overline{B_\delta(0)}, \qquad \int_{B_\delta(0)} \rho_\delta(z)\, dz = 1.$$

Remark 2.8. By properties of mollification and k_δ, we see that H_δ satisfies (H7) and $(\mathrm{H4})_{\mathrm{st}}$. Furthermore, (H1) holds with the same C_1 and (H3) holds with the same K_3. The condition (H2) is also fulfilled by H_δ, but the Lipschitz constant $A_2|x| + B_2$ must be replaced by $A_2|x| + B_2 + \delta$ since $|Dk_\delta(p)| \leqq \delta$.

By construction, H_δ converges to H locally uniformly in $\mathbb{R}^{2n+2}$. Standard stability results for viscosity solutions then imply locally uniform convergence of solutions.

Proposition 2.9 ([24, Remark 6.3])**.** *Assume that the Hamiltonian H satisfies (H1)–(H6). Let H_δ be defined as in* (2.1)*. Let u and u_δ be the viscosity solutions of* (1.1)–(1.2) *and the approximated problem* (1.11)–(1.2)*, respectively. Then, u_δ converges to u locally uniformly in $\mathbb{R}^n \times [0, T)$ as $\delta \to +0$.*

Chapter 3

PDE-based approach

In this chapter, we derive lower bound estimates for viscosity solutions to (1.1)–(1.2) using the method developed in [45]. Specifically, we utilize the concept of Barron–Jensen solutions, the properties of the inf-convolution, and the local comparison principle.

3.1 Properties of the inf-convolution

As discussed in Section 2.1, the inf-convolution plays a crucial role in establishing lower bounds of the gradient. We generalize results of [45, Lemmas 3.1 and 3.2] to (1.1)–(1.2). These results are instrumental in proving Theorems 1.2 and 2.4.

In what follows, for $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and $0 < \rho < r$, we define

$$\begin{aligned} \mathcal{A} &:= B_r(x_0) \times (t_0 - r, t_0 + r), \\ \mathcal{A}_\rho &:= B_{r-\rho}(x_0) \times (t_0 - r + \rho, t_0 + r - \rho), \\ \mathcal{M}_{\mathcal{A}} &:= \left(2 \max_{(y,s)\in\overline{\mathcal{A}}} |u(y, s)| \right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 3.1. *Assume that the Hamiltonian H satisfies (H3) and (H4). Let $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ be a locally Lipschitz continuous function that is a viscosity subsolution (resp., supersolution) of* (1.1)–(1.2)*. Then, u is a viscosity supersolution (resp., subsolution) of*

$$-u_t(x, t) - H(x, t, u(x, t), D_x u(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, T). \tag{3.1}$$

Proof. Fix $(x_0, t_0) \in \mathbb{R}^n \times (0, T)$ and $0 < \rho < r$. Then, u is Lipschitz continuous in $\mathcal{A}$. Let L denote the Lipschitz constant of u in $\mathcal{A}$. By Rademacher's theorem and [6, Proposition II.1.9], we have

$$u_t(y, s) + H(y, s, u(y, s), D_x u(y, s)) \leqq 0 \quad \text{a.e. in } \mathcal{A}. \tag{3.2}$$

Let $\delta \in (0, \frac{\rho}{2})$ and $\rho_\delta \in C^\infty(\mathbb{R}^{n+1})$ be a standard Friedrichs mollifier such that $\rho_\delta \geqq 0$ in $\mathbb{R}^{n+1}$ and

$$\operatorname{supp}(\rho_\delta) \subset \overline{B_\delta(0)} \times [-\delta, \delta], \qquad \int_{B_\delta(0)\times(-\delta,\delta)} \rho_\delta(z, \tau)\, dz d\tau = 1.$$

For $(x, t) \in \mathcal{A}_\rho$, we define the mollification

$$u_\delta(x, t) := (u * \rho_\delta)(x, t) = \int_{\mathbb{R}^n\times\mathbb{R}} u(y, s)\rho_\delta(x - y, t - s)\, dyds.$$

Using assumption (H3) and inequality (3.2), we obtain

$$u_t(y, s) + H(y, s, u_\delta(x, t), D_x u(y, s)) \leqq K_3 |u_\delta(x, t) - u(y, s)| \quad \text{a.e. in } \mathcal{A}. \tag{3.3}$$

Multiplying by $\rho_\delta(x - y, t - s)$ and integrating both sides of (3.3) over $\mathbb{R}^n \times \mathbb{R}$ yields

$$\begin{aligned} (u_\delta)_t(x, t) + \int_{\mathbb{R}^n\times\mathbb{R}} & H(y, s, u_\delta(x, t), D_x u(y, s))\rho_\delta(x - y, t - s)\, dyds \\ &\leqq K_3 \int_{\mathbb{R}^n\times\mathbb{R}} |u_\delta(x, t) - u(y, s)|\rho_\delta(x - y, t - s)\, dyds \quad \text{in } \mathcal{A}_\rho. \end{aligned} \tag{3.4}$$

The right-hand side of (3.4) is computed as follows. By the continuity of u, for any $\varepsilon > 0$, there exists a constant $\delta_0 > 0$ such that

$$|u(x-z,t-\tau) - u(y,s)| < \varepsilon$$

for all $(y,s) \in B_{\delta_0}(x) \times (t-\delta_0, t+\delta_0)$ and $(z,\tau) \in B_{\delta_0}(0) \times (-\delta_0, \delta_0)$. For $\delta \in (0,\delta_0)$, we have

$$\begin{aligned}
&|u_\delta(x,t) - u(y,s)| \\
&= \left| \int_{B_\delta(0)\times(-\delta,\delta)} u(x-z,t-\tau)\rho_\delta(z,\tau)\,dzd\tau - u(y,s)\int_{B_\delta(0)\times(-\delta,\delta)} \rho_\delta(z,\tau)\,dzd\tau \right| \\
&\leqq \int_{B_\delta(0)\times(-\delta,\delta)} |u(x-z,t-\tau) - u(y,s)|\rho_\delta(z,\tau)\,dzd\tau \\
&\leqq \varepsilon \int_{B_\delta(0)\times(-\delta,\delta)} \rho_\delta(z,\tau)\,dzd\tau = \varepsilon
\end{aligned}$$

for all $(x,t) \in \mathcal{A}_\rho$ and $(y,s) \in B_\delta(x) \times (t-\delta, t+\delta)$, and hence

$$K_3 \int_{\mathbb{R}^n\times\mathbb{R}} |u_\delta(x,t) - u(y,s)|\rho_\delta(x-y,t-s)\,dyds \leqq K_3\varepsilon. \tag{3.5}$$

The left-hand side of (3.4) is estimated in the same manner as in [45, Proof of Lemma 3.1]. Specifically, the continuity of H and Jensen's inequality yield

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}} H(y,s,u_\delta(x,t),D_xu(y,s))\rho_\delta(x-y,t-s)\,dyds \\
&\geqq \int_{\mathbb{R}^n\times\mathbb{R}} H(x,t,u_\delta(x,t),D_xu(y,s))\rho_\delta(x-y,t-s)\,dyds - \mu_{\mathcal{A}}(2\delta) \\
&\geqq H(x,t,u_\delta(x,t),D_xu_\delta(x,t)) - \mu_{\mathcal{A}}(2\delta),
\end{aligned} \tag{3.6}$$

where $\mu_{\mathcal{A}}$ is a modulus of continuity of H over $\overline{\mathcal{A}} \times \overline{B_{\frac{\mathcal{M}_{\mathcal{A}}^2}{2}}(0)} \times \overline{B_L(0)}$. Combining (3.4)–(3.6), we obtain

$$-(u_\delta)_t(x,t) - H(x,t,u_\delta(x,t),D_xu_\delta(x,t)) \geqq -K_3\varepsilon - \mu_{\mathcal{A}}(2\delta) \quad \text{in } \mathcal{A}_\rho \tag{3.7}$$

for $\delta \in (0,\delta_0)$. Since $u_\delta \in C^\infty(\mathcal{A}_\rho)$, the inequality (3.7) holds in the classical sense. In particular, u_δ is a viscosity supersolution of

$$-(u_\delta)_t(x,t) - H(x,t,u_\delta(x,t),D_xu_\delta(x,t)) = -K_3\varepsilon - \mu_{\mathcal{A}}(2\delta) \quad \text{in } \mathcal{A}_\rho.$$

Letting $\delta \to +0$ and $\varepsilon \to +0$, and applying stability of viscosity solutions [1, Theorem II.6.2] , we conclude that u is a viscosity supersolution of (3.1). ☐

Lemma 3.2. *Assume that the Hamiltonian H satisfies (H1), (H3), and (H4). Let $u \in C(\mathbb{R}^n \times [0,T))$ be a Barron–Jensen solution of (1.1)–(1.2). Fix $x_0 \in \mathbb{R}^n$ and $0 < \rho < r$. Let $\gamma \geqq (\frac{\beta}{2}+2)C_1 + K_3$, and choose $\alpha \in (0, \frac{\rho}{2\mathcal{M}_{\mathcal{A}}})$ and $\varepsilon \in (0, \frac{e^{\frac{\gamma}{2}T}\rho}{2\mathcal{M}_{\mathcal{A}}})$. Then, the inf-convolution*

$$u_{\varepsilon,\alpha}(x,t) := \inf_{(y,s)\in\overline{\mathcal{A}}} \left\{ u(y,s) + e^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\alpha^2} \right\}$$

is a viscosity subsolution of

$$\begin{aligned}
&u_t(x,t) + H(x,t,u(x,t),D_xu(x,t)) \\
&\qquad = \frac{\beta C_1}{2}e^{\gamma t}\varepsilon^2 + \mu_{\varepsilon,\mathcal{A}}(\mathcal{M}_{\mathcal{A}}\alpha) + \delta_{\varepsilon,\alpha}(t) \quad \text{in } \mathcal{A}_\rho,
\end{aligned}$$

where $\mu_{\varepsilon,\mathcal{A}}$ is a modulus of continuity of H over $\overline{\mathcal{A}} \times \overline{B_{\frac{\mathcal{M}_{\mathcal{A}}^2}{2}}(0)} \times \overline{B_{\frac{2\mathcal{M}_{\mathcal{A}}}{\varepsilon}}(0)}$, and $\delta_{\varepsilon,\alpha}$ satisfies $\delta_{\varepsilon,\alpha}(t) \to 0$ as $\varepsilon,\alpha \to +0$ for every $t \in (0,T)$.

Proof. The proof is divided into two steps.
Step 1: Fix $(\hat{x},\hat{t}) \in \mathcal{A}_\rho$, and let $(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}) \in \overline{\mathcal{A}}$ be a minimizer for the definition of $u_{\varepsilon,\alpha}(\hat{x},\hat{t})$. Let $\phi \in C^1(\mathbb{R}^n \times (0,T))$ such that $u_{\varepsilon,\alpha} - \phi$ attains a minimum at $(\hat{x},\hat{t})$. From [45, Proof of Lemma 3.2], the function

$$\Phi_{\varepsilon,\alpha}(x,t,y,s) := u(y,s) + e^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2} + \frac{|t-s|^2}{\alpha^2} - \phi(x,t)$$

attains a local minimum at $(\hat{x},\hat{t},y_{\varepsilon,\alpha},s_{\varepsilon,\alpha})$. Since u is a Barron–Jensen solution of (1.1)–(1.2), we obtain

$$\phi_t(\hat{x},\hat{t}) + H\left(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, u(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}), D_x\phi(\hat{x},\hat{t})\right) = -\gamma e^{-\gamma\hat{t}}\frac{|\hat{x}-y_{\varepsilon,\alpha}|^2}{\varepsilon^2}. \tag{3.8}$$

By the assumption (H3),

$$\begin{aligned}
&H\left(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, u(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}), D_x\phi(\hat{x},\hat{t})\right)\\
&\geqq H\left(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right) - K_3|u_{\varepsilon,\alpha}(\hat{x},\hat{t}) - u(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha})|\\
&= H\left(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right) - K_3 e^{-\gamma\hat{t}}\frac{|\hat{x}-y_{\varepsilon,\alpha}|^2}{\varepsilon^2} - K_3\frac{|\hat{t}-s_{\varepsilon,\alpha}|^2}{\alpha^2}.
\end{aligned} \tag{3.9}$$

Plugging (3.8) into (3.9), we have

$$\begin{aligned}
&\phi_t(\hat{x},\hat{t}) + H\left(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right)\\
&\qquad \leqq (K_3-\gamma)e^{-\gamma\hat{t}}\frac{|\hat{x}-y_{\varepsilon,\alpha}|^2}{\varepsilon^2} + K_3\frac{|\hat{t}-s_{\varepsilon,\alpha}|^2}{\alpha^2}.
\end{aligned} \tag{3.10}$$

The left-hand side of (3.10) is estimated in the same manner as in [45, Proof of Lemma 3.2]. Specifically, using the continuity of H, the assumption (H1), and Schwarz's inequality, we obtain

$$\begin{aligned}
&H\left(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right)\\
&\geqq H\left(y_{\varepsilon,\alpha}, \hat{t}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right) - \mu_{\varepsilon,\mathcal{A}}(\mathcal{M}_\mathcal{A}\alpha)\\
&\geqq H\left(\hat{x}, \hat{t}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right) - \mu_{\varepsilon,\mathcal{A}}(\mathcal{M}_\mathcal{A}\alpha) - \frac{\beta C_1}{2}e^{\gamma\hat{t}}\varepsilon^2\\
&\qquad - \frac{\beta C_1}{2}e^{-\gamma\hat{t}}\frac{|\hat{x}-y_{\varepsilon,\alpha}|^2}{\varepsilon^2} - 2C_1 e^{-\gamma\hat{t}}\frac{|\hat{x}-y_{\varepsilon,\alpha}|^2}{\varepsilon^2},
\end{aligned} \tag{3.11}$$

where $\mu_{\varepsilon,\mathcal{A}}$ is a modulus of continuity of H over $\overline{\mathcal{A}} \times \overline{B_{\frac{\mathcal{M}_\mathcal{A}^2}{2}}(0)} \times \overline{B_{\frac{2\mathcal{M}_\mathcal{A}}{\varepsilon}}(0)}$. Combining (3.10) and (3.11), we have

$$\begin{aligned}
&\phi_t(\hat{x},\hat{t}) + H\left(\hat{x}, \hat{t}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right)\\
&\qquad \leqq \left(\frac{\beta C_1}{2} + 2C_1 + K_3 - \gamma\right)e^{-\gamma\hat{t}}\frac{|\hat{x}-y_{\varepsilon,\alpha}|^2}{\varepsilon^2}\\
&\qquad\qquad + \frac{\beta C_1}{2}e^{\gamma\hat{t}}\varepsilon^2 + \mu_{\varepsilon,\mathcal{A}}(\mathcal{M}_\mathcal{A}\alpha) + K_3\frac{|\hat{t}-s_{\varepsilon,\alpha}|^2}{\alpha^2}.
\end{aligned}$$

We define $\delta_{\varepsilon,\alpha}(t) := K_3\frac{|t-s_{\varepsilon,\alpha}|^2}{\alpha^2}$. By the classical result for inf-convolution, we have $\delta_{\varepsilon,\alpha}(t) \to 0$ as $\varepsilon,\alpha \to +0$ for every $t \in (0,T)$. Therefore, if $\gamma \geqq (\frac{\beta}{2}+2)C_1 + K_3$, we have

$$\begin{aligned}
&\phi_t(\hat{x},\hat{t}) + H\left(\hat{x}, \hat{t}, u_{\varepsilon,\alpha}(\hat{x},\hat{t}), D_x\phi(\hat{x},\hat{t})\right)\\
&\qquad \leqq \frac{\beta C_1}{2}e^{\gamma\hat{t}}\varepsilon^2 + \mu_{\varepsilon,\mathcal{A}}(\mathcal{M}_\mathcal{A}\alpha) + \delta_{\varepsilon,\alpha}(\hat{t}).
\end{aligned}$$

Step 2: Since $u_{\varepsilon,\alpha}$ is Lipschitz continuous in $\mathcal{A}_\rho$, Lemma 3.1 implies that $u_{\varepsilon,\alpha}$ is a viscosity subsolution of

$$\begin{aligned}
&u_t(x,t) + H(x,t,u(x,t),D_xu(x,t))\\
&\qquad = \frac{\beta C_1}{2}e^{\gamma t}\varepsilon^2 + \mu_{\varepsilon,\mathcal{A}}(\mathcal{M}_\mathcal{A}\alpha) + \delta_{\varepsilon,\alpha}(t) \quad \text{in } \mathcal{A}_\rho,
\end{aligned}$$

which concludes the proof. □

Remark 3.3. If $H = H(x,t,u,p)$ is linear in u, as in (6.1), Lemmas 3.1 and 3.2 follow immediately from [45, Lemmas 3.1 and 3.2]. In the general nonlinear case, however, additional care is required: one must replace $u(y,s)$ with $u_\delta(x,t)$, or $u(y_{\varepsilon,\alpha}, s_{\varepsilon,\alpha})$ with $u_{\varepsilon,\alpha}(\hat{x},\hat{t})$, as illustrated in inequalities (3.3) and (3.10). It should also be noted that, in [45, Proof of Lemma 3.1], the right-hand side of (3.4) vanishes.

Remark 3.4. Analogously to [45, Proof of Theorem 3.1], Theorem 2.4 can be established by means of Lemmas 3.1 and 3.2.

The following result is a direct consequence of Lemma 3.2. We will apply this corollary in the proof of Theorem 1.2. For $x_0 \in \mathbb{R}^n, r > 0$, we define

$$\mathcal{M}_r := \left(2 \max_{(y,s)\in\overline{B_r(x_0)}\times[0,T]} |u(y,s)|\right)^{\frac{1}{2}}.$$

Corollary 3.5. *Assume that the Hamiltonian H satisfies (H1), (H3), and (H4). Let $u \in C(\mathbb{R}^n \times [0,T))$ be a Barron–Jensen solution of (1.1)–(1.2). Fix $x_0 \in \mathbb{R}^n$ and $0 < \rho < r$. Let $\gamma \geqq (\frac{\beta}{2}+2)C_1 + K_3$ and choose $\alpha \in (0, \frac{\rho}{2\mathcal{M}_r})$ and $\varepsilon \in (0, \frac{e^{\frac{\gamma}{2}T}\rho}{2\mathcal{M}_r})$. Then, the inf-convolution*

$$u_\varepsilon(x,t) := \inf_{y\in\overline{B_r(x_0)}} \left\{u(y,t) + e^{-\gamma t}\frac{|x-y|^2}{\varepsilon^2}\right\} \tag{3.12}$$

is a viscosity subsolution of

$$u_t(x,t) + H(x,t,u(x,t),D_x u(x,t)) = \frac{\beta C_1}{2} e^{\gamma t}\varepsilon^2 \quad \text{in } B_{r-\rho}(x_0)\times(0,T).$$

3.2 Local comparison principle

In this section, we establish a local comparison principle that extends [45, Theorem 6.1]. More precisely, we identify a larger domain on which the estimate (3.16) remains valid. This estimate plays a crucial role in deriving the main result stated in Theorem 1.2.

In the proof, we restrict our attention to the difference between the subsolution u_ε, defined in (3.12), and the supersolution u. Since $u_\varepsilon \leqq u$ in $\mathbb{R}^n \times (0,T)$, we may, without loss of generality, assume that a subsolution does not exceed the corresponding supersolution.

Theorem 3.6 (Local comparison principle)**.** *Assume that the Hamiltonian H satisfies (H1)–(H3). Fix $x_0 \in \mathbb{R}^n$ and $r > 0$. Let $f, g \in C(\mathbb{R}^n \times [0,T))$. Suppose that a function $u \in C(\mathbb{R}^n \times (0,T))$ is a viscosity subsolution of*

$$u_t(x,t) + H(x,t,u(x,t),D_x u(x,t)) = f(x,t) \quad \text{in } B_r(x_0)\times(0,T), \tag{3.13}$$

and a function $v \in C(\mathbb{R}^n \times (0,T))$ is a viscosity supersolution of

$$v_t(x,t) + H(x,t,v(x,t),D_x v(x,t)) = g(x,t) \quad \text{in } B_r(x_0)\times(0,T). \tag{3.14}$$

Additionally, we assume

$$u(x,t) \leqq v(x,t) \quad \text{in } B_r(x_0)\times(0,T). \tag{3.15}$$

Then, we have

$$\begin{aligned}(u-v)(x,t) \leqq{} & \sup_{y\in\overline{B_r(x_0)}} e^{-K_3 t}(u-v)(y,0) \\ & + e^{-K_3 t}\int_0^t e^{K_3 s} \sup_{y\in\overline{B_r(x_0)}} (f-g)(y,s)\,ds\end{aligned} \tag{3.16}$$

for all $(x,t) \in \overline{\mathcal{E}(x_0,r)}$, where $\mathcal{E}(x_0,r)$ is defined in (1.7).

The proof relies on the following auxiliary lemma, analogous to [45, Lemma 6.1].

Lemma 3.7. *Under the settings in Theorem 3.6, a function $w := u - v$ is a viscosity subsolution of*

$$\begin{aligned}w_t(x,t) + K_3 w(x,t) - (A_2|x| + B_2)|D_x w(x,t)| = (f-g)(x,t) \\ \text{in } B_r(x_0)\times(0,T).\end{aligned} \tag{3.17}$$

Proof. Fix $(\hat{x},\hat{t}) \in B_r(x_0) \times (0,T)$. Let $\phi \in C^1(B_r(x_0) \times (0,T))$ such that $w-\phi$ atains a strict local maximum at $(\hat{x},\hat{t})$. We may assume that $w-\phi$ has a strict maximum at $(\hat{x},\hat{t})$ over $\overline{B_\tau(\hat{x})} \times [\hat{t}-\tau,\hat{t}+\tau]$ for small $\tau>0$. From [45, Proof of Lemma 6.1], the function

$$\Phi_{\varepsilon,\alpha}(x,t,y,s) := u(x,t) - v(y,s) - \frac{|x-y|^2}{\varepsilon^2} - \frac{|t-s|^2}{\alpha^2} - \phi(x,t).$$

attains a local maximum at some point $(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, s_{\varepsilon,\alpha})$ over $(B_\tau(\hat{x}))^2 \times (\hat{t}-\tau,\hat{t}+\tau)^2$ for sufficiently small $\varepsilon,\alpha>0$. Moreover, there exist $x_\varepsilon, y_\varepsilon \in B_r(x_0)$ and $t_\varepsilon \in (0,T)$ such that $(x_{\varepsilon,\alpha}, y_{\varepsilon,\alpha}, t_{\varepsilon,\alpha}, s_{\varepsilon,\alpha}) \to (x_\varepsilon, y_\varepsilon, t_\varepsilon, t_\varepsilon)$ as $\alpha \to +0$. Since u is a viscosity subsolution of (3.13) and v is a viscosity supersolution of (3.14), letting $\alpha \to +0$, we have

$$\begin{aligned}\phi_t(x_\varepsilon,t_\varepsilon) &+ H\left(x_\varepsilon, t_\varepsilon, u(x_\varepsilon,t_\varepsilon), \frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2} + D_x\phi(x_\varepsilon,t_\varepsilon)\right) \\ &- H\left(y_\varepsilon, t_\varepsilon, v(y_\varepsilon,t_\varepsilon), \frac{2(x_\varepsilon-y_\varepsilon)}{\varepsilon^2}\right) \leqq f(x_\varepsilon,t_\varepsilon) - g(y_\varepsilon,t_\varepsilon).\end{aligned}$$

Similar to the proof of Lemma 3.2, using assumptions (H1)–(H3), we obtain

$$\begin{aligned}\phi_t(x_\varepsilon,t_\varepsilon) &- K_3|u(x_\varepsilon,t_\varepsilon) - v(y_\varepsilon,t_\varepsilon)| - (A_2|x_\varepsilon| + B_2)|D_x\phi(x_\varepsilon,t_\varepsilon)| \\ &\leqq f(x_\varepsilon,t_\varepsilon) - g(y_\varepsilon,t_\varepsilon) + \beta C_1|x_\varepsilon - y_\varepsilon| + 2C_1\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2}.\end{aligned}$$

Since $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (\hat{x},\hat{x},\hat{t})$ and $\frac{|x_\varepsilon-y_\varepsilon|^2}{\varepsilon^2} \to +0$ as $\varepsilon \to +0$, we have

$$\phi_t(\hat{x},\hat{t}) - K_3|w(\hat{x},\hat{t})| - (A_2|\hat{x}| + B_2)|D_x\phi(\hat{x},\hat{t})| \leqq (f-g)(\hat{x},\hat{t}),$$

which implies that w is a viscosity subsolution of (3.17) since $w \leqq 0$. □

Proof of Theorem 3.6. By Lemma 3.7, the function $w = u - v$ is a viscosity subsolution of (3.17). Moreover, by [6, Proposition II.2.5], the function $W(x,t) := e^{K_3 t} w(x,t)$ is a viscosity subsolution of

$$\begin{aligned}W_t(x,t) - (A_2|x| + B_2)|D_xW(x,t)| &= e^{K_3 t}(f-g)(x,t) \\ &\text{in } B_r(x_0) \times (0,T)\end{aligned} \tag{3.18}$$

Fix $0<\varepsilon<r$. Let $h_\varepsilon \in C^1(B_r(x_0)\times(0,T))$ be a nonnegative function to be specified below, and let $\chi_\varepsilon : [0,\infty) \to [0,\infty]$ be a nonnegative, increasing function such that $\chi_\varepsilon \in C^1((0,r))$ and

$$\chi_\varepsilon \equiv 0 \text{ on } [0, r-\varepsilon], \quad \lim_{z\to r}\chi_\varepsilon(z) = \infty, \quad \chi_\varepsilon \equiv \infty \text{ on } [r,\infty).$$

Following the method in [45, Proof of Theorem 6.1], for each $\eta>0$, the function

$$\begin{aligned}\Psi_{\varepsilon,\eta}(x,t) := W(x,t) - \eta t &- \frac{\eta}{T-t} \\ &- \int_0^t e^{K_3 s} \sup_{y\in\overline{B_r(x_0)}} (f-g)(y,s)\,ds - \chi_\varepsilon(h_\varepsilon(x,t))\end{aligned}$$

attains a maximum at some $(x_{\varepsilon,\eta}, t_{\varepsilon,\eta}) \in B_r(x_0) \times [0,T)$.

Assume, for contradiction, that $t_{\varepsilon,\eta} > 0$. Since W is a viscosity subsolution of (3.18), we obtain

$$\begin{aligned}\eta + \frac{\eta}{(T-t_{\varepsilon,\eta})^2} &+ e^{K_3 t_{\varepsilon,\eta}}\left\{\sup_{y\in\overline{B_r(x_0)}}(f-g)(y,t_{\varepsilon,\eta}) - (f-g)(x_{\varepsilon,\eta},t_{\varepsilon,\eta})\right\} \\ &+ \chi_\varepsilon'(h_\varepsilon(x_{\varepsilon,\eta},t_{\varepsilon,\eta}))\left\{(h_\varepsilon)_t(x_{\varepsilon,\eta},t_{\varepsilon,\eta}) - (A_2|x_{\varepsilon,\eta}| + B_2)|D_xh_\varepsilon(x_{\varepsilon,\eta},t_{\varepsilon,\eta})|\right\} \leqq 0.\end{aligned}$$

This leads to a contradiction if the function h_ε satisfies

$$(h_\varepsilon)_t(x_{\varepsilon,\eta},t_{\varepsilon,\eta}) - (A_2|x_{\varepsilon,\eta}| + B_2)|D_xh_\varepsilon(x_{\varepsilon,\eta},t_{\varepsilon,\eta})| \geqq 0. \tag{3.19}$$

We now define

$$h_\varepsilon(x,t) := \begin{cases} B_2 t + \sqrt{|x-x_0|^2+\varepsilon^2} & \text{if } A_2 = 0, \\ \left(\dfrac{B_2}{A_2} + \sqrt{|x|^2+\varepsilon^2}\right)(e^{A_2 t}-1) + \sqrt{|x-x_0|^2+\varepsilon^2} & \text{if } A_2 \neq 0. \end{cases} \tag{3.20}$$

It can be verified (see Remark 3.9) that $h_\varepsilon \in C^1(B_r(x_0) \times (0,T))$, $h_\varepsilon \geqq 0$ in $B_r(x_0) \times (0,T)$, and satisfies (3.19). Hence, the assumption $t_{\varepsilon,\eta} > 0$ leads to a contradiction, and we must have $t_{\varepsilon,\eta} = 0$. Consequently, we obtain

$$\Psi_\varepsilon(x,t) \leqq \Psi_\varepsilon(x_{\varepsilon,\eta}, 0) \leqq W(x_{\varepsilon,\eta}, 0) \leqq \sup_{y \in \overline{B_r(x_0)}} (u-v)(y,0).$$

This implies the inequality

$$\begin{aligned} &W(x,t) - \eta t - \frac{\eta}{T-t} \\ &\leqq \sup_{y \in \overline{B_r(x_0)}} (u-v)(y,0) + \int_0^t e^{K_3 s} \sup_{y \in \overline{B_r(x_0)}} (f-g)(y,s)\, ds + \chi_\varepsilon(h_\varepsilon(x,t)) \end{aligned} \tag{3.21}$$

for all $(x,t) \in B_r(x_0) \times (0,T)$. Now, if $h_\varepsilon(x,t) \leqq r - \varepsilon$, then $\chi_\varepsilon(h_\varepsilon(x,t)) = 0$. Taking the limit $\varepsilon \to +0$ for $h_\varepsilon(x,t) \leqq r - \varepsilon$, this yields the condition

$$R(x,t) + |x - x_0| \leqq r,$$

where $R(x,t)$ is defined in (1.8). This condition is equivalent to $(x,t) \in \mathcal{E}(x_0, r)$, the domain defined in (1.7).

Finally, letting $\eta \to +0$ in (3.21), we obtain

$$e^{K_3 t}(u-v)(x,t) \leqq \sup_{y \in \overline{B_r(x_0)}} (u-v)(y,0) + \int_0^t e^{K_3 s} \sup_{y \in \overline{B_r(x_0)}} (f-g)(y,s)\, ds$$

for all $(x,t) \in \mathcal{E}(x_0, r)$, which completes the proof. □

Remark 3.8. If $A_2 = 0$ in (H2), the local comparison principle is established in [25, Theorem V.3] without requiring the additional assumption (3.15). In our framework, tht assumption (3.15) is employed solely to derive (3.18) from (3.17), since the inequalities hold when $w \leqq 0$. When the Hamiltonian $H = H(x,t,u,p)$ is linear in u, as in (6.1), the local comparison principle can be established by direct calculation, again without invoking (3.15). Moreover, a direct proof of Theorem 3.6 under assumption (H5) has recently been given in [13, Theorem 2.2.7]. This approach provides an alternative argument for Theorem 3.6 that does not rely on Lemma 3.7.

Remark 3.9. We verify that the function h_ε defined in (3.20) satisfies the inequality (3.19). If $A_2 = 0$, then we compute

$$(h_\varepsilon)_t(x,t) = B_2, \quad D_x h_\varepsilon(x,t) = \frac{x - x_0}{\sqrt{|x-x_0|^2 + \varepsilon^2}}.$$

Hence,

$$\begin{aligned} (h_\varepsilon)_t(x,t) - B_2 |D_x h_\varepsilon(x,t)| &= B_2 - B_2 \frac{|x-x_0|}{\sqrt{|x-x_0|^2+\varepsilon^2}} \\ &\geqq B_2 - B_2 = 0. \end{aligned}$$

If $A_2 \neq 0$, then we compute

$$\begin{aligned} (h_\varepsilon)_t(x,t) &= A_2 e^{A_2 t}\left(\frac{B_2}{A_2} + \sqrt{|x|^2+\varepsilon^2}\right), \\ D_x h_\varepsilon(x,t) &= \frac{x}{\sqrt{|x|^2+\varepsilon^2}}(e^{A_2 t} - 1) + \frac{x - x_0}{\sqrt{|x-x_0|^2+\varepsilon^2}}. \end{aligned}$$

Consequently,

$$\begin{aligned} &(h_\varepsilon)_t(x,t) - (A_2|x| + B_2)|D_x h_\varepsilon(x,t)| \\ &\geqq A_2 e^{A_2 t}\left(\frac{B_2}{A_2} + \sqrt{|x|^2+\varepsilon^2}\right) \\ &\quad - (A_2|x| + B_2)\left(\frac{|x|}{\sqrt{|x|^2+\varepsilon^2}}(e^{A_2 t}-1) + \frac{|x-x_0|}{\sqrt{|x-x_0|^2+\varepsilon^2}}\right) \\ &\geqq A_2 e^{A_2 t}\left(\frac{B_2}{A_2} + |x|\right) - e^{A_2 t}(A_2|x| + B_2) = 0. \end{aligned}$$

Therefore, in both cases, the function h_ε satisfies (3.19).

Remark 3.10. In [45, Proof of Theorem 6.1], the function h_ε is introduced as

$$h_\varepsilon(x,t) = e^{(A_2+B_2+A_2|x_0|)t}(1+\sqrt{|x-x_0|^2+\varepsilon^2}) - 1,$$

and it is shown that inequality (3.16) holds in the domain $\mathcal{D}(x_0,r)$ when $K_3 = 0$, where $\mathcal{D}(x_0,r)$ is defined in (1.5). It should be emphasized that $\mathcal{D}(x_0,r)$ is strictly contained in $\mathcal{E}(x_0,r)$, as established in Theorem A.2.

3.3 Proof of Theorem 1.2

In this section, we provide the proof of Theorem 1.2. The proof relies on several previously established results: Theorem 2.4, Corollary 3.5, and Theorem 3.6. We also invoke the following lemma, which provides an initial-time estimate under assumption (U).

Lemma 3.11 ([45, Proposition 4.1])**.** *Assume that the initial data u_0 satisfies (U). Fix $\varepsilon > 0$. Let $u \in C(\mathbb{R}^n \times [0,T))$ be a viscosity solution of* (1.1)–(1.2) *and u_ε be the inf-convolution defined in* (3.12)*. Then, we have*

$$u_\varepsilon(y,0) - u(y,0) \leqq -\frac{\theta^2}{4}\varepsilon^2 \tag{3.22}$$

for all $y \in \overline{B_r(x_0)}$.

Proof of Theorem 1.2. By Theorem 2.4, a viscosity solution u of (1.1)–(1.2) is also a Barron–Jensen solution of (1.1)–(1.2). According to Corollary 3.5, for $0 < \rho < r$, $\gamma \geqq \left(\frac{\beta}{2}+2\right)C_1 + K_3$, and $\varepsilon \in (0, \frac{e^{\frac{\gamma}{2}T}\rho}{2M_r})$, the inf-convolution u_ε defined in (3.12) is a viscosity subsolution of

$$u_t(x,t) + H(x,t,u(x,t),D_x u(x,t)) = \frac{\beta C_1}{2}e^{\gamma t}\varepsilon^2 \quad \text{in } B_{r-\rho}(x_0) \times (0,T).$$

We proceed to establish both upper and lower estimates for the difference $u_\varepsilon - u$.

Applying Theorem 3.6, we obtain

$$u_\varepsilon(x,t) - u(x,t) \leqq \sup_{y \in \overline{B_{r-\rho}(x_0)}} e^{-K_3 t}(u_\varepsilon - u)(y,0) + e^{-K_3 t}\int_0^t e^{K_3 s} \cdot \frac{\beta C_1}{2}e^{\gamma s}\varepsilon^2\, ds$$

for all $(x,t) \in \overline{\mathcal{E}(x_0,r)} \cap (\overline{B_{r-\rho}(x_0)} \times [0,T])$, where $\mathcal{E}(x_0,r)$ is defined in (1.7). By (3.22) and

$$e^{-K_3 t}\int_0^t e^{K_3 s} \cdot \frac{\beta C_1}{2}e^{\gamma s}\varepsilon^2\, ds \leqq e^{\gamma t}\int_0^t \frac{\beta C_1}{2}\varepsilon^2\, ds = \frac{\beta C_1}{2}e^{\gamma t}t\varepsilon^2, \tag{3.23}$$

we obtain

$$u_\varepsilon(x,t) - u(x,t) \leqq -\frac{\theta^2}{4}e^{-K_3 t}\varepsilon^2 + \frac{\beta C_1}{2}e^{\gamma t}t\varepsilon^2. \tag{3.24}$$

On the other hand, let $(x,t) \in \mathcal{E}(x_0,r) \cap (B_{r-\rho}(x_0) \times (0,T))$ and $p \in D_x^- u(x,t)$. Let L_u denote the Lipschitz constant of u in $B_r(x_0)$. From [45, Proof of Theorem 4.2(i)], there exists a modulus of continuity $\omega \in C([0,\infty))$ with $\lim_{r\to+0}\omega(r) = 0$ such that

$$u_\varepsilon(x,t) - u(x,t) \geqq -\frac{|p|^2}{4}e^{\gamma t}\varepsilon^2 - L_u e^{\gamma T}\varepsilon^2\omega(L_u e^{\gamma T}\varepsilon^2). \tag{3.25}$$

Combining (3.24) and (3.25), we derive

$$-\frac{|p|^2}{4}e^{\gamma t}\varepsilon^2 - L_u e^{\gamma T}\varepsilon^2\omega(L_u e^{\gamma T}\varepsilon^2) \leqq -\frac{\theta^2}{4}e^{-K_3 t}\varepsilon^2 + \frac{\beta C_1}{2}e^{\gamma t}t\varepsilon^2.$$

Dividing by ε^2, we obtain

$$|p|^2 \geqq \theta^2 e^{-(\gamma+K_3)t} - 2\beta C_1 t - 4L_u e^{\gamma(T-t)}\omega(L_u e^{\gamma T}\varepsilon^2)$$

Finally, by taking the limits $\varepsilon, \rho \to +0$, we deduce

$$|p|^2 \geqq \theta^2 e^{-(\gamma+K_3)t} - 2\beta C_1 t$$

for all $(x,t) \in \mathcal{E}(x_0,r)$. Choosing $\gamma = \left(\frac{\beta}{2}+2\right)C_1 + K_3$, the proof is complete. □

Remark 3.12. A sharper gradient estimate can be obtained by explicitly evaluating the integral term in (3.23). Indeed, we have

$$\begin{aligned} e^{-K_3 t}\int_0^t e^{K_3 s}\frac{\beta C_1}{2}e^{\gamma s}\varepsilon^2\,ds &= \frac{\beta C_1}{2}\varepsilon^2 e^{-K_3 t}\int_0^t e^{(\gamma+K_3)s}\,ds \\ &= \frac{\beta C_1}{2(\gamma+K_3)}\varepsilon^2(e^{\gamma t}-e^{-K_3 t}). \end{aligned}$$

Substituting this into the upper bound of $u_\varepsilon - u$, we obtain the refined inequality in place of (3.24):

$$u_\varepsilon(x,t) - u(x,t) \leqq -\frac{\theta^2}{4}e^{-K_3 t}\varepsilon^2 + \frac{\beta C_1}{2(\gamma+K_3)}\varepsilon^2(e^{\gamma t}-e^{-K_3 t}). \tag{3.26}$$

Combining (3.26) with (3.25), and letting $\varepsilon, \rho \to +0$, we obtain

$$|p|^2 \geqq \theta^2 e^{-(\gamma+K_3)t} - \frac{2\beta C_1}{\gamma+K_3}(1-e^{-(\gamma+K_3)t}).$$

By choosing $\gamma = \left(\frac{\beta}{2}+2\right)C_1 + K_3$, we derive

$$|p|^2 \geqq \theta^2 e^{-((\frac{\beta}{2}+2)C_1+2K_3)t} - \frac{4\beta C_1}{(\beta+4)C_1+4K_3}(1-e^{-((\frac{\beta}{2}+2)C_1+2K_3)t}).$$

Chapter 4

Dynamical approach

In this chapter, we establish gradient estimates for viscosity solutions to (1.1)–(1.2) by employing a dynamical approach inspired by [32]. Specifically, we make use of Herglotz' variational principle for contact Hamiltonian systems and investigate how the initial gradients propagate along the characteristic curves governed by these systems.

4.1 Herglotz' variational principle

In the section, we introduce several results of Herglotz' variational principle and contact Hamiltonian systems, following the developments presented in [19, 20, 22]. For classical results on Hamiltonian systems, we refer the reader to [21, Chapter 6]. To utilize these tools, the following assumptions on the Hamiltonian H are commonly imposed: (H3), $(\mathrm{H4})_{\mathrm{st}}$, (H7), and

(A1) There exist superlinear functions $\theta_0, \theta_1 : [0, \infty) \to [0, \infty)$ and a constant $c_0 > 0$ such that

$$\theta_0(|p|) - c_0 \leqq H(x, t, 0, p) \leqq \theta_1(|p|)$$

for all $(x, t, p) \in \mathbb{R}^n \times [0, T] \times \mathbb{R}^n$.

(A2) There exist constants $c_1, c_2 > 0$ such that

$$\max\{|D_x H(x, t, u, p)|, |D_p H(x, t, u, p)|\} \leqq c_1 + c_2 H(x, t, u, p)$$

for all $(x, t, u, p) \in \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n$.

We say that a function $f : [0, \infty) \to [0, \infty)$ is *superlinear* if $\frac{f(r)}{r} \to \infty$ as $r \to \infty$.

Remark 4.1. The additional conditions (A1) and (A2) are not essential for our purposes, as our results are local in nature. More precisely, fix $x_0 \in \mathbb{R}^n$ and $R > 0$, and define L_0 as the Lipschitz constant of u on $\overline{B_{2R}(x_0)} \times [0, T]$, and

$$M_0 := \max\{u(x, t) \mid (x, t) \in \overline{B_{2R}(x_0)} \times [0, T]\},$$
$$L_1 := \max\{H(x, t, u, p) \mid (x, t, u, p) \in \overline{B_{2R}(x_0)} \times [0, T] \times [-M_0, M_0] \times \overline{B_{L_0}(0)}\}.$$

Let us define a new Hamiltonian $\tilde{H}$ satisfying (H3), $(\mathrm{H4})_{\mathrm{st}}$, (H7), (A1), (A2), and assume that $\tilde{H} \equiv H$ on $\overline{B_{2R}(x_0)} \times [0, T] \times [-M_0, M_0] \times \overline{B_{L_0}(x_0)}$. Let $\tilde{u}$ be the viscosity solution of (1.1)–(1.2) with H replaced by $\tilde{H}$. Then, by [25, Theorem V.3], we have $\tilde{u} \equiv u$ on $\overline{B_R(x_0)} \times [0, \frac{R}{L_1}]$. This localization argument is also employed in [2, Proof of Theorem 3.2] and [32, Remark 2.3].

In light of Remark 4.1, we assume throughout this section that H satisfies only the conditions (H1)–(H7) and $(\mathrm{H4})_{\mathrm{st}}$. We now define the *Lagrangian* $L : \mathbb{R}^n \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ associated with H by

$$L(x, t, u, q) := \sup_{p \in \mathbb{R}^n} \{\langle p, q \rangle - H(x, t, u, p)\}.$$

Next, we define the value function $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ by

$$u(x, t) := \inf_{\xi \in \mathcal{C}(x,t)} \left\{ u_0(\xi(0)) + \int_0^t L(\xi(s), s, u_\xi(s), \xi'(s))\, ds \right\}, \tag{4.1}$$

where
$$\mathcal{C}(x,t) := \{\xi : [0,t] \to \mathbb{R}^n \mid \xi \text{ is absolutely continuous and } \xi(t) = x\},$$
and u_ξ is the solution of the *Carathéodory equation*
$$u_\xi'(s) = L(\xi(s), s, u_\xi(s), \xi'(s)) \quad (\text{a.e. } s \in [0,t]) \tag{4.2}$$
with the initial condition $u_\xi(0) = u_0(\xi(0))$. The existence and uniqueness of solutions to (4.2) are guaranteed by Carathéodory's theorem (see [19, Proposition A.1]).

Proposition 4.2 ([19, Proposition 3.3], [22, Proposition 2.3(2)])**.** *Let $(x,t) \in \mathbb{R}^n \times (0,T)$. Then, the variational problem* (4.1) *admits a minimizer. Moreover, the value function $u(x,t)$ defined in* (4.1) *is a viscosity solution of* (1.1)–(1.2)*, and it is Lipschitz continuous and locally semi-concave in $\mathbb{R}^n \times (0,T)$.*

Let $\mathcal{C}_{\min}(x,t)$ be the set of all the minimizers of (4.1). For $\xi \in \mathcal{C}_{\min}(x,t)$ and the unique solution u_ξ of (4.2) with $u_\xi(0) = u_0(\xi(0))$, we define $\eta : [0,t] \to \mathbb{R}^n$ by
$$\eta(s) := D_q L(\xi(s), s, u_\xi(s), \xi'(s)) \quad (s \in [0,t]).$$
The curve η is called the *dual arc associated with* ξ.

Proposition 4.3 ([20, Theorem 1(1)], [19, Theorem 2.3])**.** *Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $\xi \in \mathcal{C}_{\min}(x,t)$. Let u_ξ be the solution of* (4.2) *with $u_\xi(0) = u_0(\xi(0))$. Then, both ξ and u_ξ are of class C^2, and satisfy Herglotz' equation*
$$\begin{aligned}&\frac{d}{ds} D_q L(\xi(s), s, u_\xi(s), \xi'(s))\\ &= D_x L_x(\xi(s), s, u_\xi(s), \xi'(s)) + D_u L(\xi(s), s, u_\xi(s), \xi'(s)) D_q L(\xi(s), s, u_\xi(s), \xi'(s))\end{aligned}$$
for almost all $s \in [0,t]$.

Proposition 4.4 ([20, Theorem 1(2)], [19, Proposition 2.1])**.** *Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $\xi \in \mathcal{C}_{\min}(x,t)$. Let u_ξ be the solution of* (4.2) *with $u_\xi(0) = u_0(\xi(0))$. Then, η is of class C^2, and the triple (ξ, u_ξ, η) satisfies contact Hamiltonian systems*
$$\begin{cases} \text{(a) } \xi'(s) = D_p H(\xi(s), s, u_\xi(s), \eta(s)), \\ \text{(b) } \eta'(s) = -D_x H(\xi(s), s, u_\xi(s), \eta(s)) - D_u H(\xi(s), s, u_\xi(s), \eta(s))\eta(s), \\ \text{(c) } u_\xi'(s) = \langle \eta(s), \xi'(s) \rangle - H(\xi(s), s, u_\xi(s), \eta(s)) \end{cases} \tag{4.3}$$
for all $s \in [0,t]$.

Let us define the regular set of differentiability of u as
$$\mathcal{R}(u) := \{(x,t) \in \mathbb{R}^n \times (0,T) \mid u \text{ is differentiable at } (x,t)\}.$$
The following results are essential in establishing lower bounds on gradients of viscosity solutions.

Proposition 4.5 ([22, Proposition 2.6])**.** *Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of* (1.1)–(1.2)*. For any $(p,\tau) \in D^* u(x,t)$, let (ξ, η, u_ξ) be the solution to* (4.3) *with the terminal condition*
$$\xi(t) = x, \quad \eta(t) = p, \quad u_\xi(t) = u(x,t). \tag{4.4}$$
Then, $\xi \in \mathcal{C}_{\min}(x,t)$, and the mapping
$$D^* u(x,t) \ni (p,\tau) \mapsto \xi \in \mathcal{C}_{\min}(x,t)$$
is bijective.

Proposition 4.6 ([22, Propositions 2.3(4) and 2.4])**.** *Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of* (1.1)–(1.2)*. Let $\xi \in \mathcal{C}_{\min}(x,t)$. Then, $(\xi(s), s) \in \mathcal{R}(u)$ for all $s \in (0,t)$. Moreover, we have*
$$\begin{aligned}&\eta(t) \in D_x^+ u(x,t),\\ &\eta(s) = D_x u(\xi(s), s) \quad (s \in (0,t)),\\ &\eta(0) \in D^- u_0(\xi(0)).\end{aligned}$$

Remark 4.7. In the cited work [22], the initial data u_0 is assumed to be smooth. However, the above statements remain valid under the weaker assumption that u_0 is merely Lipschitz continuous.

Remark 4.8. If $H = H(x,t,p)$, then [2, Theorem 3.2] establishes a stronger condition, namely $\eta(0) \in D^-_{\mathrm{pr}} u_0(\xi(0))$. See also Appendix B.

4.2 Estimates for solutions to contact Hamiltonian systems

In this section, we derive quantitative estimates for solutions (ξ, η, u_ξ) of contact Hamiltonian systems (4.3)–(4.4) under the terminal condition

$$\xi(t) = x, \quad \eta(t) = D_x u(x,t), \quad u_\xi(t) = u(x,t), \tag{4.5}$$

which is assumed to be well-defined at points $(x,t) \in \mathcal{R}(u)$. These results are analogous to those found in [32, Proposition 3.1].

Proposition 4.9. *Assume that the Hamiltonian H satisfies (H1)–(H7) and (H4)$_{\rm st}$. Let $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of (1.1)–(1.2). Let $(x,t) \in \mathcal{R}(u)$ and (ξ, η, u_ξ) be the solution of (4.3)–(4.5). If $(C_1, K_3) \neq (0,0)$, then*

$$\begin{cases} |D_x u(x,t) - \eta(0)| \leqq \left(\dfrac{\beta C_1}{C_1+K_3} + |D_x u(x,t)|\right)(e^{(C_1+K_3)t} - 1), \\ |D_x u(x,t) - \eta(0)| \leqq \left(\dfrac{\beta C_1}{C_1+K_3} + |\eta(0)|\right)(e^{(C_1+K_3)t} - 1), \end{cases} \tag{4.6}$$

where C_1, β, and K_3 are constants from (H1) and (H3). In particular, we have

$$\begin{aligned} &|\eta(0)|e^{-(C_1+K_3)t} - \frac{\beta C_1}{C_1+K_3}(1 - e^{-(C_1+K_3)t}) \\ &\leqq |D_x u(x,t)| \leqq |\eta(0)|e^{(C_1+K_3)t} + \frac{\beta C_1}{C_1+K_3}(e^{(C_1+K_3)t} - 1). \end{aligned} \tag{4.7}$$

Moreover, in the special cases: if $K_3 = 0$, then

$$|\eta(0)|e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |D_x u(x,t)| \leqq |\eta(0)|e^{C_1 t} + \beta(e^{C_1 t} - 1), \tag{4.8}$$

and

$$|\eta(0)|e^{-(C_1+K_3)t} \leqq |D_x u(x,t)| \leqq |\eta(0)|e^{(C_1+K_3)t} \quad \text{if } \beta = 0, \tag{4.9}$$

$$|\eta(0)|e^{-K_3 t} \leqq |D_x u(x,t)| \leqq |\eta(0)|e^{K_3 t} \quad \text{if } C_1 = 0, \tag{4.10}$$

$$D_x u(x,t) = \eta(0) \quad \text{if } (C_1, K_3) = (0,0). \tag{4.11}$$

Remark 4.10. The estimate (4.8) was previously derived in [32, Proposition 3.1] for the case where the Hamiltonian H is independent of u.

Proof. It suffices to prove (4.6) since (4.7)–(4.11) are immediate from (4.6). Let $\tau \in [0,t)$. Integrating both sides of (4.3)(b) over the interval $[\tau, t]$, and using assumptions (H1) and (H3), we obtain

$$\begin{aligned} &|\eta(t) - \eta(\tau)| \\ &\leqq \int_\tau^t |D_x H(\xi(s), s, u_\xi(s), \eta(s))|\,ds + \int_\tau^t |D_u H(\xi(s), s, u_\xi(s), \eta(s))| \cdot |\eta(s)|\,ds \\ &\leqq \int_\tau^t C_1(\beta + |\eta(s)|)\,ds + \int_\tau^t K_3 |\eta(s)|\,ds \\ &\leqq \int_\tau^t (\beta C_1 + (C_1+K_3)(|\eta(t)| + |\eta(t) - \eta(s)|))\,ds \\ &= (\beta C_1 + (C_1+K_3)|\eta(t)|)(t-\tau) + (C_1+K_3)\int_\tau^t |\eta(t) - \eta(s)|\,ds. \end{aligned} \tag{4.12}$$

By Gronwall's lemma, and taking $\tau = 0$, we have

$$\begin{aligned} &|\eta(t) - \eta(0)| \\ &\leqq (\beta C_1 + (C_1+K_3)|\eta(t)|)t \\ &\quad + e^{(C_1+K_3)t}\int_0^t e^{-(C_1+K_3)(t-s)}(C_1+K_3)(\beta C_1 + (C_1+K_3)|\eta(t)|)(t-s)\,ds \\ &= (\beta C_1 + (C_1+K_3)|\eta(t)|)\left\{t + (C_1+K_3)\int_0^t (t-s)e^{(C_1+K_3)s}\,ds\right\}. \end{aligned}$$

If $(C_1, K_3) \neq (0,0)$, then

$$\begin{aligned}(C_1+K_3)\int_0^t (t-s)e^{(C_1+K_3)s}\,ds &= \Big[(t-s)e^{(C_1+K_3)s}\Big]_0^t + \int_0^t e^{(C_1+K_3)s}\,ds \\ &= -t + \frac{e^{(C_1+K_3)t}-1}{C_1+K_3}.\end{aligned}$$

Hence, we obtain

$$|\eta(t)-\eta(0)| \leqq \left(\frac{\beta C_1}{C_1+K_3} + |\eta(t)|\right)(e^{(C_1+K_3)t}-1), \tag{4.13}$$

which proves the first estimate in (4.6) since $\eta(t) = D_x u(x,t)$.

For the second estimate in (4.6), fix $\sigma \in (0,t]$, integrate both sides of (4.3)(b) over the interval $[0,\sigma]$, and then set $\sigma = t$. Applying the same argument as above yields the desired inequality. □

We also utilize the following result, originally established in [32], which remains valid under the current assumptions.

Proposition 4.11 ([32, Proposition 3.2])**.** *Assume that the Hamiltonian H satisfies (H1)–(H7) and (H4)$_{\mathrm{st}}$. Let $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of* (1.1)–(1.2)*. Let $(x,t) \in \mathcal{R}(u)$ and (ξ, η, u_ξ) be the solution of* (4.3)–(4.5)*. Then, we have*

$$\begin{cases} |x-\xi(0)| \leqq \left(\dfrac{B_2}{A_2} + |x|\right)(e^{A_2 t}-1) & \text{if } A_2 > 0, \\ |x-\xi(0)| \leqq B_2 t & \text{if } A_2 = 0, \end{cases} \tag{4.14}$$

where A_2 and B_2 are constants from (H2).

Proof. The proof of (4.14) proceeds analogously to that of the first inequality in (4.6). Specifically, integrating both sides of (4.3)(a) over the interval $[\tau, t]$ for any $\tau \in [0,t)$, we obtain

$$\begin{aligned}|\xi(t)-\xi(\tau)| = \left|\int_\tau^t D_pH(\xi(s), s, u_\xi(s), \eta(s))\,ds\right| &\leqq \int_\tau^t |D_pH(\xi(s), s, u_\xi(s), \eta(s))|\,ds \\ &\leqq \int_\tau^t (A_2|\xi(s)| + B_2)\,ds.\end{aligned}$$

If $A_2 = 0$, the estimate (4.14) follows immediately. If $A_2 > 0$, the structure of the integral is analogous to that in (4.12). We see that the resulting estimate for $|\xi(t) - \xi(0)|$ is obtained by letting $K_3 = 0$ and replacing C_1 and β by A_2 and $\frac{B_2}{A_2}$ in (4.13), respectively. Hence,

$$|\xi(t)-\xi(0)| \leqq \left(\frac{B_2}{A_2} + |\xi(t)|\right)(e^{A_2 t}-1).$$

Since $\xi(t) = x$, this completes the proof. □

As a consequence of Propositions 4.6, 4.9, and 4.11, we now derive the following theorem under the additional assumptions (H7) and (H4)$_{\mathrm{st}}$.

Theorem 4.12. *Assume that the Hamiltonian H satisfies (H1)–(H7) and (H4)$_{\mathrm{st}}$. Let $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of* (1.1)–(1.2)*. Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $p \in D_x^- u(x,t)$. If $(C_1, K_3) \neq (0,0)$, then*

$$\begin{aligned}&I(x,t;u_0)e^{-(C_1+K_3)t} - \frac{\beta C_1}{C_1+K_3}(1-e^{-(C_1+K_3)t}) \\ &\leqq |p| \leqq S(x,t;u_0)e^{(C_1+K_3)t} + \frac{\beta C_1}{C_1+K_3}(e^{(C_1+K_3)t}-1),\end{aligned}$$

and if $(C_1, K_3) = (0,0)$, then

$$I(x,t;u_0) \leqq |p| \leqq S(x,t;u_0).$$

Here, β, C_1, and K_3 are constants from (H1) and (H3), and

$$\begin{aligned}S(x,t;u_0) &:= \sup\left\{|p| \;\middle|\; p \in D^-u_0(y),\, y \in \overline{B_{R(x,t)}(x)}\right\}, \\ I(x,t;u_0) &:= \inf\left\{|p| \;\middle|\; p \in D^-u_0(y),\, y \in \overline{B_{R(x,t)}(x)}\right\},\end{aligned}$$

where $R(x,t)$ is defined in (1.8)*.*

Proof. By Proposition 4.2, the function u is locally semi-concave in $\mathbb{R}^n \times (0,T)$. If $(x,t) \notin \mathcal{R}(u)$, then $D^-u(x,t) = \emptyset$ by Proposition 2.7(1), and the conclusion is trivially satisfied. If $(x,t) \in \mathcal{R}(u)$, then $D_x^-u(x,t) = \{D_x u(x,t)\}$ by Proposition 2.7(2). The estimates follow directly from Propositions 4.6, 4.9, and 4.11. □

4.3 Proof of Theorem 1.3

In this section, we provide the proof of Theorem 1.3. Following the way in Section 2.2, we approximate H by a smooth Hamiltonian H_δ satisfying (H7) and $(\mathrm{H4})_{\mathrm{st}}$. We also recall that H_δ fulfills (H2) with the Lipschitz constant $A_2|x| + B_2 + \delta$ by Remark 2.8. For this reason, we prepare the following notations:

$$R_\delta(x,t) := \begin{cases} \left(\dfrac{B_2+\delta}{A_2} + |x|\right)(e^{A_2 t} - 1) & \text{if } A_2 > 0, \\ (B_2+\delta)t & \text{if } A_2 = 0, \end{cases}$$

and

$$\begin{aligned} S_\delta(x,t;u_0) &:= \sup\left\{|p| \;\middle|\; p \in D^-u_0(y),\, y \in \overline{B_{R_\delta(x,t)}(x)}\right\}, \\ I_\delta(x,t;u_0) &:= \inf\left\{|p| \;\middle|\; p \in D^-u_0(y),\, y \in \overline{B_{R_\delta(x,t)}(x)}\right\}. \end{aligned}$$

Proof of Theorem 1.3. Let $\delta \in (0,1]$ and H_δ defined in (2.1). Let u_δ be the viscosity solution of (1.11)–(1.2). By Proposition 2.9, u_δ converges to u locally uniformly in $\mathbb{R}^n \times [0,T)$ as $\delta \to +0$.

Let $p \in D_x^-u(x,t)$. By [6, Lemma II.2.4], there exist sequences $\{(x_\delta,t_\delta)\}_{\delta\in(0,1]} \subset \mathbb{R}^n \times (0,T)$ and $\{p_\delta\}_{\delta\in(0,1]} \subset \mathbb{R}^n$ such that

$$p_\delta \in D_x^-u_\delta(x_\delta,t_\delta), \qquad \lim_{\delta\to+0}(x_\delta,t_\delta,p_\delta) = (x,t,p).$$

We apply Theorem 4.12 to obtain

$$\begin{aligned} &I_\delta(x_\delta,t_\delta;u_0)e^{-(C_1+K_3)t_\delta} - \frac{\beta C_1}{C_1+K_3}(1 - e^{-(C_1+K_3)t_\delta}) \\ &\leqq |p_\delta| \leqq S_\delta(x_\delta,t_\delta;u_0)e^{(C_1+K_3)t_\delta} + \frac{\beta C_1}{C_1+K_3}(e^{(C_1+K_3)t_\delta} - 1). \end{aligned}$$

Now fix any $\alpha > 0$. For sufficiently small $\delta > 0$, it follows from the definition of R_δ that $R_\delta(x_\delta,t_\delta) < R(x,t) + \alpha$, and hence

$$\begin{aligned} &\inf\left\{|q| \;\middle|\; q \in D^-u_0(y),\; y \in \overline{B_{R(x,t)+\alpha}(x)}\right\} \cdot e^{-(C_1+K_3)t_\delta} \\ &\qquad - \frac{\beta C_1}{C_1+K_3}(1 - e^{-(C_1+K_3)t_\delta}) \\ &\leqq |p_\delta| \leqq \sup\left\{|q| \;\middle|\; q \in D^-u_0(y),\; y \in \overline{B_{R(x,t)+\alpha}(x)}\right\} \cdot e^{(C_1+K_3)t_\delta} \\ &\qquad + \frac{\beta C_1}{C_1+K_3}(1 - e^{(C_1+K_3)t_\delta}). \end{aligned}$$

Finally, taking the limit $\delta \to +0$ and $\alpha \to +0$, we deduce desired estimates. □

4.4 Comparison with Theorem 1.2

In this section, we compare lower bounds provided by (1.9) in Theorem 1.2 and by (1.15) in Theorem 1.4, under the assumption (U). We first note that both estimates are valid on the same domain of dependence, namely $\mathcal{E}(x_0,r)$, which allows for a direct comparison. When $(C_1,K_3) = (0,0)$, the two lower bounds coincide. However, when $(C_1,K_3) \neq (0,0)$, the two lower bounds differ in general. The purpose of this section is to identify which of the two lower bounds is sharper. To this end, we restrict our attention to the case $(C_1,K_3) \neq (0,0)$, and introduce the following notations.

Definition 4.13. Let $\theta > 0$ and assume $(C_1, K_3) \neq (0,0)$. We define

$$l(t) := \sqrt{\theta^2 e^{-((\frac{\beta}{2}+2)C_1+2K_3)t} - 2\beta C_1 t},$$
$$L(t) := \theta e^{-(C_1+K_3)t} - \frac{\beta C_1}{C_1+K_3}(1 - e^{-(C_1+K_3)t}),$$

where β, C_1, and K_3 are constants from (H1) and (H3). Furthermore, in the special case $\beta = 1$, we define the critical time $t_l > 0$ by

$$t_l := \inf\{t \in [0,\infty) \mid l(t) = 0\}.$$

The next theorem demonstrates that the lower bound in Theorem 1.4, given by (1.15), is better than the one in Theorem 1.2, given by (1.9) when $(C_1, K_3) \neq (0,0)$.

Theorem 4.14. *Let $\theta > 0$ and assume $(C_1, K_3) \neq (0,0)$.*

(1) If $\beta = 0$, then $l(t) = L(t)$ for all $t \in [0,\infty)$.

(2) If $\beta = 1$, then $l(0) = L(0)$ and $l(t) < L(t)$ for all $t \in (0, t_l]$.

Proof. (1) It is obvious since $l(t) = L(t) = \theta e^{-(C_1+K_3)t}$.
(2) Suppose $\beta = 1$. Then, we have

$$l(t) = \sqrt{\theta^2 e^{-(\frac{5}{2}C_1+2K_3)t} - 2C_1 t},$$
$$L(t) = \theta e^{-(C_1+K_3)t} - \frac{C_1}{C_1+K_3}(1 - e^{-(C_1+K_3)t}).$$

It is clear that $l(0) = L(0) = \theta$. To establish the desired inequality on $(0, t_l]$, it suffices to show that $\{l(t)\}^2 < \{L(t)\}^2$ for all $t \in (0, t_l]$ since $l(t), L(t) \geqq 0$ on this interval. Let us define

$$F(t) := \{L(t)\}^2 - \{l(t)\}^2.$$

A direct computation yields

$$\begin{aligned}
\{l(t)\}^2 &= \theta^2 e^{-(\frac{5}{2}C_1+2K_3)t} - 2C_1 t,\\
\{L(t)\}^2 &= \theta^2 e^{-2(C_1+K_3)t} - \frac{2C_1\theta}{C_1+K_3} e^{-(C_1+K_3)t}(1 - e^{-(C_1+K_3)t})\\
&\qquad + \frac{C_1^2}{(C_1+K_3)^2}(1 - e^{-(C_1+K_3)t})^2,
\end{aligned}$$

and hence

$$\begin{aligned}
F(t) &= \theta^2 e^{-2(C_1+K_3)t}(1 - e^{-\frac{1}{2}C_1 t}) - \frac{2C_1\theta}{C_1+K_3} e^{-(C_1+K_3)t}(1 - e^{-(C_1+K_3)t})\\
&\qquad + \frac{C_1^2}{(C_1+K_3)^2}(1 - e^{-(C_1+K_3)t})^2 + 2C_1 t\\
&= e^{-2(C_1+K_3)t}(1 - e^{-\frac{1}{2}C_1 t}) \left\{\theta - \frac{C_1}{C_1+K_3} e^{(C_1+K_3)t} \frac{1 - e^{-(C_1+K_3)t}}{1 - e^{-\frac{1}{2}C_1 t}}\right\}^2\\
&\qquad - \frac{C_1^2}{(C_1+K_3)^2}\left(\frac{(1 - e^{-(C_1+K_3)t})^2}{1 - e^{-\frac{1}{2}C_1 t}} + (1 - e^{-(C_1+K_3)t})^2\right) + 2C_1 t\\
&\geqq -\frac{C_1^2}{(C_1+K_3)^2} \cdot \frac{e^{-\frac{1}{2}C_1 t}(1 - e^{-(C_1+K_3)t})^2}{1 - e^{-\frac{1}{2}C_1 t}} + 2C_1 t =: G(t).
\end{aligned}$$

Since $e^X > X + 1$ $(X \neq 0)$, we observe that

$$\begin{aligned}
(1 - e^{-\frac{1}{2}C_1 t})G(t) &= -\frac{C_1^2}{(C_1+K_3)^2} \cdot e^{-\frac{1}{2}C_1 t}(1 - e^{-(C_1+K_3)t})^2 + 2C_1 t(1 - e^{-\frac{1}{2}C_1 t})\\
&> -\frac{C_1^2}{(C_1+K_3)^2} \cdot e^{-\frac{1}{2}C_1 t}(C_1+K_3)^2 t^2 + 2C_1 t(1 - e^{-\frac{1}{2}C_1 t})\\
&= 2C_1 t\left\{-\left(\frac{1}{2}C_1 t + 1\right) e^{-\frac{1}{2}C_1 t} + 1\right\}\\
&> 2C_1 t(-e^{\frac{1}{2}C_1 t} \cdot e^{-\frac{1}{2}C_1 t} + 1) = 0.
\end{aligned}$$

Since $1 - e^{-\frac{1}{2}C_1 t} > 0$ in $(0, t_l]$, we conclude that

$$F(t) \geqq G(t) > 0,$$

which completes the proof. □

Remark 4.15. Figure 4.1 illustrates the functions $l(t)$ and $L(t)$ in the case where $\beta = 1$. Specifically, when $C_1 = \beta = K_3 = \theta = 1$, the respective lower bounds become

$$l(t) = \sqrt{e^{-\frac{9}{2}t} - 2t}, \quad L(t) = \frac{3}{2}e^{-2t} - \frac{1}{2}.$$

We define $t_L := \inf\{t \in [0, \infty) \mid L(t) = 0\}$, and compute

$$t_l \approx 0.2014, \quad t_L = \frac{1}{2}\log 3 \approx 0.5493.$$

These quantities help visualize and compare the effectiveness of the two gradient estimates, especially for small time. For further discussion related to eikonal-type equations, we refer to Example 6.7.

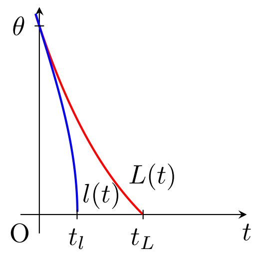


Figure 4.1: $L(t)$ and $l(t)$.

Remark 4.16. Lower bound estimates can be applied to reinitialization algorithms in numerical computations. Numerical errors may occur when the gradient of the solution to (1.1)–(1.2) on the zero level-set is extremely large or small. To avoid this problem, we stop solving the equation periodically at regular time intervals, replace the initial values with good ones, and solve it again so that the modified function approximates the signed distance function. This is called "reinitialization algorithms". For more details, see [34] and references therein. By Theorem 4.14, the number of times the calculation needs to be stopped can be reduced, thereby leading to a potential reduction in computational cost.

Chapter 5

Homogeneous Hamiltonians

In this chapter, we analyze the viscosity solution of (1.3)–(1.2) with the Hamiltonian $H = H(x,t,p)$ satisfying the assumption (H8), utilizing the dynamical approach developed in Chapter 4. We further extend our investigation to m-homogeneous Hamiltonians, which will be introduced precisely as the assumption $(\mathrm{H8})_m$ in Section 5.3.

5.1 Heuristic discussion

We begin by observing that, for the Hamiltonian $H = H(x,t,p)$, contact Hamiltonian systems (4.3) associated with (1.3) reduce to classical Hamiltonian systems:

$$\begin{cases} \text{(a) } \xi'(s) = D_pH(\xi(s), s, \eta(s)), \\ \text{(b) } \eta'(s) = -D_xH(\xi(s), s, \eta(s)), \end{cases} \tag{5.1}$$

which describes the evolution of characteristic curves in the absence of the contact variable u. All results established in Chapter 4 continue to hold for the solution pair (ξ, η) of this reduced system.

Assume now that H is smooth and satisfies (H8). Then, we have

$$\langle D_pH(x,t,p), p\rangle = H(x,t,p) \tag{5.2}$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$. Indeed, differentiating both sides of $H(x,t,\mu p) = \mu H(x,t,p)$ with respect to $\mu > 0$, we obtain $\langle D_pH(x,t,\mu p), p\rangle = H(x,t,p)$, and evaluating at $\mu = 1$ yields (5.2).

This relation has a significant implication: the viscosity solution u of (1.3)–(1.2) remains constant along $\xi \in \mathcal{C}_{\min}(x,t)$ for any $(x,t) \in \mathbb{R}^n \times (0,T)$. To see this, fix any $(p,\tau) \in D^*u(x,t)$ and let $(\xi,\eta) \in C^1([0,1])^2$ be the solution of (5.1) with the terminal conditions

$$\xi(t) = x, \quad \eta(t) = p.$$

We claim that

$$u(x,t) = u_0(\xi(0)). \tag{5.3}$$

By Proposition 4.5, we see that $\xi \in \mathcal{C}_{\min}(x,t)$. It follows from Proposition 4.6 that $(\xi(s), s) \in \mathcal{R}(u)$ for all $s \in (0,t)$. Using (5.2), we compute

$$\begin{aligned} \frac{d}{ds}u(\xi(s),s) &= \langle D_xu(\xi(s),s), \xi'(s)\rangle + u_t(\xi(s),s) \\ &= \langle \eta(s), D_pH(\xi(s),s,\eta(s))\rangle + u_t(\xi(s),s) \\ &= H(\xi(s),s,\eta(s)) + u_t(\xi(s),s) \\ &= H(\xi(s),s,D_xu(\xi(s),s)) + u_t(\xi(s),s). \end{aligned} \tag{5.4}$$

Since u solves (1.3) classically at $(\xi(s),s)$, the right-hand side of (5.4) is equal to 0. Thus, we have

$$u(\xi(s),s) = u(\xi(0),0) = u_0(\xi(0))$$

for all $s \in [0,t]$. Letting $s = t$, we obtain (5.3).

Moreover, by Proposition 4.6, we have $\eta(0) \in D^-u_0(\xi(0))$. This observation suggests that the generalized gradients of u at (x,t) depend only on $D^-u_0(y)$ at those points $y \in \mathbb{R}^n$ such that $u(x,t) = u_0(y)$; in other words, the gradients depend on the initial gradients on the same level-set.

In the next section, we will rigorously justify this observation by approximating H by the smooth Hamiltonian H_δ, as defined in (2.1). Note that if H is smooth and homogeneous, then it must be linear in p, hence the approximation plays a key role to study the general (possibly non-smooth) homogeneous Hamiltonian.

Remark 5.1. It is natural to ask whether similar estimates can be obtained even without assuming homogeneity (H8). Let L be the Lagrangian associated with H. Then, it follows from convex duality (see, e.g., [21, Theorem A.2.5 and Equation (A.21)]) that

$$H(x,t,p) + L(x,t,D_pH(x,t,p)) = \langle D_pH(x,t,p), p\rangle.$$

In the same manner as (5.4), we deduce that

$$\frac{d}{ds}u(\xi(s),s) = L(\xi(s),s,\xi'(s)).$$

Thus, if there exists a bound $C > 0$ for $|L(\xi(s),s,\xi'(s))|$, we obtain the estimate $|u(x,t) - u_0(\xi(0))| \leqq Ct$. This inequality provides a way to control the gradients of u at (x,t) in terms of initial data gradients within a sublevel set where $|u(x,t) - u_0(y)| \leqq Ct$. While this direction of analysis is promising, particularly when (H8) does not hold, we do not pursue it further in this chapter. Instead, we will revisit similar calculations under the assumption of m-homogeneous in Section 5.3.

5.2 Proof of Theorem 1.7

By virtue of (5.2), it is natural to expect that the approximate Hamiltonian H_δ satisfies

$$H_\delta(x,t,p) \approx \langle D_pH_\delta(x,t,p), p\rangle.$$

In this section, we provide a quantitative estimate for the deviation from this identity.

Since our assumptions do not impose Lipschitz regularity in t, we additionally assume condition (1.19) to ensure the existence of $D_pH(x,t,p)$ almost everywhere in $\mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

Lemma 5.2. *Assume that the Hamiltonian $H = H(x,t,p)$ satisfies (H2), (H8), and* (1.19). *Let $\delta \in (0,1]$ and H_δ be the approximate Hamiltonian defined in* (2.1). *Let us define the approximation error:*

$$E_\delta(x,t,p) := H_\delta(x,t,p) - \langle D_pH_\delta(x,t,p), p\rangle. \tag{5.5}$$

Then,

$$|E_\delta(x,t,p)| \leqq (A_2|x| + A_2\delta + B_2 + 1)\delta \tag{5.6}$$

for all $(x,t,p) \in \mathbb{R}^n \times [0,T] \times \mathbb{R}^n$.

Proof. Let us begin with the auxiliary term $k_\delta(p) = \delta\sqrt{|p|^2+1}$. Then, a direct calculation yields

$$k_\delta(p) - \langle Dk_\delta(p), p\rangle = \delta\sqrt{|p|^2+1} - \left\langle \frac{\delta p}{\sqrt{|p|^2+1}}, p \right\rangle = \frac{\delta}{\sqrt{|p|^2+1}}.$$

We next study the mollified Hamiltonian $(H * \rho_\delta)(x,t,p)$, where we write $z = (x,t,p) \in \mathbb{R}^{2n+1}$ and $w = (y,s,q) \in \mathbb{R}^{2n+1}$. Using the local Lipschitz continuity of H in (1.19), it follows that

$$D_p(H * \rho_\delta)(z) = D_p\left(\int_{B_\delta(0)} H(z-w)\rho_\delta(w)\,dw\right) = \int_{B_\delta(0)} D_pH(z-w)\rho_\delta(w)\,dw.$$

Thus, we compute

$$\begin{aligned}
&\langle D_p(H * \rho_\delta)(z), p\rangle \\
&= \int_{B_\delta(0)} \langle D_pH(z-w), p\rangle \rho_\delta(w)\,dw \\
&= \int_{B_\delta(0)} \langle D_pH(z-w), p-q\rangle \rho_\delta(w)\,dw + \int_{B_\delta(0)} \langle D_pH(z-w), q\rangle \rho_\delta(w)\,dw \\
&=: J_1 + J_2.
\end{aligned}$$

Since (5.2) holds almost everywhere in $\mathbb{R}^{2n+1}$, we recognize that

$$J_1 = \int_{B_\delta(0)} H(z-w)\rho_\delta(w)\,dw = (H * \rho_\delta)(z).$$

Next, to estimate J_2, observe that for any $w = (y, s, q) \in B_\delta(0)$,

$$\begin{aligned}|\langle D_pH(z-w), q\rangle| &\leqq |D_pH(z-w)|\cdot|q| \leqq (A_2|x-y| + B_2)\delta\\ &\leqq (A_2|x| + A_2\delta + B_2)\delta.\end{aligned}$$

Therefore,

$$\begin{aligned}|J_2| &\leqq \int_{B_\delta(0)} |\langle D_pH(z-w), q\rangle|\rho_\delta(w)\,dw\\ &\leqq (A_2|x| + A_2\delta + B_2)\delta\int_{B_\delta(0)} \rho_\delta(w)\,dw\\ &= (A_2|x| + A_2\delta + B_2)\delta. \end{aligned}\tag{5.7}$$

Finally, we combine these components. Since

$$\begin{aligned}E_\delta(x,t,p) &= (H*\rho_\delta)(x,t,p) + k_\delta(p) - \langle D_p(H*\rho_\delta)(x,t,p), p\rangle - \langle Dk_\delta(p), p\rangle\\ &= \frac{\delta}{\sqrt{|p|^2+1}} - J_2,\end{aligned}$$

we conclude that

$$\begin{aligned}|E_\delta(x,t,p)| &\leqq \frac{\delta}{\sqrt{|p|^2+1}} + |J_2| \leqq \delta + (A_2|x| + A_2\delta + B_2)\delta\\ &= (A_2|x| + A_2\delta + B_2 + 1)\delta,\end{aligned}$$

which establishes the desired inequality (5.6). $\square$

We are now ready to prove Theorem 1.7. In the proof, we solve approximate Hamiltonian systems associated with H_δ:

$$\begin{cases}\text{(a) } \xi_\delta'(s) = D_pH_\delta(\xi_\delta(s), s, \eta_\delta(s)),\\ \text{(b) } \eta_\delta'(s) = -D_xH_\delta(\xi_\delta(s), s, \eta_\delta(s)).\end{cases}\tag{5.8}$$

To carry out the analysis, we also invoke the following inclusion result.

Lemma 5.3. *Assume that the Hamiltonian $H = H(x,t,p)$ satisfies (H1), (H2), (H4)$_{\mathrm{st}}$, (H6), and (H7). Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $\xi \in \mathcal{C}_{\min}(x,t)$. Then,*

$$B_{R(\xi(s),s)}(\xi(s)) \subset B_{R(x,t)}(x)$$

for all $s \in (0,t)$, where $R(x,t)$ is defined in (1.8).

Proof. Let $s \in (0,t)$. We aim to prove

$$|x - \xi(s)| + R(\xi(s), s) \leqq R(x,t).$$

First, consider the case $A_2 > 0$. By the computation alanogue to the proof of (4.14), we have

$$|x - \xi(s)| \leqq \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2(t-s)} - 1).$$

Using this, we compute

$$\begin{aligned}|x-\xi(s)| + R(\xi(s),s) &= |x-\xi(s)| + \left(\frac{B_2}{A_2} + |\xi(s)|\right)(e^{A_2 s} - 1)\\ &\leqq |x-\xi(s)| + \left(\frac{B_2}{A_2} + |x-\xi(s)| + |x|\right)(e^{A_2 s} - 1)\\ &= |x-\xi(s)|e^{A_2 s} + \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2 s} - 1)\\ &\leqq \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2(t-s)} - 1)e^{A_2 s} + \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2 s} - 1)\\ &= \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2 t} - 1) = R(x,t).\end{aligned}$$

If $A_2 = 0$, the proof is more straightforward. Indeed, by the computation alanogue to the proof of (4.14), we have

$$|x - \xi(s)| \leqq B_2(t - s),$$

and hence

$$|x - \xi(s)| + R(\xi(s), s) \leqq B_2(t - s) + B_2 s = B_2 t = R(x, t).$$

Therefore, in both cases, we obtain the desired conclusion. □

We now proceed to the proof of Theorem 1.7. The overall strategy is analogous to that employed in Theorem 1.3, with a key difference being the need for a precise error estimate of the form: $|u_\delta(x_\delta, t_\delta) - u(\xi_\delta(0))|$, where ξ_δ solves approximate Hamiltonian systems.

Proof of Theorem 1.7. Let $\delta \in (0, 1]$ and H_δ be the approximate Hamiltonian defined in (2.1). Let u_δ be the viscosity solution of (1.11)–(1.2). Fix any $p \in D_x^- u(x, t)$. Then, by [6, Lemma II.2.4], there exist sequences $\{(x_\delta, t_\delta)\}_{\delta\in(0,1]} \subset \mathbb{R}^n \times (0, T)$ and $\{p_\delta\}_{\delta\in(0,1]} \subset \mathbb{R}^n$ such that

$$p_\delta \in D_x^- u_\delta(x_\delta, t_\delta), \quad \lim_{\delta\to+0} (x_\delta, t_\delta, p_\delta) = (x, t, p).$$

Since u_δ is semi-concave, it is differentiable at (x_δ, t_δ); that is, $(x_\delta, t_\delta) \in \mathcal{R}(u_\delta)$ by Proposition 2.7(1). Now, let $(\xi_\delta, \eta_\delta) \in C^1([0, 1])^2$ be the solution of the approximate system (5.8) with the terminal condition

$$\xi_\delta(t_\delta) = x_\delta, \quad \eta_\delta(t_\delta) = p_\delta = D_x u_\delta(x_\delta, t_\delta).$$

By Proposition 4.6, we know that $(\xi_\delta(s), s) \in \mathcal{R}(u_\delta)$ for all $s \in (0, t_\delta)$, and that $\eta_\delta(0) \in D^- u_0(\xi_\delta(0))$. In particular,

$$(u_\delta)_t(\xi_\delta(s), s) + H_\delta(\xi_\delta(s), s, D_x u_\delta(\xi_\delta(s), s)) = 0.$$

Let E_δ be defined as in (5.5). Then, by a computation analogous to (5.4), we have

$$\begin{aligned}
\frac{d}{ds} u_\delta(\xi_\delta(s), s) &= \langle D_x u_\delta(\xi_\delta(s), s), \xi_\delta'(s)\rangle + (u_\delta)_t(\xi_\delta(s), s) \\
&= \langle \eta_\delta(s), D_p H_\delta(\xi_\delta(s), s, \eta_\delta(s))\rangle + (u_\delta)_t(\xi_\delta(s), s) \\
&= -E_\delta(\xi_\delta(s), s, \eta_\delta(s)) + H_\delta(\xi_\delta(s), s, \eta_\delta(s)) + (u_\delta)_t(\xi_\delta(s), s) \\
&= -E_\delta(\xi_\delta(s), s, \eta_\delta(s)) + H_\delta(\xi_\delta(s), s, D_x u_\delta(\xi_\delta(s), s)) + (u_\delta)_t(\xi_\delta(s), s) \\
&= -E_\delta(\xi_\delta(s), s, \eta_\delta(s)).
\end{aligned} \tag{5.9}$$

Integrating both sides over $[0, t_\delta]$, we find that

$$u_\delta(x_\delta, t_\delta) = u_0(\xi_\delta(0)) - \int_0^{t_\delta} E_\delta(\xi_\delta(s), s, \eta_\delta(s))\, ds.$$

By Lemma 5.3, we have

$$\xi_\delta(s) \in \overline{B_{R_\delta(x_\delta,t_\delta)}(x_\delta)} \tag{5.10}$$

for all $s \in [0, t_\delta]$, which implies that the existence of a constant $K > 0$, independent of δ and s, such that $|\xi_\delta(s)| \leqq K$ for all $\delta \in (0, 1]$ and $s \in [0, t_\delta]$. Hence, Lemma 5.2 yields

$$|E_\delta(\xi_\delta(s), s, \eta_\delta(s))| \leqq (A_2|\xi_\delta(s)| + A_2\delta + B_2 + 1)\delta \leqq (A_2 K + A_2\delta + B_2 + 1)\delta,$$

and therefore

$$|u_\delta(x_\delta, t_\delta) - u_0(\xi_\delta(0))| \leqq \int_0^{t_\delta} |E_\delta(\xi_\delta(s), s, \eta_\delta(s))|\, ds \leqq (A_2 K + A_2\delta + B_2 + 1)\delta t_\delta. \tag{5.11}$$

Now, fix any $\alpha > 0$ and choose $\delta > 0$ small enough so that

$$|x - x_\delta| < \frac{\alpha}{2}, \quad R_\delta(x_\delta, t_\delta) < R(x, t) + \frac{\alpha}{2} \tag{5.12}$$

and

$$|u_\delta(x_\delta, t_\delta) - u(x, t)| < \frac{\alpha}{2}, \quad (A_2 K + A_2\delta + B_2 + 1)\delta t_\delta < \frac{\alpha}{2}. \tag{5.13}$$

From (5.10) and (5.12), it follows that

$$|x - \xi_\delta(0)| \leqq |x - x_\delta| + |x_\delta - \xi_\delta(0)| < \frac{\alpha}{2} + R_\delta(x_\delta, t_\delta) < R(x,t) + \alpha, \tag{5.14}$$

and from (5.11) and (5.13), we obtain

$$|u_0(\xi_\delta(0)) - \gamma| \leqq |u_0(\xi_\delta(0)) - u_\delta(x_\delta, t_\delta)| + |u_\delta(x_\delta, t_\delta) - u(x,t)| < \alpha. \tag{5.15}$$

Now, using (4.8), we have

$$|\eta_\delta(0)|e^{-C_1 t_\delta} - \beta(1 - e^{-C_1 t_\delta}) \leqq |p_\delta| \leqq |\eta_\delta(0)|e^{C_1 t_\delta} + \beta(e^{C_1 t_\delta} - 1).$$

Recalling that $\eta_\delta(0) \in D^- u_0(\xi_\delta(0))$, we deduce from (5.14) and (5.15) that

$$\begin{aligned}
&\inf\left\{|q| \;\middle|\; q \in D^- u_0(y),\, y \in \overline{B_{R(x,t)+\alpha}(x)},\, |u_0(y) - \gamma| \leqq \alpha\right\} \cdot e^{-C_1 t_\delta} - \beta(1 - e^{-C_1 t_\delta}) \\
&\leqq |p_\delta| \leqq \sup\left\{|q| \;\middle|\; q \in D^- u_0(y),\, y \in \overline{B_{R(x,t)+\delta}(x)},\, |u_0(y) - \gamma| \leqq \alpha\right\} \cdot e^{C_1 t_\delta} + \beta(e^{C_1 t_\delta} - 1).
\end{aligned}$$

Finally, taking the limits $\delta \to +0$ and $\alpha \to +0$, we obtain the desired conclusion. □

5.3 m-homogeneous Hamiltonian

The method developed in the previous section can also apply to a Hamiltonian which is independent of t, and is positively homogeneous of degree $m > 1$ with respect to p. In this section, we impose the following assumption:

(H8)$_m$ The Hamiltonian H is independent of t, and there exists $m > 1$ such that $H(x, \mu p) = \mu^m H(x,p)$ for all $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\mu \geqq 0$.

Note that no Hamiltonian H can satisfy both (H2) and (H8)$_m$ simultaneously. Therefore, we introduce a localized version of (H2) as follows:

(H2)$_L$ There exist constants $L > 0$ and $A_2, B_2 \geqq 0$ such that

$$|H(x,p) - H(x,q)| \leqq (A_2|x| + B_2)|p - q|$$

for all $x \in \mathbb{R}^n$ and $p, q \in \overline{B_L(0)}$.

Throughout this section, the function $R(x,t)$ is defined as in (1.8) using constants A_2 and B_2 appearing in (H2)$_L$.

Assume now that H is smooth and satisfies (H8)$_m$. Then, in a similar way to (5.2), we have

$$\langle D_p H(x,p), p\rangle = mH(x,p) \tag{5.16}$$

for all $(x,p) \in \mathbb{R}^n \times \mathbb{R}^n$. By a similar calculation to (5.4), we deduce from (5.16) that

$$\begin{aligned}
\frac{d}{ds} u(\xi(s), s) &= \langle D_x u(\xi(s), s), \xi'(s)\rangle + u_t(\xi(s), s) \\
&= \langle \eta(s), D_p H(\xi(s), \eta(s))\rangle + u_t(\xi(s), s) \\
&= mH(\xi(s), \eta(s)) + u_t(\xi(s), s) \\
&= (m-1)H(\xi(s), \eta(s)) + \{H(\xi(s), D_x u(\xi(s), s)) + u_t(\xi(s), s)\} \\
&= (m-1)H(\xi(s), \eta(s)).
\end{aligned} \tag{5.17}$$

Note that the final term is generally nonzero, but it is constant in s. In fact, since the Hamiltonian H is now independent of t, it is constant along $(\xi(s), \eta(s))$. Therefore, we have $H(\xi(s), \eta(s)) = H(\xi(0), \eta(0))$, and

$$u(x,t) = u_0(\xi(0)) + (m-1)tH(\xi(0), \eta(0)).$$

This observation implies that the gradients of u depend on the initial gradients *near* the same level-set of the initial data. If the Hamiltonian H is smooth and strictly convex, the above reasoning holds without further modification or approximation. For a general (possibly nonsmooth) m-homogeneous H, the arguments can be made rigorous via mollification and careful error estimates, following the ideas in the previous section. We omit the full details, as they proceed in a completely analogous fashion.

We first state the result for a smooth and strictly convex Hamiltonian.

Theorem 5.4 (Gradient estimates for an m-homogeneous, smooth, and strictly convex Hamiltonian)**.** *Assume that the Hamiltonian $H = H(x, p)$ satisfies (H1), (H2)$_L$, (H4)$_{\mathrm{st}}$, (H6), (H7), and (H8)$_m$. Let $u \in C(\mathbb{R}^n \times [0, T))$ be the viscosity solution of* (1.3)–(1.2)*. Let $(x, t) \in \mathbb{R}^n \times (0, T)$ and set $\gamma := u(x, t)$. Assume that the constant L appearing in (H2)$_L$ satisfies*

$$L \geqq e^{C_1 t}\|Du_0\|_{L^\infty(\mathbb{R}^n)} + \beta(e^{C_1 t} - 1). \tag{5.18}$$

Then,

$$I^m(x, t; u_0, \gamma)e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq S^m(x, t; u_0, \gamma)e^{C_1 t} + \beta(e^{C_1 t} - 1)$$

for all $p \in D_x^- u(x, t)$, where

$$S^m(x, t; u_0, \gamma) := \sup\left\{ |p| \;\middle|\; \begin{array}{l} p \in D^- u_0(y),\ y \in \overline{B_{R(x,t)}(x)}, \\ u_0(y) + (m-1)tH(y, p) = \gamma \end{array} \right\},$$

$$I^m(x, t; u_0, \gamma) := \inf\left\{ |p| \;\middle|\; \begin{array}{l} p \in D^- u_0(y),\ y \in \overline{B_{R(x,t)}(x)}, \\ u_0(y) + (m-1)tH(y, p) = \gamma \end{array} \right\}.$$

Proof. It suffices to check that the estimate (4.14) in Proposition 4.11 still holds under assumptions of this theorem. Let (ξ, η) be the same one as Proposition 4.11. Then, by (1.16) and (5.18), we see that

$$|\eta(s)| = |D_x u(\xi(s), s)| \leqq e^{C_1 t}\|Du_0\|_{L^\infty(\mathbb{R}^n)} + \beta(e^{C_1 t} - 1) \leqq L$$

for all $s \in (0, t)$. Now, applying (H2)$_L$, we have

$$|D_p H(\xi(s), \eta(s))| \leqq A_2|\xi(s)| + B_2.$$

This estimate enables us to derive the same bound as in (4.14). Therefore, the desired result follows as a direct consequence of Theorem 4.12, applied to the m-homogeneous case. □

We now turn our attention to the case where the Hamiltonian is not necessarily smooth. In this setting, we establish an error estimate that plays a role analogous to that in Lemma 5.2.

Lemma 5.5. *Assume that the Hamiltonian $H = H(x, p)$ satisfies (H2)$_L$, (H8)$_m$, and* (1.19)*. Let $\delta \in (0, 1]$, and H_δ be the approximate Hamiltonian defined in* (2.1)*. Let us define the error term*

$$E_\delta^m(x, p) := mH_\delta(x, p) - \langle D_p H_\delta(x, p), p\rangle.$$

Fix any $L' \in (0, L)$. Then,

$$|E_\delta^m(x, p)| \leqq (A_2|x| + A_2\delta + B_2 + m + (m-1)|p|)\delta \tag{5.19}$$

for all $\delta \in (0, L - L')$ and $(x, p) \in \mathbb{R}^n \times B_{L'}(0)$.

Proof. We begin by estimating $k_\delta(p) := \delta\sqrt{|p|^2 + 1}$. Then, a direct computation yields

$$mk_\delta(p) - \langle Dk_\delta(p), p\rangle = m\delta\sqrt{|p|^2 + 1} - \left\langle \frac{\delta p}{\sqrt{|p|^2 + 1}}, p \right\rangle = \frac{m\delta}{\sqrt{|p|^2 + 1}} + \frac{(m-1)\delta|p|^2}{\sqrt{|p|^2 + 1}}.$$

Next, fix $\delta \in (0, L - L')$ and set $z = (x, p) \in \mathbb{R}^n \times B_{L'}(0)$. Let $w = (y, q) \in B_\delta(0)$. Then, since $z - w \in \mathbb{R}^n \times B_L(0)$, we may apply (H2)$_L$ to obtain

$$|D_p H(z - w)| \leqq A_2|x - y| + B_2$$

almost everywhere. Following a similar calculation to the proof of Lemma 5.2, we have

$$\begin{aligned} E_\delta^m(x, p) &= m(H * \rho_\delta)(x, p) + mk_\delta(p) - \langle D_p(H * \rho_\delta)(x, p), p\rangle - \langle Dk_\delta(p), p\rangle \\ &= \frac{m\delta}{\sqrt{|p|^2 + 1}} + \frac{(m-1)\delta|p|^2}{\sqrt{|p|^2 + 1}} - J_2, \end{aligned}$$

where J_2 is the same integral term as in the proof of Lemma 5.2. Hence, by (5.7),

$$\begin{aligned} |E_\delta^m(x, p)| &\leqq \frac{m\delta}{\sqrt{|p|^2 + 1}} + \frac{(m-1)\delta|p|^2}{\sqrt{|p|^2 + 1}} + |J_2| \\ &\leqq m\delta + (m-1)\delta|p| + (A_2|x| + A_2\delta + B_2)\delta \\ &= \{m + (m-1)|p| + A_2|x| + A_2\delta + B_2\}\delta, \end{aligned}$$

which completes the proof. □

Theorem 5.6 (Gradient estimates for an m-homogeneous Hamiltonian)**.** *Assume that the Hamiltonian $H = H(x,p)$ satisfies (H1), (H2)$_L$, (H4), (H6), and (H8)$_m$. Let $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of* (1.3)–(1.2). *Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and set $\gamma := u(x,t)$. Assume that the constant L appearing in (H2)$_L$ satisfies*

$$L > e^{C_1 t}\|Du_0\|_{L^\infty(\mathbb{R}^n)} + \beta(e^{C_1 t} - 1).$$

Then,

$$\underline{I}^m(x,t;u_0,\gamma)e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq \overline{S}^m(x,t;u_0,\gamma)e^{C_1 t} + \beta(e^{C_1 t} - 1)$$

for all $p \in D_x^- u(x,t)$, where

$$\overline{S}^m(x,t;u_0,\gamma) := \lim_{\alpha\to+0} \sup\left\{ |p| \;\middle|\; \begin{array}{l} p \in D^-u_0(y), \quad y \in \overline{B_{R(x,t)+\alpha}(x)}, \\ |u_0(y) + (m-1)tH(y,p) - \gamma| \leqq \alpha \end{array} \right\},$$

$$\underline{I}^m(x,t;u_0,\gamma) := \lim_{\alpha\to+0} \inf\left\{ |p| \;\middle|\; \begin{array}{l} p \in D^-u_0(y), \quad y \in \overline{B_{R(x,t)+\alpha}(x)}, \\ |u_0(y) + (m-1)tH(y,p) - \gamma| \leqq \alpha \end{array} \right\}.$$

Proof. We follow the notation and the strategy employed in the proof of Theorem 1.7. From the derivation analogous to (5.9) and (5.17), we obtain

$$\begin{aligned} \frac{d}{ds}u_\delta(\xi_\delta(s),s) &= -E_\delta^m(\xi_\delta(s),\eta_\delta(s)) + (m-1)H_\delta(\xi_\delta(s),\eta_\delta(s)) \\ &= -E_\delta^m(\xi_\delta(s),\eta_\delta(s)) + (m-1)H_\delta(\xi_\delta(0),\eta_\delta(0)), \end{aligned}$$

where we have used the fact that H_δ remains constant along $(\xi_\delta(s),\eta_\delta(s))$. Integrating both sides over $[0,t_\delta]$, we find

$$u_\delta(x_\delta,t_\delta) = u_0(\xi_\delta(0)) + (m-1)t_\delta H_\delta(\xi_\delta(0),\eta_\delta(0)) - \int_0^{t_\delta} E_\delta^m(\xi_\delta(s),\eta_\delta(s))\,ds. \tag{5.20}$$

Now, let $L' \in (0,L)$ be a constant such that

$$L' > e^{C_1 t}\|Du_0\|_{L^\infty(\mathbb{R}^n)} + \beta(e^{C_1 t} - 1).$$

Applying (1.16) to the solution u_δ of (1.11)–(1.2), we deduce that

$$|\eta_\delta(s)| = |D_x u_\delta(\xi_\delta(s),s)| \leqq e^{C_1 t_\delta}\|Du_0\|_{L^\infty(\mathbb{R}^n)} + \beta(e^{C_1 t_\delta} - 1)$$

for all $s \in (0,t_\delta)$. Since $t_\delta \to t$ as $\delta \to +0$, we have

$$|\eta_\delta(s)| < L'$$

for all $s \in (0,t_\delta)$ provided that $\delta > 0$ is sufficiently small. Thus, the estimate (5.19) in Lemma 5.5 is applicable at $(x,p) = (\xi_\delta(s),\eta_\delta(s))$. Consequently, we obtain

$$\begin{aligned} |E_\delta^m(\xi_\delta(s),\eta_\delta(s))| &\leqq (A_2|\xi_\delta(s)| + A_2\delta + B_2 + m + (m-1)|\eta_\delta(s)|)\delta \\ &\leqq (A_2K + A_2\delta + B_2 + m + (m-1)L')\delta, \end{aligned}$$

where K is a constant independent of δ that provides a uniform bound on $|\xi_\delta(s)|$, as established earlier. Substituting this estimate into (5.20), and proceeding with arguments parallel to those used in the proof of Theorem 1.7, we obtain the desired conclusion. □

Chapter 6

Examples

In this chapter, we provide several illustrative examples to demonstrate the sharpness and applicability of our main results. In particular, we highlight cases where the derived gradient estimates are optimal or cannot be improved under the given assumptions.

6.1 Linear Hamiltonian

We begin by considering a Hamiltonian that is linear in the variable u, namely,

$$u_t(x,t) + \lambda u(x,t) + H_0(x,t,D_x u(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T). \tag{6.1}$$

Here, the Hamiltonian is defined by $H(x,t,u,p) = \lambda u + H_0(x,t,p)$, where $\lambda \in \mathbb{R}$, and $H_0 \in C(\mathbb{R}^n \times [0,T] \times \mathbb{R}^n)$ satisfies (H1), (H2), (H4), and (H6). Under this formulation, H satisfies (H3) with $K_3 = |\lambda|$, and also satisfies (H5). However, as demonstrated in Example 6.6(iv), the estimate in Theorem 1.3 may not be optimal when $\lambda < 0$. In such cases, a more precise gradient estimate can be obtained through a direct computation. The following proposition provides an analogue of Proposition 4.9 for the linear Hamiltonian in (6.1).

Proposition 6.1. *Assume that H_0 satisfies (H1), (H2), (H4)$_{\rm st}$, (H6), and (H7). Let $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of (6.1)–(1.2). Let $(x,t) \in \mathcal{R}(u)$ and let $(\xi,\eta) \in C^1([0,t])^2$ be the solution of (4.3)–(4.5). If $\lambda \neq \pm C_1$, then*

$$\begin{cases} |D_x u(x,t) - e^{-\lambda t}\eta(0)| \leqq \dfrac{\beta C_1}{C_1+\lambda}(e^{C_1 t} - e^{-\lambda t}) + |D_x u(x,t)|(e^{C_1 t} - 1), \\ |D_x u(x,t) - e^{-\lambda t}\eta(0)| \leqq \dfrac{\beta C_1}{C_1-\lambda}(e^{(C_1-\lambda)t} - 1) + |\eta(0)|(e^{(C_1-\lambda)t} - e^{-\lambda t}), \end{cases} \tag{6.2}$$

and

$$\begin{aligned} &|D_x u(x,t) - e^{C_1 t}\eta(0)| \leqq \beta C_1 t e^{C_1 t} + |D_x u(x,t)|(e^{C_1 t} - 1) && \text{if } \lambda = -C_1, \\ &|D_x u(x,t) - e^{-C_1 t}\eta(0)| \leqq \beta C_1 t + |\eta(0)|(1 - e^{-C_1 t}) && \text{if } \lambda = C_1, \end{aligned}$$

where C_1 and β are constants from (H1). Moreover, if $\lambda \neq \pm C_1$,

$$\begin{aligned} &|\eta(0)|e^{-(C_1+\lambda)t} - \frac{\beta C_1}{C_1+\lambda}(1 - e^{-(C_1+\lambda)t}) \\ &\qquad \leqq |D_x u(x,t)| \leqq |\eta(0)|e^{(C_1-\lambda)t} + \frac{\beta C_1}{C_1-\lambda}(e^{(C_1-\lambda)t} - 1) \end{aligned}$$

and

$$\begin{aligned} |D_x u(x,t)| &\geqq |\eta(0)| - \beta C_1 t \quad \text{if } \lambda = -C_1, \\ |D_x u(x,t)| &\leqq |\eta(0)| + \beta C_1 t \quad \text{if } \lambda = C_1. \end{aligned}$$

Proof. The estimates can be derived in a manner analogous to those in Proposition 4.9. We provide the proof of the first inequality in (6.2). Throughout the argument, we assume $\lambda \neq 0$. From (4.3)(b), we obtain

$$\begin{aligned}\eta'(s) &= -D_xH(\xi(s), s, u_\xi(s), \eta(s)) - D_uH(\xi(s), s, u_\xi(s), \eta(s))\eta(s) \\ &= -D_xH_0(\xi(s), s, \eta(s)) - \lambda\eta(s).\end{aligned}$$

Let us define $\zeta(s) := e^{\lambda s}\eta(s)$. Then, ζ is a solution of

$$\zeta'(s) = -e^{\lambda s} D_xH_0(\xi(s), s, e^{-\lambda s}\zeta(s)). \tag{6.3}$$

Fix $\tau \in [0, t)$. Integrating both sides of (6.3) over the interval $[\tau, t]$ and applying (H1), we obtain

$$\begin{aligned}|\zeta(t) - \zeta(\tau)| &\leqq \int_\tau^t e^{\lambda s}|D_xH_0(\xi(s), s, e^{-\lambda s}\zeta(s))|\, ds \\ &\leqq C_1 \int_\tau^t (\beta e^{\lambda s} + |\zeta(s)|)\, ds \\ &\leqq \frac{\beta C_1}{\lambda}(e^{\lambda t} - e^{\lambda \tau}) + C_1|\zeta(t)|(t - \tau) + C_1 \int_\tau^t |\zeta(t) - \zeta(s)|\, ds.\end{aligned}$$

Applying Gronwall's lemma and taking $\tau = 0$, we have

$$\begin{aligned}|\zeta(t) - \zeta(0)| \leqq\ & \frac{\beta C_1}{\lambda}(e^{\lambda t} - 1) + C_1|\zeta(t)|t \\ &+ \frac{\beta {C_1}^2}{\lambda} \int_0^t (e^{\lambda t}e^{C_1 s} - e^{(C_1+\lambda)s})\, ds + {C_1}^2|\zeta(t)| \int_0^t (t - s)e^{C_1 s}\, ds.\end{aligned}$$

If $\lambda \neq -C_1$, we compute

$$\begin{aligned}&\frac{\beta {C_1}^2}{\lambda} \int_0^t (e^{\lambda t}e^{C_1 s} - e^{(C_1+\lambda)s})\, ds \\ &= \frac{\beta C_1}{\lambda}\left(e^{\lambda t}(e^{C_1 t} - 1) - \frac{C_1}{C_1 + \lambda}(e^{(C_1+\lambda)t} - 1)\right) \\ &= \frac{\beta C_1}{\lambda}(e^{(C_1+\lambda)t} - e^{\lambda t}) - \frac{\beta {C_1}^2}{\lambda(C_1 + \lambda)}(e^{(C_1+\lambda)t} - 1)\end{aligned}$$

and

$$\begin{aligned}{C_1}^2|\zeta(t)| \int_0^t (t - s)e^{C_1 s}\, ds &= C_1|\zeta(t)| \left[(t - s)e^{C_1 s}\right]_0^t + C_1|\zeta(t)| \int_0^t e^{C_1 s}\, ds \\ &= -C_1|\zeta(t)|t + |\zeta(t)|(e^{C_1 t} - 1).\end{aligned}$$

Combining them, we obtain

$$\begin{aligned}|\zeta(t) - \zeta(0)| &\leqq \frac{\beta C_1}{\lambda}(e^{\lambda t} - 1) + C_1|\zeta(t)|t + \frac{\beta C_1}{\lambda}(e^{(C_1+\lambda)t} - e^{\lambda t}) \\ &\qquad - \frac{\beta {C_1}^2}{\lambda(C_1 + \lambda)}(e^{(C_1+\lambda)t} - 1) - C_1|\zeta(t)|t + |\zeta(t)|(e^{C_1 t} - 1) \\ &= \frac{\beta C_1}{C_1 + \lambda}(e^{(C_1+\lambda)t} - 1) + |\zeta(t)|(e^{C_1 t} - 1).\end{aligned}$$

Since $\zeta(s) = e^{\lambda s}\eta(s)$, we have

$$|e^{\lambda t}\eta(t) - \eta(0)| \leqq \frac{\beta C_1}{C_1 + \lambda}(e^{(C_1+\lambda)t} - 1) + |e^{\lambda t}\eta(t)|(e^{C_1 t} - 1),$$

which is the first estimate in (6.2) since $\eta(t) = D_xu(x, t)$. □

By Propositions 4.6, 4.11, and 6.1, we obtain the following theorem. We emphasize that, in contrast to the setting of Theorem 1.3, the parameter λ is allowed to be negative.

Theorem 6.2. *Assume that H_0 satisfies (H1), (H2), (H4), and (H6). Let $u \in C(\mathbb{R}^n \times [0,T))$ be the viscosity solution of* (6.1)–(1.2)*. Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and $p \in D_x^- u(x,t)$. If $\lambda \neq \pm C_1$, then*

$$\begin{aligned}&\underline{I}(x,t;u_0)e^{-(C_1+\lambda)t} - \frac{\beta C_1}{C_1+\lambda}(1-e^{-(C_1+\lambda)t})\\ &\leqq |p| \leqq \overline{S}(x,t;u_0)e^{(C_1-\lambda)t} + \frac{\beta C_1}{C_1-\lambda}(e^{(C_1-\lambda)t}-1)\end{aligned}$$

and

$$\begin{aligned}\underline{I}(x,t;u_0) - \beta C_1 t \leqq |p| &\leqq \overline{S}(x,t;u_0)e^{2C_1t} + \frac{\beta}{2}(e^{2C_1t}-1) \qquad &\text{if } \lambda = -C_1,\\ \underline{I}(x,t;u_0)e^{-2C_1t} - \frac{\beta}{2}(1-e^{-2C_1t}) \leqq |p| &\leqq \overline{S}(x,t;u_0) + \beta C_1 t \qquad &\text{if } \lambda = C_1.\end{aligned}$$

Remark 6.3. If, in addition, the Hamiltonian H_0 satisfies the homogeneity condition (H8), then sharper gradient estimates for solutions to (6.1)–(1.2) can be obtained by directly applying the result in Corollary 1.5. Indeed, let us define $v(x,t) := e^{\lambda t}u(x,t)$. Then, by [6, Exercise 2.3] and (H8), the function v is a viscosity solution of

$$v_t(x,t) + H_0(x,t,D_x v(x,t)) = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Applying the estimate (1.16) to v and using the identity $D_x^- v(x,t) = e^{\lambda t} D_x^- u(x,t)$, we obtain

$$|p| \geqq \underline{I}(x,t;u_0)e^{-(C_1+\lambda)t} - \beta(e^{-\lambda t} - e^{-(C_1+\lambda)t}) \tag{6.4}$$

for all $(x,t) \in \mathbb{R}^n \times (0,T)$ and $p \in D_x^- u(x,t)$. A similar argument yields an analogue of the result in Theorem 1.7, providing both lower and upper bounds for the gradient when the Hamiltonian H_0 satisfies further structural properties.

We now compare lower bounds established in Theorem 6.2 and inequality (6.4). For $\theta > 0$, we define

$$\begin{aligned}M(t) &:= \begin{cases}\theta e^{-(C_1+\lambda)t} - \dfrac{\beta C_1}{C_1+\lambda}(1-e^{-(C_1+\lambda)t}) & \text{if } \lambda \neq -C_1,\\ \theta - \beta C_1 t & \text{if } \lambda = -C_1,\end{cases}\\ m(t) &:= \theta e^{-(C_1+\lambda)t} - \beta(e^{-\lambda t} - e^{-(C_1+\lambda)t}).\end{aligned}$$

Note that $M(0) = m(0) = \theta$, and $M(t) = m(t)$ for all $t \in (0,T)$ if any of the following conditions holds: $C_1 = 0$, $\beta = 0$, or $\lambda = 0$.

Now suppose that $(C_1, \beta, \lambda) \neq (0,0,0)$. It then follows that

(1) If $\lambda > 0$, then $M(t) < m(t)$ for all $t \in (0,T)$.

(2) If $\lambda < 0$, then $M(t) > m(t)$ for all $t \in (0,T)$.

To verify this, first consider the case $\lambda = -C_1 < 0$. Then

$$M(t) = \theta - C_1 t, \quad m(t) = \theta - (e^{C_1 t} - 1).$$

Since $e^{C_1 t} - 1 > C_1 t$ for all $t \in (0,T)$, it follows that $M(t) > m(t)$.

Now consider the case $\lambda \neq -C_1$ and define the function

$$f(t) := M(t) - m(t) = -\frac{\lambda}{C_1+\lambda}e^{-(C_1+\lambda)t} + e^{-\lambda t} - \frac{C_1}{C_1+\lambda}.$$

A direct computation shows that

$$f'(t) = \lambda e^{-(C_1+\lambda)t} - \lambda e^{-\lambda t} = \lambda e^{-\lambda t}(e^{-C_1 t} - 1).$$

Since $e^{-C_1 t} - 1 < 0$ for all $t \in (0,T)$, we have

$$f'(t) < 0 \quad \text{if } \lambda > 0, \qquad f'(t) > 0 \quad \text{if } \lambda < 0 \text{ and } \lambda \neq -C_1.$$

Together with $f(0) = 0$, we conclude

$$M(t) < m(t) \quad \text{if } \lambda > 0, \qquad M(t) > m(t) \quad \text{if } \lambda < 0 \text{ and } \lambda \neq -C_1.$$

Remark 6.4. In [18, 31], the authors study modified level-set equations of the form

$$u_t(x,t) + H_0(x,t,D_x u(x,t)) = u(x,t)G(x,t,D_x u(x,t)) \quad \text{in } \mathbb{R}^n \times (0,T),$$

where H_0 satisfies (H8), and G is a suitable auxiliary function. They demonstrate that such equations preserve the zero level-set of the standard level-set formulation, while simultaneously regulating the gradient magnitude of u near the interface so that it remains close to 1. In the special case of the transport Hamiltonian $H_0(x,p) = \langle x,p\rangle$, the function G is given as $G \equiv 1$, which corresponds to the linear Hamiltonian (6.1) with $\lambda = -1$ (see also (6.6)). However, it is unclear whether the results in this thesis can be extended to a general function G, and this remains an interesting direction for future investigation.

Remark 6.5. In [34], the following equation is studied as an approximation to the signed distance function (see also Remark 4.16):

$$(u^\varepsilon)_t(x,t) = H_0(x,t,D_x u^\varepsilon(x,t)) + \varepsilon\beta(u^\varepsilon(x,t))h(D_x u^\varepsilon(x,t)) \quad \text{in } \mathbb{R}^n \times (0,T), \tag{6.5}$$

where $\varepsilon > 0$ is a small parameter, H_0 satisfies (H8), $\beta(u) := \frac{u}{\sqrt{u^2+\varepsilon_0^2}}$ with $\varepsilon_0 > 0$, and $h(p) := 1 - |p|$. Despite the convexity of H_0, the full Hamiltonian in (6.5) is not necessarily convex in the gradient variable p, which presents a serious obstacle for applying the results developed in this thesis. Further analysis is required to understand how these nonconvex effects influence the gradient structure of the solution.

6.2 Concrete equations

In this section, we show several examples to illustrate our results.

Example 6.6 (Transport equation)**.** (i) We begin with the transport equation

$$u_t(x,t) + \lambda u(x,t) + \langle x, D_x u(x,t)\rangle = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{6.6}$$

for $\lambda \in \mathbb{R}$, which arises in the context of modified level-set equations proposed in [18, 31] (see also Remark 6.4). The associated linear Hamiltonian $H(x,u,p) = \lambda u + \langle x,p\rangle$ satisfies (H1) with $C_1 = 1$, $\beta = 0$, (H2) with $A_2 = 1$, $B_2 = 0$, (H3) with $K_3 = |\lambda|$, and also satisfies (H4), (H5), (H7), (H8), and (1.19). We thus have $R(x,t) = |x|(e^t - 1)$. Although H does not satisfy (H6), the results developed in this thesis remain applicable by suitably modifying H outside the set $\overline{B_M(0)} \times \overline{B_M(0)} \times \mathbb{R}^n$ for a sufficiently large constant $M > 0$, so that the modified Hamiltonian fulfills (H6). Nevertheless, to maintain clarity and simplicity in the presentation, we continue to work with the original Hamiltonian H throughout the remainder of this discussion.

By a direct calculation, the viscosity solution of (6.6)–(1.2) is given by

$$u(x,t) = e^{-\lambda t}u_0(xe^{-t}). \tag{6.7}$$

Therefore, the spatial subgradients of u are given by

$$D_x^- u(x,t) = \{e^{-(1+\lambda)t}p \mid p \in D^- u_0(xe^{-t})\}$$

for all $(x,t) \in \mathbb{R}^n \times (0,T)$. Since

$$|xe^{-t} - x| = e^{-t}|x|(e^t - 1) = e^{-t}R(x,t) \leqq R(x,t),$$

we have $xe^{-t} \in \overline{B_{R(x,t)}(x)}$. It follows that

$$I(x,t;u_0)e^{-(1+\lambda)t} \leqq |p| \leqq S(x,t;u_0)e^{-(1+\lambda)t} \leqq S(x,t;u_0)e^{(1-\lambda)t}$$

for all $p \in D_x^- u(x,t)$. These inequalities provide exactly the same upper and lower bounds as those established in Theorems 1.3 and 6.2. Moreover, under the assumption (U), we obtain

$$|p| \geqq \theta e^{-(1+\lambda)t}$$

for all $(x,t) \in \mathcal{E}(x_0,r)$ and $p \in D_x^- u(x,t)$, which corresponds to the lower bound in Theorems 1.2, 1.4, and 6.2.

If $\lambda = 0$, then (6.7) implies that $u(x,t) = u_0(xe^{-t})$. It follows that

$$\underline{I}(x,t;u_0,u(x,t))e^{-t} \leqq |p| \leqq \overline{S}(x,t;u_0,u(x,t))e^{-t} \leqq \overline{S}(x,t;u_0,u(x,t))e^{t}.$$

These inequalities provide exactly the same upper and lower bound as those established in Theorem 1.7.
(ii) We now consider the initial data

$$u_0(x) = \max\{1 - |x|, 0\}. \tag{6.8}$$

The subgradients of u_0 are given explicitly by

$$D^- u_0(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \left\{ -\dfrac{x}{|x|} \right\} & \text{if } 0 < |x| < 1, \\ \left\{ -\dfrac{sx}{|x|} \;\middle|\; s \in [0,1] \right\} & \text{if } |x| = 1, \\ \{0\} & \text{if } |x| > 1. \end{cases}$$

Thus, the assumption (U) reduces to the condition

$$|p| = 1 \tag{6.9}$$

for all $x \in B_1(0)$ and $p \in D^- u_0(x)$. Substituting (6.8) into (6.7), we obtain

$$u(x,t) = \max\{e^{-\lambda t} - e^{-(1+\lambda)t}|x|, 0\}.$$

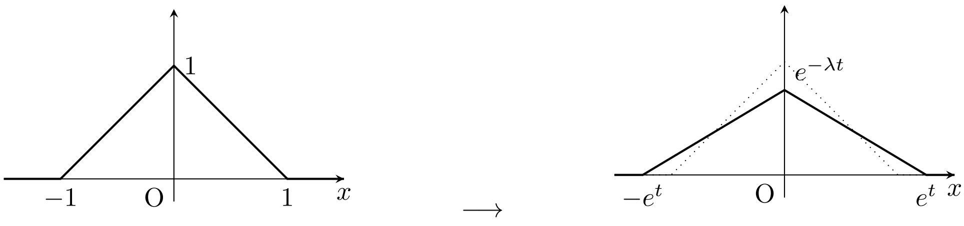
The graphs of $u_0(x)$ and $u(x,t)$ are illustrated in Figure 6.1. We can explicitly describe (1.7) as

$$\begin{aligned} \mathcal{E}(0,1) &= \{(x,t) \in B_1(0) \times (0,T) \mid |x| + |x|(e^t - 1) < 1\} \\ &= B_{e^{-t}}(0) \times (0,T). \end{aligned}$$

From Figure 6.1(B), we observe that

$$|p| = e^{-(1+\lambda)t}$$

for all $(x,t) \in B_{e^t}(0) \times (0,T)$ and $p \in D_x^- u(x,t)$, which shows that the lower bound in Theorems 1.2, 1.4, and 6.2 is sharp. However, the domain of dependence (1.7) is not optimal in this example.



(a) $u_0(x) = \max\{1 - |x|, 0\}$.

(b) $u(x,t) = \max\{e^{-\lambda t} - e^{-(1+\lambda)t}|x|, 0\}$.

Figure 6.1: Solution to (6.6)–(1.2).

(iii) When $\lambda = 0$, one may wonder if it is possible to replace $\overline{S}(x,t;u_0,\gamma)$ and $\underline{I}(x,t;u_0,\gamma)$ in Theorem 1.7 by

$$\begin{aligned} S(x,t;u_0,\gamma) &:= \sup\left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_{R(x,t)}(x)},\, u_0(y) = \gamma \right\}, \\ I(x,t;u_0,\gamma) &:= \inf\left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_{R(x,t)}(x)},\, u_0(y) = \gamma \right\}, \end{aligned}$$

respectively, but it is impossible. In fact, let $\alpha \in (0,1)$, $K > 1$, and

$$u_0(x) = \max\{-|x|^{1+\alpha}, -K\}.$$

Let u be the solution of (6.6)–(1.2) with $\lambda = 0$, which is given by (6.7). Fix $t \in (0, T)$ and we consider the subgradients $D_x^- u(0,t)$. By (6.7), we have $u(0,t) = u_0(0) = 0$. From the definition of u_0, we deduce that if $y \in \overline{B_{R(0,t)}(0)}$ and $u_0(y) = 0$, then $y = 0$. However, we have $D^- u_0(0) = \emptyset$ and therefore

$$\left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_{R(x,t)}(x)},\, u_0(y) = 0 \right\} = \emptyset.$$

Thus, the definitions of $S(x,t;u_0,0)$ and $I(x,t;u_0,0)$ do not make sense.

(iv) Next, we consider the equation

$$u_t(x,t) + \lambda u(x,t) - \langle x, D_x u(x,t) \rangle = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \tag{6.10}$$

whose associated linear Hamiltonian is given by $H(x,u,p) = \lambda u - \langle x, p \rangle$. This Hamiltonian satisfies (H1)–(H5), (H7), (H8), and (1.19) with same constants as (i). The viscosity solution of (6.10)–(1.2) is given by

$$u(x,t) = e^{-\lambda t} u_0(xe^t).$$

Therefore, the spatial subgradients of u are given by

$$D_x^- u(x,t) = \{ e^{(1-\lambda)t} p \mid p \in D^- u_0(xe^t) \}.$$

Since

$$|xe^t - x| = |x|(e^t - 1) = R(x,t),$$

we have $xe^t \in \overline{B_{R(x,t)}(x)}$. It follows that

$$I(x,t;u_0) e^{-(1+\lambda)t} \leqq I(x,t;u_0) e^{(1-\lambda)t} \leqq |p| \leqq S(x,t;u_0) e^{(1-\lambda)t}$$

for all $p \in D_x^- u(x,t)$. These inequalities provide exactly the same upper and lower bounds as those established in Theorems 1.3 and 6.2.

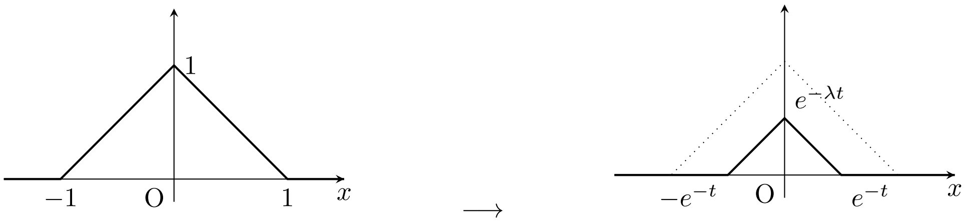
For the initial data specified in (6.8), we obtain

$$u(x,t) = \max\{ e^{-\lambda t} - e^{(1-\lambda)t}|x|, 0 \}.$$

The graphs of $u_0(x)$ and $u(x,t)$ are shown in Figure 6.2. From Figure 6.2(B), we observe that

$$|p| = e^{(1-\lambda)t}$$

for all $(x,t) \in B_{e^{-t}}(0) \times (0,T)$ and $p \in D_x^- u(x,t)$. This implies that the domain (1.7) is optimal in this case, whereas the estimate (1.15) is not.



(a) $u_0(x) = \max\{1 - |x|, 0\}$.

(b) $u(x,t) = \max\{e^{-\lambda t} - e^{(1-\lambda)t}|x|, 0\}$.

Figure 6.2: Solution to (6.10)–(1.2).

Example 6.7 (Eikonal equation)**.** We next consider the eikonal-type equation

$$u_t(x,t) + \lambda u(x,t) + c|D_x u(x,t)| = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \tag{6.11}$$

for $\lambda \in \mathbb{R}$ and $c > 0$. The corresponding Hamiltonian is given by $H(u,p) = \lambda u + c|p|$, which satisfies (H1) with $C_1 = 0$, $\beta = 0$, (H2) with $A_2 = 0$, $B_2 = c$, (H3) with $K_3 = |\lambda|$, and also satisfies (H4)–(H6), (H8), and (1.19). We thus have $R(x,t) = ct$. By applying the Hopf-Lax formula, the viscosity solution of (6.11)–(1.2) is given by

$$u(x,t) = \min_{y \in \overline{B_{ct}(x)}} e^{-\lambda t} u_0(y). \tag{6.12}$$

(i) Let us consider the initial data u_0 as (6.8). Substituting (6.8) into (6.12), we compute

$$u(x,t) = \max\{e^{-\lambda t}(1 - ct - |x|), 0\}.$$

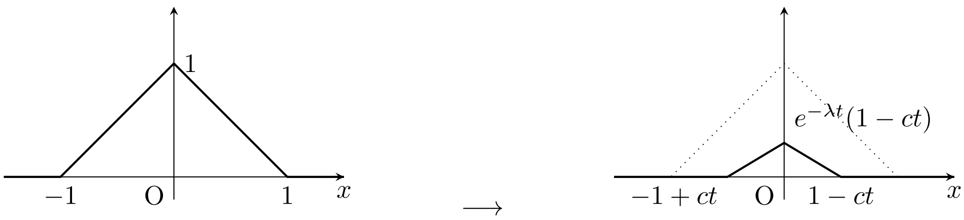
The graphs of $u_0(x)$ and $u(x,t)$ are shown in Figure 6.3. In this setting, the condition (1.7) becomes

$$\begin{aligned}\mathcal{E}(0,1) &= \{(x,t) \in B_1(0) \times (0,T) \mid |x| + ct < 1\}\\ &= B_{1-ct}(0) \times (0,T).\end{aligned}$$

By a direct computation from the graph in Figure 6.3(b), it follows that

$$|p| = e^{-\lambda t}$$

for all $(x,t) \in B_{1-ct}(0) \times (0,T)$ and $p \in D_x^- u(x,t)$, which confirms that both the lower bound in Theorems 1.2, 1.4, and 6.2 and the domain (1.7) are optimal in this case.



(a) $u_0(x) = \max\{1 - |x|, 0\}$. (b) $u(x,t) = \max\{e^{-\lambda t}(1 - ct - |x|), 0\}$.

Figure 6.3: Solution to (6.11)–(1.2).

Moreover, assume $\lambda = 0$. Let us consider the subgradients $D_x^- u(a, \frac{1}{2c})$ for $|a| = \frac{1}{2}$. It is clear that

$$D_x^- u\left(a, \frac{1}{2c}\right) = \left\{ -\frac{sa}{|a|} \;\middle|\; s \in [0,1] \right\}.$$

Furthermore, we notice that $R(a, \frac{1}{2c}) = \frac{1}{2}$ and

$$\underline{I}\left(a, \frac{1}{2c}; u_0\right) = \lim_{\alpha \to +0} \inf \left\{ |p| \;\middle|\; p \in D^- u_0(y),\ y \in \overline{B_{\frac{1}{2}+\alpha}(a)} \right\} = 0.$$

Therefore, we deduce that

$$|p| \geqq \underline{I}\left(a, \frac{1}{2c}; u_0\right) \tag{6.13}$$

for all $p \in D_x^- u(a, \frac{1}{2c})$, which is the assertion of Theorem 1.3. In addition, (6.13) is an optimal estimate in the sense that there is $p \in D_x^- u(a, \frac{1}{2c})$ such that $|p| = 0$. Also, we easily see that

$$\underline{I}\left(a, \frac{1}{2c}; u_0, u\left(a, \frac{1}{2c}\right)\right) = \underline{I}\left(a, \frac{1}{2c}; u_0, 0\right) = 0,$$

and hence $\underline{I}(a, \frac{1}{2c}; u_0)$ in (6.13) can be replaced by $\underline{I}(a, \frac{1}{2c}; u_0, 0)$. This is what Theorem 1.7 asserts.

We note that H satisfies neither (H7) nor $(\mathrm{H4})_{\mathrm{st}}$, and thus Theorem 4.12 cannot be applied. In fact, it is not possible to replace $\underline{I}(a, \frac{1}{2c}; u_0)$ in (6.13) by $I(a, \frac{1}{2c}; u_0)$ since we have $I(a, \frac{1}{2c}; u_0) = 1$. This means that the assertion of Theorem 4.12 fails in the present case.

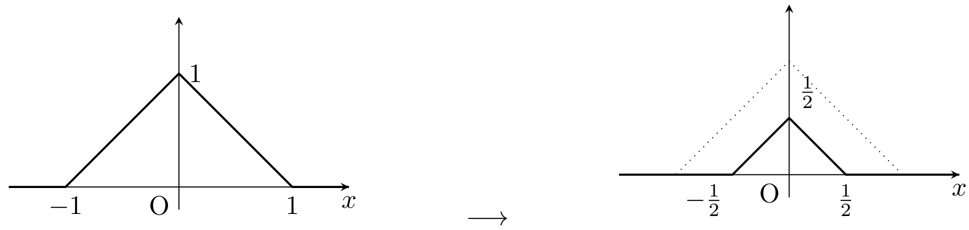
Finally, we mention the optimality of $R(a, \frac{1}{2c}) = \frac{1}{2}$. If $R \in (0, \frac{1}{2})$, then

$$\underline{I}_R := \lim_{\alpha \to +0} \inf \left\{ |p| \;\middle|\; p \in D^- u_0(y),\ y \in \overline{B_{R+\alpha}(a)} \right\} = 1,$$

which shows that (6.13) does not hold if we replace $\underline{I}(a, \frac{1}{2c}; u_0)$ by $\underline{I}_R$ above. The graphs of $u_0(x)$ and $u(x, \frac{1}{2c})$ are shown in Figure 6.4.

(ii) When $\lambda = 0$, we change the initial data to

$$u_0(x) = \max\{1 - |x|, 0\} + f(x),$$



(a) $u_0(x) = \max\{1 - |x|, 0\}$. (b) $u(x, \frac{1}{2c}) = \max\{\frac{1}{2} - |x|, 0\}$.

Figure 6.4: Solution to (6.11)–(1.2).

where $f \in C^1(\mathbb{R}^n)$ is a nonnegative function such that

$$\operatorname{supp}(f) \subset B_{\frac{1}{2}}(a), \qquad \inf_{x \in B_{\frac{1}{2}}(a)} |Df(x)| > 1.$$

for $|a| = \frac{1}{2}$. Let u be the viscosity solution of (6.11)–(1.2). Then, we have $u(x, \frac{1}{2c}) = \max\{\frac{1}{2} - |x|, 0\}$ again, but the estimate (6.13), which is the result of Theorem 1.3, becomes worse. Indeed, due to the additional term f, we have

$$\underline{I}\left(a, \frac{1}{2c}; u_0\right) = \inf_{x \in B_{\frac{1}{2}}(a)} |Df(x)| > 1.$$

In contrast, the estimate in Theorem 1.7 still keeps the optimality. In fact, since we only need to see the level-sets of u_0 close to 0-level, we find that $\underline{I}(a, \frac{1}{2c}; u_0, 0) = 0$.

(iii) We next consider (6.11) with a spatial source term

$$u_t(x,t) + \lambda u(x,t) + c|D_x u(x,t)| = f(x) \quad \text{in } \mathbb{R}^n \times (0,T), \tag{6.14}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous with a Lipschitz constant denoted by L_f. The corresponding Hamiltonian is given by $H(x,u,p) = -f(x) + \lambda u + c|p|$, which satisfies (H1) with $C_1 = L_f$, $\beta = 1$, (H2) with $A_2 = 0$, $B_2 = c$, (H3) with $K_3 = |\lambda|$, and also satisfies (H4)–(H6), (H8), and (1.19). The viscosity solution of (6.14) can be obtained through the representation formula in optimal control theory (see [6, Proposition III.3.5]).

To simplify the exposition, let us consider the one-dimensional case $n = 1$, with $\lambda = 0$, $f(x) = x$, and initial data

$$u_0(x) = 1 - |x|. \tag{6.15}$$

For any $t \in [0, 1)$, the viscosity solution of (6.14)–(1.2) is given by

$$\begin{aligned} u(x,t) &= \inf_{y \in B_{ct}(x)} \left(u_0(y) + \frac{x+y}{2} t \right) \\ &= \begin{cases} 1 + (t-1)x - ct + \dfrac{c}{2}t^2 & \left(x \geqq \dfrac{c}{2}t^2 \right), \\ 1 + (t+1)x - ct - \dfrac{c}{2}t^2 & \left(x \leqq \dfrac{c}{2}t^2 \right). \end{cases} \end{aligned}$$

The graphs of $u_0(x)$ and $u(x,t)$ are shown in Figure 6.5. Due to the presence of the source term f, the solution u no longer exhibits symmetry with respect to $x = 0$. From the explicit formula and the graph in Figure 6.5(b), we obtain

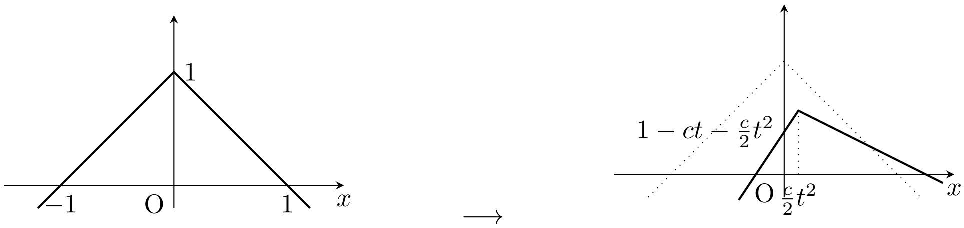
$$|p| \geqq 1 - t$$

for all $(x,t) \in \mathbb{R}^n \times (0,1)$ and $p \in D_x^- u(x,t)$. On the other hand, lower bounds predicted by Theorems 1.2 and 1.4 are given respectively by

$$l(t) = \sqrt{e^{-\frac{5}{2}t} - 2t}, \quad L(t) = 2e^{-t} - 1.$$

It is readily verified that $1 - t > L(t) > l(t)$ for all $t \in [0, t_l)$, where the thresholds are given numerically by

$$t_l \approx 0.2606, \quad t_L = \log 2 \approx 0.3010.$$



(a) $u_0(x) = 1 - |x|$. (b) $u(x,t) = \inf_{y \in B_{ct}(x)} \left(u_0(y) + \frac{x+y}{2}t \right)$.

Figure 6.5: Solution to (6.14)–(1.2).

Example 6.8. Let us next study the case where the Hamiltonian H is smooth and strictly convex. Let $H(p) = \frac{1}{2}|p|^2$. Then, H satisfies (H1) with $C_1 = 0$, $\beta = 0$, $(\mathrm{H4})_{\mathrm{st}}$, (H6), (H7), and $(\mathrm{H8})_m$ with $m = 2$. Since (H2) is not fulfilled, we will modify H or localize the assumption later.

The equation associated with H is

$$u_t(x,t) + \frac{1}{2}|D_x u(x,t)|^2 = 0 \quad \text{in } \mathbb{R}^n \times (0,T). \tag{6.16}$$

The viscosity solution of (6.16)–(1.2) is given by the *inf-convolution*:

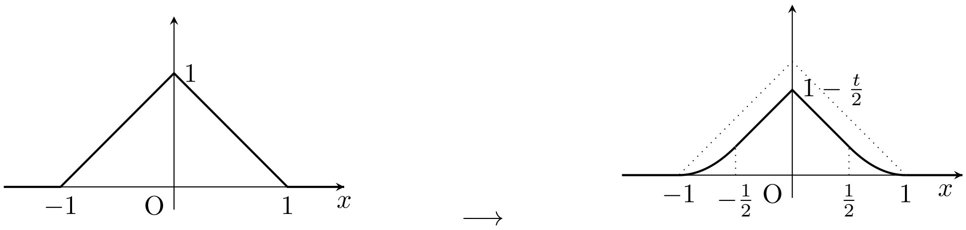
$$u(x,t) = \inf_{y \in \mathbb{R}^n} \left\{ u_0(y) + \frac{1}{2t}|x-y|^2 \right\}. \tag{6.17}$$

This is a special case of the Hopf-Lax formula ([21], [26, Chapters 3 and 10]).

Let us consider the initial data u_0 given by (6.8). By a straightforward calculation, we find that all $y \in B_1(0)$ can be removed from the range of the infimum in (6.17). Namely,

$$u(x,t) = \inf_{y \in \mathbb{R}^n,\, |y| \geqq 1} \left\{ u_0(y) + \frac{1}{2t}|x-y|^2 \right\} = \begin{cases} 1 - |x| - \dfrac{t}{2} & \text{if } |x| \leqq 1 - t, \\ \dfrac{1}{2t}(|x|-1)^2 & \text{if } 1 - t \leqq |x| \leqq 1, \\ 0 & \text{if } |x| \geqq 1. \end{cases}$$

See Figure 6.6 for the graphs of $u_0(x)$ and $u(x,t)$.



(a) $u_0(x) = \max\{1 - |x|, 0\}$. (b) $u(x,t) = \inf_{|y| \geqq 1} \left\{ u_0(y) + \frac{1}{2t}|x-y|^2 \right\}$.

Figure 6.6: Solution of (6.16)–(1.2).

(i) We now have $\|Du_0\|_{L^\infty(\mathbb{R}^n)} = 1$. By a property of the inf-convolution, the Lipschitz constant of $u(\cdot,t)$ does not increase. Namely, we have $\|D_x u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leqq 1$ for all $t \in (0,T)$. The above observation implies that we may change the value of $H(p)$ for $|p| > 1$. For example, for any $\varepsilon > 0$ small, we take a convex Hamiltonian $\tilde{H} \in C^2(\mathbb{R}^n)$ such that

$$\tilde{H} = H \quad \text{in } \overline{B_1(0)}, \quad |D_p \tilde{H}| \leqq 1 + \varepsilon \quad \text{in } \mathbb{R}^n.$$

Then, $\tilde{H}$ satisfies (H2) with $A_2 = 0$, $B_2 = 1 + \varepsilon$, and we have $R(x,t) = (1+\varepsilon)t$.

We investigate the subgradients $D_x^- u(a, \frac{1}{2})$ for $|a| = \frac{1}{2}$. We see that $u(a, \frac{1}{2}) = \frac{1}{4}$ and $D_x^- u(a, \frac{1}{2}) = 1$. Moreover, we deduce that

$$I\left(a, \frac{1}{2}; u_0\right) = \inf\left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_{\frac{1}{2}+\frac{\varepsilon}{2}}(a)} \right\} = 0.$$

In consequence, we find that

$$|p| \geqq I\left(a, \frac{1}{2}; u_0\right)$$

for all $p \in D_x^- u\left(a, \frac{1}{2}\right)$, which is the assertion of Theorem 4.12. Unlike (6.13) in Example 6.7(i), we do not need $\underline{I}(a, \frac{1}{2}; u_0)$ for the estimate.

(ii) For any $L > 0$, the present Hamiltonian H satisfies $(\mathrm{H2})_L$. Since $\|Du_0\|_{L^\infty(\mathbb{R}^n)} = 1$, we see that

$$e^{C_1 t}\|Du_0\|_{L^\infty(\mathbb{R}^n)} + \beta(e^{C_1 t} - 1) = 1.$$

Let us now take $L = 1$ and apply Theorem 5.4. When $L = 1$, the assumption $(\mathrm{H2})_L$ is fulfilled for $A_2 = 0$, $B_2 = 1$. This yields $R(x,t) = t$, which is smaller than the previous one in (i).

Recall that

$$I^2(x, t; u_0, u(x,t)) = \inf\left\{|p| \;\middle|\; p \in D^- u_0(y),\ y \in \overline{B_{R(x,t)}(x)},\ u_0(y) + \frac{1}{2}t|p|^2 = u(x,t)\right\}$$

when $H(p) = \frac{1}{2}|p|^2$ and $m = 2$ (see Theorem 5.4). At a point $(x,t) = (a, \frac{1}{2})$ for $|a| = \frac{1}{2}$, we have

$$I^2\left(a, \frac{1}{2}; u_0, \frac{1}{4}\right) = \inf\left\{|p| \;\middle|\; p \in D^- u_0(y),\ y \in \overline{B_{\frac{1}{2}}(a)},\ u_0(y) + \frac{1}{4}|p|^2 = \frac{1}{4}\right\}.$$

When $|y| = 1$, there exists $p \in D^- u_0(y)$ such that $u_0(y) + \frac{1}{4}|p|^2 = \frac{1}{4}$, i.e., $|p| = 1$. This shows that $I^2(a, \frac{1}{2}; u_0, \frac{1}{4}) = 1$ and therefore

$$|p| \geqq I^2\left(a, \frac{1}{2}; u_0, \frac{1}{4}\right)$$

for all $p \in D_x^- u\left(a, \frac{1}{2}\right)$, which is the assertion of Theorem 5.4.

Remark 6.9. For the Hamiltonian $H(p) = \frac{1}{2}|p|^2$, gradient estimates similar to Theorem 5.4 are derived from several properties of the inf-convolution. Let $(x,t) \in \mathbb{R}^n \times (0,T)$ and u be the viscosity solution of (6.16)–(1.2), which is given by (6.17).

Assume that $p \in D_x^- u(x,t)$. Then, it is known from [24, Lemma A.5] and [6, Lemmas II.4.11, II.4.12, and their proofs] that

$$p \in D^- u_0(x + tp), \quad u_0(x + tp) + \frac{1}{2}t|p|^2 = u(x,t).$$

Let us write $L = \|Du_0\|_{L^\infty(\mathbb{R}^n)}$. As already mentioned, we have $\|D_x u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leqq L$, and therefore $|p| \leqq L$. This implies that $x + tp \in \overline{B_{Lt}(x)}$. Hence,

$$\begin{aligned}
&\inf\left\{|q| \;\middle|\; q \in D^- u_0(y),\, y \in \overline{B_{Lt}(x)},\, u_0(y) + \frac{1}{2}t|q|^2 = u(x,t)\right\} \\
&\leqq |p| \leqq \sup\left\{|q| \;\middle|\; q \in D^- u_0(y),\, y \in \overline{B_{Lt}(x)},\, u_0(y) + \frac{1}{2}t|q|^2 = u(x,t)\right\}.
\end{aligned}$$

Example 6.10. This example illustrates that the lower bound gradient estimate does not, in general, hold when the Hamiltonian fails to be convex. A similar observation is discussed in [32, Example 6.5].

We consider the Hamiltonian $H(u,p) = u - c|p|$, where $c > 0$ is a constant. Clearly, this Hamiltonian is not convex with respect to p. The corresponding equation is

$$u_t(x,t) + u(x,t) - c|D_x u(x,t)| = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \tag{6.18}$$

which is the eikonal equation with the opposite sign compared to (6.11) with $\lambda = 1$. By the Hopf-Lax formula, the viscosity solution of (6.18)–(1.2) is given by

$$u(x,t) = \max_{y \in \overline{B_{ct}(x)}} e^{-t} u_0(y).$$

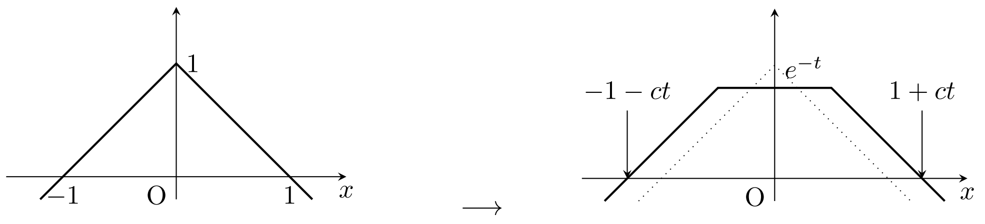
Let us now take the initial data as (6.15). The subgradients of u_0 are then given by

$$D^- u_0(x) = \begin{cases} \emptyset & \text{if } x = 0, \\ \left\{-\dfrac{x}{|x|}\right\} & \text{if } x \neq 0. \end{cases}$$

For this initial data, the solution u becomes

$$u(x,t) = \min\{e^{-t}(1 + ct - |x|), e^{-t}\}.$$

The graphs of $u_0(x)$ and $u(x,t)$ are shown in Figure 6.7. We observe that the solution u immediately develops a flat part around $x = 0$, and in particular, it holds that $D_x u(0,t) = 0$ for all $t > 0$.



(a) $u_0(x) = 1 - |x|$.

(b) $u(x,t) = \min\{e^{-t}(1 + ct - |x|), e^{-t}\}$.

Figure 6.7: Solution of (6.18)–(1.2).

Now, fix any $t > 0$. Suppose, for the sake of contradiction, that the gradient estimate in Theorem 1.3 were applicable at the point $(0,t)$. Then, it would follow that

$$|p| \geqq \underline{I}(0,t;u_0)$$

for all $p \in D_x^- u(0,t)$. Since $D_x^- u(0,t) = \{0\}$, this would imply

$$0 \geqq \underline{I}(0,t;u_0). \tag{6.19}$$

However, this inequality does not hold. Indeed, for any $R > 0$, we define

$$I_R := \inf\left\{ |p| \;\middle|\; p \in D^- u_0(y),\, y \in \overline{B_R(0)} \right\}.$$

Then, we observe that $I_R = 1$ for all $R > 0$, in particular, we have $\underline{I}(0,t;u_0) = 1$. This implies that and the inequality (6.19) is violated. Moreover, even if we attempt to replace $\underline{I}(0,t;u_0)$ with I_R for some $R > 0$, the inequality (6.19) still fails. This example thus clearly demonstrates that the convexity of the Hamiltonian is essential for the validity of the lower bound gradient estimate in Theorem 1.3.

Appendix A

Domain of dependence

In this appendix, we investigate the domains $\mathcal{D}(x_0, r)$ and $\mathcal{E}(x_0, r)$ introduced in (1.5) and (1.7), respectively, within which the gradient estimates are valid. These results are based on the analysis in [32, Section 6.3].

Let us define

$$r(x,t) = e^{(A_2+B_2+A_2|x|)t} - 1. \tag{A.1}$$

Then, we can describe $\mathcal{D}(x_0, r)$ as

$$\mathcal{D}(x_0, r) = \{(x,t) \in B_r(x_0) \times (0,T) \mid (r(x_0,t)+1)(|x-x_0|+1) - 1 < r\}.$$

Let us first begin with comparison between $r(x,t)$ and $R(x,t)$.

Proposition A.1. *Let $(x,t) \in \mathbb{R}^n \times (0,T)$. Then,*

$$R(x,t) = r(x,t) = 0 \quad \text{if } (A_2, B_2) = (0,0), \qquad R(x,t) < r(x,t) \quad \text{if } (A_2, B_2) \neq (0,0).$$

Proof. By definition, we easily see that $R(x,t) = r(x,t) = 0$ when $(A_2, B_2) = (0,0)$. Assume that $(A_2, B_2) \neq (0,0)$. First, if $A_2 = 0$ and $B_2 \neq 0$, we have

$$r(x,t) = e^{B_2 t} - 1 > B_2 t = R(x,t).$$

We next assume that $A_2 \neq 0$. Observe

$$e^{A_2 t} = \frac{A_2 t}{A_2 t e^{-A_2 t}} > \frac{1 - e^{-A_2 t}}{A_2 t e^{-A_2 t}} = \frac{e^{A_2 t} - 1}{A_2 t}.$$

Then,

$$\begin{aligned} r(x,t) &= e^{A_2 t} \cdot e^{(B_2 + A_2|x|)t} - 1 > \frac{e^{A_2 t} - 1}{A_2 t} \cdot \{(B_2 + A_2|x|)t + 1\} - 1 \\ &= (e^{A_2 t} - 1)\left\{\left(\frac{B_2}{A_2} + |x|\right) + \frac{1}{A_2 t}\right\} - 1 = R(x,t) + \frac{e^{A_2 t} - 1 - A_2 t}{A_2 t} > R(x,t). \end{aligned}$$

The proof is complete. □

Theorem A.2. *Let $x_0 \in \mathbb{R}^n$ and $r > 0$. Then,*

$$\mathcal{D}(x_0, r) = \mathcal{E}(x_0, r) \quad \text{if } (A_2, B_2) = (0,0), \qquad \mathcal{D}(x_0, r) \subsetneq \mathcal{E}(x_0, r) \quad \text{if } (A_2, B_2) \neq (0,0).$$

Proof. If $(A_2, B_2) = (0,0)$, then we easily see that $\mathcal{D}(x_0, r) = \mathcal{E}(x_0, r) = B_r(x_0) \times (0,T)$. Assume that $(A_2, B_2) \neq (0,0)$. It suffices to prove that

$$\{(x,t) \in B_r(x_0) \times (0,T) \mid (r(x_0,t)+1)(|x-x_0|+1) - 1 \leqq r\} \subset \mathcal{E}(x_0, r),$$

where "$< r$" in (1.5) was replaced by "$\leqq r$". Let us take $(x,t) \in B_r(x_0) \times (0,T)$ satisfying

$$(r(x_0,t)+1)(|x-x_0|+1) - 1 \leqq r. \tag{A.2}$$

We have to demonstrate that $R(x,t) + |x - x_0| < r$. By (A.2), we have

$$r - (R(x,t) + |x - x_0|) \geqq r(x_0,t)(|x - x_0| + 1) - R(x,t) =: J.$$

Let us prove that $J > 0$. If $A_2 = 0$, we have

$$J = (e^{B_2 t} - 1)(|x - x_0| + 1) - B_2 t > B_2 t(|x - x_0| + 1) - B_2 t \geqq 0.$$

Assume next that $A_2 \neq 0$. When $|x| \leqq |x_0|$, we have $r(x,t) \leqq r(x_0,t)$ by the definition (A.1). Thus, Proposition A.1 implies that

$$J \geqq r(x,t)(|x - x_0| + 1) - R(x,t) > R(x,t)(|x - x_0| + 1) - R(x,t) \geqq 0.$$

Suppose that $|x| > |x_0|$. Observe

$$\begin{aligned} J &= (e^{(A_2+B_2+A_2|x_0|)t} - 1)(|x - x_0| + 1) - \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2 t} - 1) \\ &\geqq (e^{(A_2+B_2+A_2|x_0|)t} - 1)(|x| - |x_0| + 1) - \left(\frac{B_2}{A_2} + |x|\right)(e^{A_2 t} - 1). \end{aligned}$$

Let us define $j(\rho)$ as the right-hand side above with $\rho = |x|$. Then, $j(\rho)$ is an affine function of ρ, and the coefficient of ρ is

$$e^{(A_2+B_2+A_2|x_0|)t} - e^{A_2 t} = e^{A_2 t}(e^{(B_2+A_2|x_0|)t} - 1) \geqq 0.$$

Hence, $j(\rho)$ is nondecreasing. Moreover,

$$j(|x_0|) = (e^{(A_2+B_2+A_2|x_0|)t} - 1) - \left(\frac{B_2}{A_2} + |x_0|\right)(e^{A_2 t} - 1) = r(x_0,t) - R(x_0,t) > 0.$$

Summarizing the above, we see that $J \geqq j(|x|) \geqq j(|x_0|) > 0$, which completes the proof. $\square$

We next investigate the geometric properties of the domains $\mathcal{D}(x_0,r)$ and $\mathcal{E}(x_0,r)$. To do so, we fix $x_0 \in \mathbb{R}^n$, $r > 0$, and assume that $(A_2, B_2) \neq (0,0)$. Figures A.1 to A.3 illustrate typical shapes of $\mathcal{D}(x_0,r)$ and $\mathcal{E}(x_0,r)$ under various parameter settings, in the case where $|x_0| < r$. The remainder of this chapter provides a detailed explanation for the constructions shown in the figures.

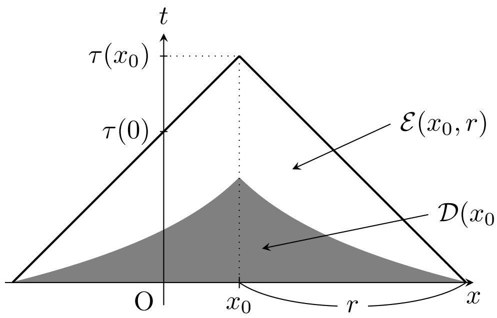


Figure A.1: $A_2 = 0$.

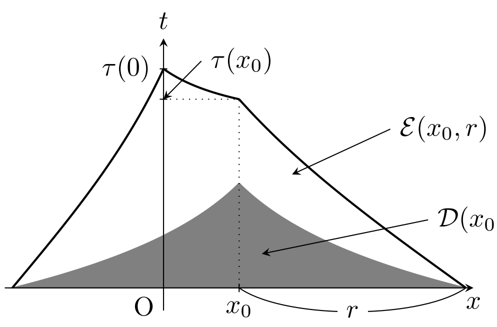


Figure A.2: $A_2(r - |x_0|) > B_2$.

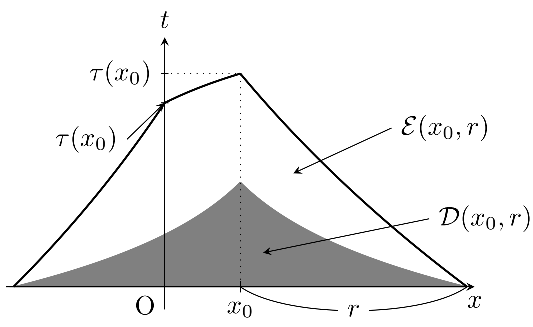


Figure A.3: $A_2(r - |x_0|) < B_2$.

We first observe that both domains share the same base, meaning their intersection with the plane $t = 0$ is given by the ball $B_r(x_0)$. However, the vertex of each domain differs. Specifically, we define a vertex of $\mathcal{D}(x_0, r)$ to be a point $(\bar{x}, \bar{t}) \in \partial\mathcal{D}(x_0, r)$ such that $\bar{t} \geqq t$ for all $(x, t) \in \mathcal{D}(x_0, r)$. A vertex of $\mathcal{E}(x_0, r)$ is defined analogously.

We begin by analyzing the domain $\mathcal{D}(x_0, r)$; see also [45, Remark 6.1]. This domain is cone-like in shape. Indeed, the defining condition $e^{(A_2+B_2+A_2|x_0|)t}(1 + |x - x_0|) < r + 1$ in (1.5) is equivalent to

$$|x - x_0| < (r + 1)e^{-(A_2+B_2+A_2|x_0|)t} - 1.$$

This implies that $\mathcal{D}(x_0, r)$ is axially symmetric around the line $x = x_0$, with the vertex located at

$$\left(x_0, \frac{1}{A_2 + B_2 + A_2|x_0|} \log(r + 1)\right).$$

In contrast, the domain $\mathcal{E}(x_0, r)$ is generally not axially symmetric. Moreover, the x-coordinate of its vertex depends on the values of A_2 and B_2, and it may not be x_0. We now analyze this structure in more detail.

Suppose first that $A_2 = 0$. Then, the function $R(x, t)$ simplifies to $R(x, t) = B_2 t$, and the condition $(x, t) \in \mathcal{E}(x_0, r)$ becomes

$$|x - x_0| < r - B_2 t.$$

Thus, $\mathcal{E}(x_0, r)$ is a cone with the vertex located at $(x_0, \frac{r}{B_2})$. See Figure A.1 for an illustration.

Now consider the case $A_2 > 0$. To identify the vertex of $\mathcal{E}(x_0, r)$, let us define $\tau : B_r(x_0) \to [0, \infty)$ such that $\tau(x)$ is the unique value for which $R(x, \tau(x)) + |x - x_0| = r$. That is,

$$\tau(x) = \frac{1}{A_2} \log\left(\frac{A_2(r - |x - x_0|)}{B_2 + A_2|x|} + 1\right). \tag{A.3}$$

Since $\mathcal{E}(x_0, r)$ is defined by the inequality $R(x, \tau(x)) + |x - x_0| < r$, it follows that the vertex lies on the boundary $\partial\mathcal{E}(x_0, r)$, specifically at a point $(x, \tau(x))$ for some $x \in B_r(x_0)$. Hence, the vertex corresponds to the point maximizing $\tau(x)$, and our task reduces to solving the optimization problem

$$\max_{x \in B_r(x_0)} \tau(x). \tag{A.4}$$

When $x_0 = 0$, the definition (1.7) implies that $\mathcal{E}(x_0, r)$ is axially symmetric around the line $x = 0$. Moreover, from (A.3), it is readily seen that the maximum of (A.4) is uniquely attained at 0. Consequently, the vertex of $\mathcal{E}(x_0, r)$ is located at $(0, \tau(0))$.

In the case where $x_0 \neq 0$, $\mathcal{E}(x_0, r)$ no longer exhibits axial symmetry, as can be inferred from (1.7). To analyze the location of the vertex, we consider the geometric observation that, for every $\rho > 0$, the point on the sphere $\partial B_\rho(0)$ that is closest to x_0 is uniquely given by $y_\rho := \frac{\rho x_0}{|x_0|}$. This fact, combined with the formula (A.3), implies that the maximum of $\tau(x)$ over the sphere $\partial B_\rho(0)$ is attained at $x = y_\rho$. The optimization problem (A.4) reduces to a one-dimensional maximization problem over the interval $I := [(|x_0| - r)_+, |x_0| + r]$, where $a_+ = \max\{a, 0\}$ denotes the plus part of $a \in \mathbb{R}$. That is,

$$\max_{\rho \in I} \tau(y_\rho), \tag{A.5}$$

where we also define $y_0 := 0$ for notational convenience.

Now, for any $\rho \in I$, we observe that $|y_\rho| = \rho$ and $|y_\rho - x_0| = |\rho - |x_0||$. Substituting these into (A.3), we obtain

$$\begin{aligned}
\tau(y_\rho) &= \frac{1}{A_2} \log\left(\frac{A_2(r - |\rho - |x_0||)}{B_2 + A_2\rho} + 1\right) \\
&= \begin{cases} \dfrac{1}{A_2} \log\left(\dfrac{-B_2 + A_2(r - |x_0|)}{B_2 + A_2\rho} + 2\right) & \text{if } \rho \in [(|x_0| - r)_+, |x_0|], \\[2ex] \dfrac{1}{A_2} \log \dfrac{B_2 + A_2(r + |x_0|)}{B_2 + A_2\rho} & \text{if } \rho \in [|x_0|, |x_0| + r]. \end{cases}
\end{aligned}$$

In what follows, we examine the monotonicity of $\tau(y_\rho)$ as a function of ρ within the interval I.

Case 1: $|x_0| \geqq r$, i.e., $0 \notin B_r(x_0)$. In this case, the function $\tau(y_\rho)$ is strictly increasing on the interval $[(|x_0| - r)_+, |x_0|]$, and strictly decreasing on $[|x_0|, |x_0| + r]$. Thus, the maximum of (A.5) over I is uniquely attained at $\rho = |x_0|$. Consequently, the vertex of $\mathcal{E}(x_0, r)$ is located at $(x_0, \tau(x_0))$.

Case 2: $|x_0| < r$, i.e., $0 \in B_r(x_0)$. In this case, the function $\tau(y_\rho)$ is still strictly decreasing on $[|x_0|, |x_0| + r]$, while its behavior on the interval $[(|x_0| - r)_+, |x_0|]$ depends on the relative size of the quantities $A_2(r - |x_0|)$ and B_2. Specifically, $\tau(y_\rho)$ satisfies the following monotonicity properties in $[(|x_0| - r)_+, |x_0|]$:

$$\tau(y_\rho) \text{ is } \begin{cases} \text{strictly decreasing} & \text{if } A_2(r - |x_0|) > B_2, \\ \text{constant} & \text{if } A_2(r - |x_0|) = B_2, \\ \text{strictly increasing} & \text{if } A_2(r - |x_0|) < B_2. \end{cases}$$

Thus, it follows that the maximum of (A.5) is attained at

$$\begin{cases} \rho = 0 & \text{if } A_2(r - |x_0|) > B_2, \\ \rho \in [0, |x_0|] & \text{if } A_2(r - |x_0|) = B_2, \\ \rho = |x_0| & \text{if } A_2(r - |x_0|) < B_2. \end{cases}$$

We therefore conclude that the vertex of $\mathcal{E}(x_0, r)$ is given by

$$\begin{cases} (0, \tau(0)) & \text{if } A_2(r - |x_0|) > B_2, \\ (1 - \mu)(0, \tau(0)) + \mu(x_0, \tau(x_0)) & \text{if } A_2(r - |x_0|) = B_2 \text{ for any } \mu \in [0, 1], \\ (x_0, \tau(x_0)) & \text{if } A_2(r - |x_0|) < B_2. \end{cases}$$

See Figures A.2 and A.3 for visual illustrations of the cases $A_2(r - |x_0|) > B_2$ and $A_2(r - |x_0|) < B_2$, respectively.

Appendix B

Comparison with the results in [32]

In this appendix, we clarify the relationship between the results obtained in this thesis and those presented in [32]. We begin by introducing some notation.

Definition B.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$. We define the *proximal subgradients* $D^-_{\mathrm{pr}} f(x)$ of f at x by

$$D^-_{\mathrm{pr}} f(x) := \left\{ p \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \text{there exist constants } K, r > 0 \text{ such that} \\ f(y) \geqq f(x) + \langle p, y - x\rangle - \dfrac{K}{2}|y-x|^2 \\ \text{for all } y \in B_r(x) \end{array} \right\}.$$

Remark B.2. (1) Assume that f is Lipschitz continuous on $\mathbb{R}^n$. Then, for any $p \in \mathbb{R}^n$ belonging to one of the sets $D^{\pm} f(x)$, $D^-_{\mathrm{pr}} f(x)$ and $D^* f(x)$, we have $|p| \leqq \|Df\|_{L^\infty(\Omega)}$.

(2) By definition, we have $D^-_{\mathrm{pr}} f(x) \subset D^- f(x)$, while the reverse inclusion does not hold in general. Indeed, for $f(y) = -|y|^{1+\alpha}$ with $\alpha \in (0, 1)$, one finds $D^- f(0) = \{0\}$ and $D^-_{\mathrm{pr}} f(0) = \emptyset$.

Definition B.3. Let $(x, t) \in \mathbb{R}^n \times (0, T)$. We define

$$\overline{S}_{\mathrm{pr}}(x, t; u_0) := \lim_{\alpha \to +0} \sup \left\{ |p| \;\middle|\; p \in D^-_{\mathrm{pr}} u_0(y),\ y \in \overline{B_{R(x,t)+\alpha}(x)} \right\},$$
$$\underline{I}_{\mathrm{pr}}(x, t; u_0) := \lim_{\alpha \to +0} \inf \left\{ |p| \;\middle|\; p \in D^-_{\mathrm{pr}} u_0(y),\ y \in \overline{B_{R(x,t)+\alpha}(x)} \right\}.$$

The following result from [32] can be stated as follows.

Proposition B.4 ([32, Theorems 1.2])**.** *Assume that the Hamiltonian $H = H(x, t, p)$ satisfies (H1), (H2), (H4), and (H6). Let $u \in C(\mathbb{R}^n \times [0, T))$ be the viscosity solution of* (1.3)–(1.2)*. Let $(x, t) \in \mathbb{R}^n \times (0, T)$ and $p \in D^-_x u(x, t)$. Then,*

$$\underline{I}_{\mathrm{pr}}(x, t; u_0) e^{-C_1 t} - \beta(1 - e^{-C_1 t}) \leqq |p| \leqq \overline{S}_{\mathrm{pr}}(x, t; u_0) e^{C_1 t} + \beta(e^{C_1 t} - 1).$$

Here, β and C_1 are constants from (H1).

The same replacement of $D^- u_0(x)$ by $D^-_{\mathrm{pr}} u_0(x)$ can be applied to Theorems 1.7 and 4.12. The proof of Proposition B.4 relies on the fact, established in [2, Theorem 3.2], that

$$\eta(0) \in D^-_{\mathrm{pr}} u_0(\xi(0)).$$

This condition is stronger than that in Proposition 4.6. However, as shown below, any element of $D^- f(x)$ can be approximated by elements of $D^-_{\mathrm{pr}} f(x_\varepsilon)$ for nearby points x_ε.

Proposition B.5. *Let $\Omega \subset \mathbb{R}^n$ be an nonempty open set, and let $x \in \mathbb{R}^n$ and $p \in D^- f(x)$. Then, there exist sequences $\{x_\varepsilon\}_{\varepsilon \in (0,1]} \subset \mathbb{R}^n$ and $\{p_\varepsilon\}_{\varepsilon \in (0,1]} \subset \mathbb{R}^n$ such that*

$$p_\varepsilon \in D^-_{\mathrm{pr}} f(x_\varepsilon), \quad x_\varepsilon \to x, \quad p_\varepsilon \to p \quad \text{as } \varepsilon \to +0.$$

Proof. Take $\phi \in C^1(\Omega)$ such that $p = D\phi(x)$ and $f - \phi$ attains a local minimum at x. We may assume that $f(x) = \phi(x)$. Since ϕ is differentiable at x, there exists an increasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\lim_{r\to+0} \omega(r) = \omega(0) = 0$ and

$$\phi(y) \geqq \phi(x) + \langle p, y - x\rangle - \omega(|y-x|)|y-x| \quad \text{for all } y \in \Omega.$$

Choose $\varepsilon_0 \in (0,1]$ so small that $\omega(\varepsilon_0) < 1$, $\overline{B_{\varepsilon_0}(x)} \subset \Omega$, and that x is a minimizer of $f - \phi$ over $\overline{B_{\varepsilon_0}(x)}$. Let $\varepsilon \in (0, \varepsilon_0]$ and define $r_\varepsilon := \varepsilon\omega(\varepsilon)$. Note that $0 < r_\varepsilon < \varepsilon$ by the choice of ε_0. We further define

$$\zeta_\varepsilon(y) := \phi(x) + \langle p, y - x\rangle - \frac{|y-x|^2}{\varepsilon} \quad (y \in \Omega).$$

Let x_ε be a minimum point of $f - \zeta_\varepsilon$ over $\overline{B_{r_\varepsilon}(x)} \subset \Omega$.

We now claim that $x_\varepsilon \in B_{r_\varepsilon}(x)$. First, by the definition of ζ_ε, we notice that

$$f(x) - \zeta_\varepsilon(x) = f(x) - \phi(x) = 0. \tag{B.1}$$

Next, for $y \in \partial B_{r_\varepsilon}(x)$, we observe

$$\begin{aligned} f(y) - \zeta_\varepsilon(y) &\geqq \phi(y) - \zeta_\varepsilon(y) \geqq -\omega(|y-x|)|y-x| + \frac{|y-x|^2}{\varepsilon} \\ &= -\omega(r_\varepsilon)r_\varepsilon + \frac{r_\varepsilon^2}{\varepsilon} = r_\varepsilon(-\omega(r_\varepsilon) + \omega(\varepsilon)) > 0. \end{aligned}$$

This and (B.1) imply that the minimizer x_ε lies in $B_{r_\varepsilon}(x)$.

By the claim, we have $p_\varepsilon := D\zeta_\varepsilon(x_\varepsilon) \in D^-_{\mathrm{pr}} f(x_\varepsilon)$. Since $|x_\varepsilon - x| < r_\varepsilon$, we easily see that $x_\varepsilon \to x$ as $\varepsilon \to +0$. Moreover, we have

$$p_\varepsilon = p - \frac{2(x_\varepsilon - x)}{\varepsilon}, \quad \frac{|x_\varepsilon - x|}{\varepsilon} < \frac{r_\varepsilon}{\varepsilon} = \omega(\varepsilon).$$

This shows that $p_\varepsilon \to p$ as $\varepsilon \to +0$. □

From Proposition B.5, we immediately obtain the following corollary.

Corollary B.6. *Under the settings in Proposition B.5, we have*

$$\begin{aligned} \sup\{|p| \mid p \in D^- f(y),\ y \in \Omega\} &= \sup\{|p| \mid p \in D^-_{\mathrm{pr}} f(y),\ y \in \Omega\}, \\ \inf\{|p| \mid p \in D^- f(y),\ y \in \Omega\} &= \inf\{|p| \mid p \in D^-_{\mathrm{pr}} f(y),\ y \in \Omega\}. \end{aligned}$$

In particular,

$$\overline{S}_{\mathrm{pr}}(x,t;u_0) = \overline{S}(x,t;u_0), \quad \underline{I}_{\mathrm{pr}}(x,t;u_0) = \underline{I}(x,t;u_0).$$

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