Stability of Metric Viscosity Solutions

（距離粘性解の安定性に関して）

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Abstract

Part I examines the stability of metric viscosity solutions under Hausdorff convergence. Let (X, d) be a metric space and let X_n, $n \in \mathbb{N} \cup \{\infty\}$, be subsets endowed with metrics d_n. Assume that the sequence of spaces (X_n, d_n) converges to a limit space (X_∞, d_∞) in Hausdorff sense. We investigate whether the relaxed half semilimit u^∞ of metric viscosity solutions u^n to Hamilton–Jacobi equations on X_n remains a metric viscosity solution of the corresponding equation on X_∞. This type of stability problem arises in domain perturbation problems and numerical constructions of solutions of PDEs on fractals.

First, when both X_n and X_∞ are compact, we establish the stability of metric viscosity solutions to the stationary Hamilton–Jacobi equation. The proof relies on a characterization of metric viscosity solutions via squared distance functions. Thus, to prove the stability result, we need to assume the following condition (H2): for arbitrary sequences $a_n \in X_n \to a_\infty \in X_\infty$ and $b_n \in X_n \to b_\infty \in X_\infty$ with respect to d, the convergence

$$d_n(a_n, b_n) \to d_\infty(a_\infty, b_\infty).$$

holds. We therefore analyze several examples that illustrate when condition (H2) holds and when it fails. Specifically, condition (H2) fails for the Koch curve, but it is satisfied by certain other fractals—such as the Sierpiński gasket and the Vicsek fractal—as well as by a junction of shrinking tubes and by lattice lines endowed with the Manhattan distance.

Finally, we extend our analysis to time-dependent Hamilton–Jacobi equations and to the case where X_n, X_∞ are noncompact. In the noncompact case, Ekeland's variational principle is used to control the maximizer of $u^n - \varphi^n$, where φ^n is an appropriate test function.

Part II is concerned with the stability of metric viscosity solutions under Gromov–Hausdorff convergence. We study the stability of metric viscosity solutions to the initial-value problem for time-dependent Hamilton–Jacobi equations when a sequence of metric spaces (X_n, d_n) converges in Gromov–Hausdorff sense to a limit space (X_∞, d_∞). Working within Gromov–Hausdorff framework removes the need to embed the spaces into a common ambient metric space and dispenses with the need for (H2).

The proof combines two main ingredients: a characterization of metric viscosity solutions via squared distance functions, and a doubling variable argument that exploits

ϵ-isometries to capture Gromov–Hausdorff convergence.

When the metric spaces X_n and X_∞ are compact, Hausdorff convergence implies Gromov–Hausdorff convergence. Hence, in this case, the stability result can be regarded as an extension of the stability result for metric viscosity solutions of time-dependent Hamilton–Jacobi equations under Hausdorff convergence given in Part I.

As an application of the stability under Gromov–Hausdorff convergence, we give a PDE proof of the stability of maximizers of the dual Kantorovich problem under measured-Gromov–Hausdorff convergence.

Each of Parts I and II of this doctoral thesis corresponds to the references [20, 19] respectively. Since all the parts are independent, there are some common definitions and similar arguments in them.

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Contents

Part I

Stability of metric viscosity solutions under Hausdorff convergence

1 Introduction

Consider the Hamilton–Jacobi equation of the generalized form

$$H(x, u(x), |\nabla_d u|(x)) = 0 \quad \text{for } x \in X. \tag{1.1}$$

Here, (X, d) is a complete geodesic space embedded in a whole space (X, d). The symbol $|\nabla_d u|$ represents a so-called local slope of an unknown function $u : X \to \mathbb{R}$:

$$|\nabla_d u|(x) := \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(x, y)} \quad \text{for } x \in X. \tag{1.2}$$

The local slope is well defined as a non-negative real number in $\mathbb{R}_+ := [0, \infty)$ as long as u is locally Lipschitz continuous. Also, H is a real-valued function on $X \times \mathbb{R} \times \mathbb{R}_+$ called a Hamiltonian.

This study mainly evaluates the stability of the Hamilton–Jacobi equation (1.1) with respect to the space (X, d). That is, we study the sequence of solutions of the Hamilton–Jacobi equation on a variable space (X_n, d_n) in (X, d) of the form

$$H_n(x, u^n(x), |\nabla_{d_n} u^n|(x)) = 0 \quad \text{for } x \in X_n \tag{1.3}$$

with the indexes $n \in \mathbb{N}$ and show that the solution u^n converges to a solution of the limit equation

$$H_\infty(x, u^\infty(x), |\nabla_{d_\infty} u^\infty|(x)) = 0 \quad \text{for } x \in X_\infty \tag{1.4}$$

under a suitable convergence of X_n, d_n and H_n.

The problem of stability with respect to the domain on which the equation is posed is strongly related to problems in domain perturbation, homogenization, and the construction of solutions to PDEs on fractals, which have been extensively studied. Due to the large number of results, we only mention those relating to the Hamilton–Jacobi equation. The stability of viscosity solutions of the Hamilton–Jacobi equations with respect to a sequence of domains appears in the proof of the existence of solutions by domain approximation. In [1], the solution of the Hamilton–Jacobi equation on the junction is constructed as the graphical limit of the optimal control in the region where the junction is fattened in two dimensions. In [5], the viscosity solution of the eikonal equation on the Sierpiński gasket is constructed using the graphical approximation of the Sierpiński gasket. To generalize the results of [5], we investigated the stability of viscosity solutions of the Hamilton–Jacobi equations when the increasing sequence of networks in a Euclidean space converges in Hausdorff sense [18].

This study aims to extend the results of the previous work [18] and to investigate for which types of a sequence of metric spaces the stability holds. In [18], the uniform

Lipschitz estimate of the family of viscosity solutions is required to extract the limit function at the Hausdorff limit. In this study, the relaxed half semilimit is used rather than the uniform Lipschitz estimate. The relaxed half semilimit is a method used to study large time behavior and homogenization, and the definition itself does not require any regularity property of the solution. This allows us to consider the stability of viscosity solutions for a sequence of metric spaces that is not necessarily monotonically increasing.

The stability of the heat equation has been studied in the celebrated paper [12]. The heat equation requires a measure structure on the space, while the Hamilton–Jacobi equation requires only the metric structure. This makes our argument much simpler than the argument for the heat equation.

As in [18], we adopted the metric viscosity solutions of Gangbo and Święch [9, 10] as a notion of solutions of the Hamilton–Jacobi equations. The metric viscosity solution is one of the notions of viscosity solutions on metric spaces to deal with fractals and networks, introduced in [11, 21, 2, 9, 10]. In particular, the metric viscosity solution in [9, 10] is very useful because the comparison theorem holds for a wide class of Hamiltonians. On the equivalence between metric viscosity solutions and other weak solutions of the eikonal equation, we refer the readers to [16].

Let us recall the essence of the metric viscosity solution of Gangbo and Święch. The idea is the same as in the classical theory of viscosity solutions. Consider the situation where the unknown function u is touched by a smooth ($C^{1,\pm}$) function ϕ called a test function at a point. Then, we use the local slope of ϕ instead of that of u in (1.1) at the touching point $\hat{x}$. The precise definition will be given in Section 2. Under such a setting, the existence and uniqueness theorem of the solution is shown, and the stability of standard type holds [22], where standard stability means that the space (X, d) does not vary.

However, the argument of standard stability cannot be applied to the stability with the space perturbation directly. The difficulty is that since both the touching point $\hat{x}$ and the test function ϕ may be perturbed, we should investigate the convergence of

$$|\nabla_{d_n}\phi_n|(x_n) \to |\nabla_{d_\infty}\phi|(\hat{x}). \tag{1.5}$$

The critical idea of the present paper to overcome this difficulty is to restrict the class of test functions to the squared distance function

$$\phi(x) = \frac{k}{2}d(\hat{a}, x)^2$$

with $k \geq 0$. Then, the condition (1.5) can be reduced to

$$d_n(a_n, x_n) \to d_\infty(\hat{a}, \hat{x}).$$

In order to restrict the class of test functions, Proposition 2.6 is established using a so-called doubling variable method.

Throughout this paper, we assume that the Hamiltonian $H_n : X_n \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous for each $n \in \mathbb{N} \cup \{\infty\}$. For $x \in X_\infty$, we define

$$S(x) := \{\text{a sequence } (x_n) \text{ converging to } x \text{ in } (\mathsf{X}, \mathsf{d}) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \cup \{\infty\}\}.$$

In view of the argument in the preceding paragraph we show the stability of the metric viscosity solutions under the assumptions

(H1) $X_n \to X_\infty$ in d-Hausdorff sense:

$$\mathsf{d}_H(X_\infty, X_n) = \max\left\{\sup_{x \in X_n} \mathsf{d}(X_\infty, x), \sup_{x \in X_\infty} \mathsf{d}(X_n, x)\right\} \to 0,$$

where $\mathsf{d}(A, x) := \inf_{a \in A} \mathsf{d}(a, x)$ is a distance function from a subset A of X.

(H2) For arbitrary $a_\infty, b_\infty \in X_\infty$ and sequences $(a_n) \in S(a_\infty)$, $(b_n) \in S(b_\infty)$, the convergence

$$d_n(a_n, b_n) \to d_\infty(a_\infty, b_\infty)$$

holds.

(H3) For arbitrary $x_\infty \in X_\infty$, $u_\infty \in \mathbb{R}$, $p_\infty \in \mathbb{R}_+$ and sequences $(x_n) \in S(x_\infty)$, $u_n \in \mathbb{R} \to u_\infty$, $p_n \in \mathbb{R}_+ \to p_\infty$, the convergence

$$H_n(x_n, u_n, p_n) \to H_\infty(x_\infty, u_\infty, p_\infty)$$

holds.

The crucial assumption (H2) means that the sequence of the distance functions d_n locally uniformly converges.

Remark 1.1. Conditions (H1) and (H2) are independent. For example, consider the arcs

$$X_n = \{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi - n^{-1}\}, \quad X_\infty = \{(\cos\theta, \sin\theta) \mid 0 \le \theta \le 2\pi\}$$

embedded in the plane $\mathbb{R}^2$. Then, condition (H2) does not hold. We remark that the geodesic distance between $(0, 0)$ and $(\cos(2\pi - n^{-1}), \sin(2\pi - n^{-1}))$ over X_n is given by $2\pi - n^{-1}$ while the geodesic distance over X_∞ is n^{-1}. This example satisfies condition (H1). An example which satisfies (H2) and does not fulfill (H1) will be given in Remark 8.4.

Now, which type of sequence of metric spaces satisfies condition (H2)? In this study, we give three main examples. The first is a pre-fractal approximation of self-similar sets. Generally, since it is difficult to find a solution of the Hamilton–Jacobi equation on fractals, our approach allows spatial discretization. However, not all fractals and their approximations satisfy condition (H2). We show positive results for the Vicsek fractal and the Sierpiński gasket. However, the Koch curve does not satisfy condition (H2). The second one is shrinking tubes with a junction as a singularly perturbed problem [1]. The third one is the shrinking lattice lines on a Euclidean plane. The lattice lines converge to the plane in Hausdorff sense and the limit metric becomes the so-called Manhattan distance (or ℓ^1-distance) reflecting the intrinsic metric on the lattice lines.

We extend our stability theorem to time-dependent problems and to noncompact metric spaces. In the noncompact case, a key tool of the proof of stability is Ekeland's variational principle [7], which corresponds to the maximum principle in general metric spaces. In this study, we establish the stability in the noncompact case on the basis of the discussion in [18, 22].

In Section 2, we recall the definition of the metric viscosity solution and discuss the equivalent condition critical to prove the stability. In Section 3, we establish the stability of the metric viscosity solutions in the locally compact case. In Sections 4, 5, and 6, we give examples satisfying assumption (H2) of the stability. In Section 7, we consider the stability of metric viscosity solutions of time-dependent problems. In Section 8, we discuss the generalization of the stability to the case when the metric spaces are not necessarily locally compact.

2 Metric viscosity solutions

The theory of metric viscosity solutions to the Hamilton–Jacobi equations must be reviewed on a general geodesic space (1.1) proposed by Gangbo and Święch. See [9, 10]. Let (X, d) be a complete geodesic space.

A central notion in this theory is the local slope as defined in (1.2). The upper and lower local slopes are introduced by

$$|\nabla_d^{\pm} u|(x) := \limsup_{y \to x} \frac{[u(y) - u(x)]_{\pm}}{d(x, y)},$$

where $[\cdot]_{\pm}$ is the plus or minus part: $[a]_{\pm} = \max\{\pm a, 0\}$. A locally Lipschitz continuous function u is called $C^{1,\pm}$ if $|\nabla_d u|$ is continuous and

$$|\nabla_d^{\pm} u| = |\nabla_d u|$$

holds on its domain.

The following lemmas are important.

Lemma 2.1 (Maximum principle)**.** *Let u, v, w be locally Lipschitz continuous functions.*

(i) Assume that $u - v$ attains a local maximum at $\hat{x}$.

- *If v is $C^{1,-}$, then*
$$|\nabla_d u|(\hat{x}) \geq |\nabla_d v|(\hat{x})$$
holds.
- *If u is $C^{1,+}$, then*
$$|\nabla_d u|(\hat{x}) \leq |\nabla_d v|(\hat{x})$$
holds.
- *If u is $C^{1,+}$ and v is $C^{1,-}$, then*
$$|\nabla_d u|(\hat{x}) = |\nabla_d v|(\hat{x})$$
holds.

(ii) Assume that $u - v + w$ attains a local maximum at $\hat{x}$.

- *If u is $C^{1,+}$ and v is $C^{1,-}$, then*
$$||\nabla_d u|(\hat{x}) - |\nabla_d v|(\hat{x})| \leq |\nabla_d w|(\hat{x})$$
holds.

Proof. We prove the first assertion in (i). The second assertion in (i) can be deduced in the same way. The remaining cases can be established by combining the first and the second argument.

By the assumption of $u - v$, we have
$$u(\hat{x}) - u(x) \geq v(\hat{x}) - v(x).$$
Taking the positive part of the inequality, we obtain
$$|u(\hat{x}) - u(x)| \geq [u(\hat{x}) - u(x)]_+ \geq [v(\hat{x}) - v(x)]_+ = [v(x) - v(\hat{x})]_-.$$
Consequently, we have
$$\frac{|u(\hat{x}) - u(x)|}{d(\hat{x}, x)} \geq \frac{[v(x) - v(\hat{x})]_-}{d(\hat{x}, x)}.$$
Since v is of class $C^{1,-}$, by passing to the $\liminf_{x \to \hat{x}}$, we have
$$|\nabla_d u|(\hat{x}) \geq |\nabla_d v|(\hat{x}),$$
which completes the proof. $\square$

Lemma 2.2 (Squared distance function). *For $a \in X$ the function*

$$\phi(x) = \frac{1}{2} d(a,x)^2$$

is $C^{1,-}$ and

$$|\nabla_d^- \phi|(x) = |\nabla_d \phi|(x) = d(a,x)$$

for all $x \in X$.

Proof. The proof of this lemma is a direct application of [10, Lemma 2.3]. □

A locally Lipschitz function $\psi(x) = \psi_1(x) + \psi_2(x)$ is a test function for subsolutions if $\psi_1 \in C^{1,-}$ and ψ_2 is a locally Lipschitz function and for supersolutions if $\psi_1 \in C^{1,+}$ and ψ_2 is a locally Lipschitz function. For a function $u : X \to \mathbb{R}$, we define its upper and lower semicontinuous envelopes u^* and u_* by

$$u^*(x) := \limsup_{y \to x} u(y)$$

and

$$u_*(x) := \liminf_{y \to x} u(y).$$

Let $|\nabla_d u|^*$ be the upper semicontinuous envelope of $|\nabla_d u|$.

Before introducing metric viscosity solutions, we extend the domain of the Hamiltonian H to $X \times \mathbb{R} \times \mathbb{R}$ by a constant extension.

Definition 2.3 (Metric viscosity solutions). Let $u : X \to \mathbb{R}$.

- u is a metric viscosity subsolution of (1.1) if for every test function $\psi = \psi_1 + \psi_2$ for subsolutions such that $u^* - \psi$ attains a local maximum at $\hat{x} \in X$ we have

  $$H_{|\nabla_d \psi_2|^*(\hat{x})}(\hat{x}, u^*(\hat{x}), |\nabla_d \psi_1|(\hat{x})) \leq 0,$$

  where $H_a(x,u,p) = \inf_{|p'-p| \leq a} H(x,u,p')$.

- u is a metric viscosity supersolution of (1.1) if for every test function $\psi = \psi_1 + \psi_2$ for supersolutions such that $u_* - \psi$ attains a local minimum at $\hat{x} \in X$ we have

  $$H^{|\nabla_d \psi_2|^*(\hat{x})}(\hat{x}, u_*(\hat{x}), |\nabla_d \psi_1|(\hat{x})) \geq 0,$$

  where $H^a(x,u,p) = \sup_{|p'-p| \leq a} H(x,u,p')$.

- u is a metric viscosity solution of (1.1) if u is both a metric viscosity subsolution and supersolution.

To obtain uniqueness of the metric viscosity solutions, we need the following assumptions introduced in [10]. Let x_0 be a fixed point in X.

(A1) H is uniformly continuous on any bounded subsets of $X \times \mathbb{R} \times \mathbb{R}$.

(A2) There exists $\nu > 0$ such that for $(x, u_1, u_2, p) \in X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with $u_1 \geq u_2$, the inequality

$$H(x, u_1, p) - H(x, u_2, p) \geq \nu(u_1 - u_2)$$

holds.

(A3) For $R > 0$, there exists a modulus of continuity ω_R depending on R such that for $(x, y, u, p) \in X \times X \times \mathbb{R} \times \mathbb{R}$, the inequality

$$|H(x, u, p) - H(y, u, p)| \leq \omega_R \left(d(x, y)(1 + |p|)\right)$$

holds if $\max\{d(x, x_0), d(y, x_0), |p|\} \leq R$.

(A4) There exists a nonnegative constant L such that for $(x, u, p_1, p_2) \in X \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the inequality

$$|H(x, u, p_1) - H(x, u, p_2)| \leq L \left(1 + d(x, x_0)\right) |p_1 - p_2|$$

holds.

Under these settings the following theorems are established in [10].

Theorem 2.4 (Comparison principle). *Assume (A1), (A2), (A3), and (A4). Let u be a metric viscosity subsolution of* (1.1) *and v be a metric viscosity supersolution of* (1.1). *Let $u, -v$ be bounded from above. Then, $u \leq v$ in X holds.*

Theorem 2.5 (Unique existence). *Assume H is continuous and there exists $\lambda > 0$ such that $H(x, u, p) - \lambda u$ is non-decreasing with respect to u variable. We also assure*

$$\sup_{x \in X} H(x, 0, 0) < \infty.$$

Then, a metric viscosity solution of (1.1) *exists. Furthermore, if we assume (A1), (A3), and (A4), the metric viscosity solution is unique.*

Here is a key proposition to show the stability, suggesting that ψ_2 can be dropped and ψ_1 reduced to the squared distance function in the test functions.

Proposition 2.6. *Let (X, d) be a locally compact geodesic space. Then, the following conditions are equivalent.*

(i) A function $u : X \to \mathbb{R}$ is a metric viscosity subsolution of (1.1).

(ii) For all $\hat{a}, \hat{x} \in X$, $k \geq 0$, if $u^*(x) - \frac{k}{2}d(\hat{a}, x)^2$ attains a local maximum at $x = \hat{x}$, then the inequality

$$H(\hat{x}, u^*(\hat{x}), kd(\hat{a}, \hat{x})) \leq 0$$

holds.

Similarly, the following conditions are equivalent.

(i') A function $u : X \to \mathbb{R}$ is a metric viscosity supersolution of (1.1).

(ii') For all $\hat{a}, \hat{x} \in X$, $k \geq 0$, if $u_*(x) + \frac{k}{2}d(\hat{a}, x)^2$ attains a local minimum at $x = \hat{x}$, then the inequality

$$H(\hat{x}, u_*(\hat{x}), kd(\hat{a}, \hat{x})) \geq 0$$

holds.

Proof. We only show the assertion for subsolutions.

To show that condition (ii) implies condition (i), test function $\psi = \psi_1 + \psi_2$ for subsolutions should be fixed such that $u^* - \psi$ attains a local maximum at $\hat{x} \in X$. Since X is locally compact, there is a neighborhood B of $\hat{x}$ such that the closure $\overline{B}$ is compact and $u^* - \psi$ attains the maximum on $\overline{B}$ at $\hat{x}$. Consider

$$(x, y) \in \overline{B} \times \overline{B} \mapsto u^*(x) - \psi(y) - \frac{1}{2\varepsilon}d(x, y)^2 - \frac{\alpha}{2}d(\hat{x}, y)^2$$

with positive ε, α. We can take a maximum point $(x_\varepsilon, y_\varepsilon) \in \overline{B} \times \overline{B}$. Then,

$$u^*(x_\varepsilon) - \psi(y_\varepsilon) - \frac{1}{2\varepsilon}d(x_\varepsilon, y_\varepsilon)^2 - \frac{\alpha}{2}d(\hat{x}, y_\varepsilon)^2 \geq u^*(\hat{x}) - \psi(\hat{x}) \geq u^*(x_\varepsilon) - \psi(x_\varepsilon)$$

and hence

$$\frac{1}{2\varepsilon}d(x_\varepsilon, y_\varepsilon)^2 + \frac{\alpha}{2}d(\hat{x}, y_\varepsilon)^2 \leq Ld(x_\varepsilon, y_\varepsilon)$$

for the (local) Lipschitz constant L of ψ. Therefore, $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{x})$ as $\varepsilon \to 0$. In particular, $x_\varepsilon, y_\varepsilon \in B$ for sufficiently small ε. Now, by the assumption of u we have

$$H(x_\varepsilon, u^*(x_\varepsilon), \frac{1}{\varepsilon}d(x_\varepsilon, y_\varepsilon)) \leq 0.$$

By Lemma 2.1 we obtain

$$||\nabla_d \psi_1|(y_\varepsilon) - \frac{1}{\varepsilon}d(x_\varepsilon, y_\varepsilon)| \leq |\nabla_d \psi_2|(y_\varepsilon) + \alpha d(\hat{x}, y_\varepsilon).$$

Combining them,

$$H_{|\nabla_d \psi_2|(y_\varepsilon) + \alpha d(\hat{x}, y_\varepsilon)}(x_\varepsilon, u^*(x_\varepsilon), |\nabla_d \psi_1|(y_\varepsilon)) \leq 0$$

and therefore

$$H_{|\nabla_d \psi_2|^*(\hat{x})}(\hat{x}, u^*(\hat{x}), |\nabla_d \psi_1|(\hat{x})) \leq 0.$$

Thus, u is a metric viscosity subsolution.

The proof of the other part of this proposition is trivial. $\square$

Remark 2.7. An analogous equivalence holds when one requires the local maximum in (ii) and the local minimum in (ii') to be strict; see Proposition 8.2 in Part I and Proposition 3.1 in Part II.

Remark 2.8. Proposition 2.6 reveals that metric viscosity solutions and classical viscosity solutions of the Hamilton-Jacobi equation are equivalent when the space is a Euclidean space $\mathbb{R}^d$ with the Euclidean norm $|\cdot|$ and the Euclidean distance $d^E(a,b) = |b-a|$. Let us illustrate the equivalence with the example of the stationary Hamilton-Jacobi equation of the form

$$H(x, u(x), |\nabla u(x)|) = 0 \quad \text{for } x \in \mathbb{R}^d \tag{2.1}$$

where the Hamiltonian $H = H(x,u,p) : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. A function u is a viscosity subsolution of (2.1) if $u^* - \phi$ with $\phi \in C^1(\mathbb{R}^d)$ has a local maximum at $x \in \mathbb{R}^d$, then

$$H(x, u^*(x), |\nabla \phi(x)|) \leq 0.$$

Similarly, a function u is a viscosity supersolution of (2.1) if $u_* - \phi$ with $\phi \in C^1(\mathbb{R}^d)$ has a local minimum at $x \in \mathbb{R}^d$, then

$$H(x, u_*(x), |\nabla \phi(x)|) \geq 0.$$

Note that $|\nabla \phi(x)|$ is the norm of the gradient $\nabla \phi$, which equals the local slope $|\nabla_{d^E} \phi|(x)$. To prove the equivalence of the viscosity solution and the metric viscosity solution of (2.1) in Euclidean space, it is sufficient to show that if property (ii) of Proposition 2.6 for u holds, then the function u is a viscosity solution. The converse is obvious from the definition of the viscosity solution. Assume u satisfies the following condition. For all $\hat{a}, \hat{x} \in \mathbb{R}^d$, $k \geq 0$, if $u^*(x) - \frac{k}{2} d(\hat{a}, x)^2$ attains a local maximum at $x = \hat{x}$, then the inequality

$$H(\hat{x}, u^*(\hat{x}), k d^E(\hat{a}, \hat{x})) \leq 0$$

holds. In the same way in Proposition 2.6, we consider

$$(x,y) \mapsto u^*(x) - \phi(y) - \frac{1}{2\varepsilon}|x-y|^2 - \frac{\alpha}{2}|y - \hat{x}|^2$$

with positive ε, α and this function has a local maximum at some $(x_\varepsilon, y_\varepsilon) \in \mathbb{R}^d \times \mathbb{R}^d$. The same calculation yields

$$H(x_\varepsilon, u^*(x_\varepsilon), \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|) \leq 0$$

and $x_\varepsilon, y_\varepsilon \to \hat{x}$ as $\varepsilon \to 0$. Since, in Euclidean space, the classical maximum principle holds, we get

$$\nabla\phi(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + \alpha(y_\varepsilon - \hat{x}) = 0.$$

Thus we can prove u is a viscosity subsolution of (2.1). Similar conclusions can be obtained for viscosity supersolutions. In conclusion, we see that the viscosity solution can be characterized using the distance d^E, as in Proposition 2.6. Since, for Euclidean space $\mathbb{R}^d$, the Euclidean distance and the distance induced by the length of the curve are equal, so the viscosity solution and the metric viscosity solution of (2.1) are equivalent.

3 Main results on stability

Let (X, d) be a proper metric space and for $n \in \mathbb{N} \cup \{\infty\}$ let X_n be a compact subset of X equipped with the intrinsic metric d_n from d. Thus (X_n, d_n) is a compact metric space. Here we say that (X, d) is proper if any closed ball in X is compact. We assume that (X_n, d_n) forms a geodesic space.

Theorem 3.1 (Stability)**.** *Assume (H1), (H2), and (H3).*

- *Let u^n be a sequence of metric viscosity subsolutions of* (1.3)*. If its upper semilimit*

$$\overline{u}^\infty(x) := \limsup_{n\to\infty,\,(x_n)\in S(x)} u^n(x_n) = \sup_{(x_n)\in S(x)} \limsup_{n\to\infty} u^n(x_n)$$

  is upper semicontinuous in d_∞ in X_∞, then $\overline{u}^\infty$ is a metric viscosity subsolution of (1.4)*.*

- *Similarly, let u^n be a sequence of metric viscosity supersolutions of* (1.3)*. If its lower semilimit*

$$\underline{u}^\infty(x) := \liminf_{n\to\infty,\,(x_n)\in S(x)} u^n(x_n) = \inf_{(x_n)\in S(x)} \liminf_{n\to\infty} u^n(x_n)$$

  is lower semicontinuous in d_∞ in X_∞, then $\underline{u}^\infty$ is a metric viscosity supersolution of (1.4)*.*

Proof. We only show the theorem for the subsolution since the proof for the supersolution is similar. Fix $\hat{a}, \hat{x} \in X_\infty$, $k \geq 0$ and assume that $\overline{u}^\infty(x) - \frac{k}{2}d_\infty(\hat{a}, x)^2$ attains a local maximum at $x = \hat{x}$ over X_∞. Note that the function

$$x \in X_\infty \mapsto \overline{u}^\infty(x) - \frac{k}{2}d_\infty(\hat{a}, x)^2 - \frac{\eta}{2}d_\infty(\hat{x}, x)^2$$

attains its strict local maximum at $x = \hat{x}$ for $\eta > 0$. By the assumption (H1), there exist sequences $a_n, \tilde{x}_n \in X_n$ satisfying $a_n \to \hat{a}$, $\tilde{x}_n \to \hat{x}$ with respect to d and $u^n(\tilde{x}_n) \to \overline{u}^\infty(\hat{x})$.

For $n \in \mathbb{N}$ consider the function

$$x \in X_n \mapsto u_n(x) - \frac{k}{2} d_n(a_n, x)^2 - \frac{\eta}{2} d_n(\tilde{x}_n, x)^2.$$

Since this is an upper semicontinuous function on (X_n, d_n), there is a local maximum point $x = x_n$. Note that

$$u^n(\tilde{x}_n) - \frac{k}{2} d_n(a_n, \tilde{x}_n)^2 \leq u^n(x_n) - \frac{k}{2} d_n(a_n, x_n)^2 - \frac{\eta}{2} d_n(\tilde{x}_n, x_n)^2$$

and the left-hand side converges to $\overline{u}^\infty(\hat{x}) - \frac{k}{2} d_\infty(\hat{a}, \hat{x})^2$. Therefore, $x_n \to \hat{x}$ in d and $u^n(x_n) \to \overline{u}^\infty(\hat{x})$. Now, take the limit of

$$(H_n)_{\eta d_n(\tilde{x}_n, x_n)}(x_n, u^n(x_n), k d_n(a_n, x_n)) \leq 0.$$

In view of (H2) and (H3) we have

$$H_\infty(\hat{x}, \overline{u}^\infty(\hat{x}), k d_\infty(\hat{a}, \hat{x})) \leq 0,$$

which completes the proof. □

Corollary 3.2. *Assume (H1), (H2), (H3) and that there exists $\lambda > 0$ such that $H_n(x, u, p) - \lambda u$ is non-decreasing for every $n \in \mathbb{N} \cup \{\infty\}$. Let u^n be the unique metric viscosity solution of* (1.3)*. We assume that $\overline{u}^\infty$ and $\underline{u}^\infty$ are upper semicontinuous and lower semicontinuous on X_∞, respectively. Furthermore, we assume (A1), (A3), (A4) for H_∞. Then, u^n converges to the unique metric viscosity solution u^∞ of* (1.4) *uniformly.*

Proof. Using stability, the upper semilimit $\overline{u}^\infty$ is a metric viscosity subsolution and the lower semilimit $\underline{u}^\infty$ is a metric viscosity supersolution of the limit equation (1.4). Now, by the comparison principle, we see that $\overline{u}^\infty \leq \underline{u}^\infty$ and therefore $\overline{u}^\infty = \underline{u}^\infty$ is a metric viscosity solution. □

Remark 3.3. Basically, we mention that the existence and uniqueness of viscosity solutions are guaranteed when the comparison theorem and Perron's method can be used. For the detailed conditions required for the Hamiltonian, see [10, 22]. For example, the unique existence of the viscosity solution of (1.1) is guaranteed for the quadratic Hamiltonian.

4 Example: Self-similar sets

4.1 Iterated function systems

Iterated function systems are self-similar sets in which for a given family of contraction mappings $F_1, \cdots, F_N$ with $N = 1, 2, 3, \cdots$, the Hutchinson operator

$$\mathcal{F}(K) = F_1(K) \cup \cdots \cup F_N(K)$$

is a contraction mapping on the space of non-empty compact subsets of X under the Hausdorff metric d_H. Hence, a unique non-empty compact set K is a fixed point of the Hutchinson operator $\mathcal{F}$, and for any non-empty compact set K_0 the iteration

$$K_{n+1} = \mathcal{F}(K_n) = F_1(K_n) \cup \cdots \cup F_N(K_n)$$

converges to K in the sense of

$$K = \bigcap_{n=0}^{\infty} \mathrm{cl}\left(\bigcup_{l=n}^{\infty} K_l\right),$$

where cl means the closure with respect to the topology d. We call K_n pre-fractals of the fractal K. For details on the iterated function systems, see [8].

4.2 The Vicsek fractal

Consider the iterated function system on two dimensional Euclidean space $(\mathbb{R}^2, d^E)$ defined by

$$F_1(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y\right), \quad F_2(x,y) = \left(\frac{1}{3}x + 2, \frac{1}{3}y\right), \quad F_3(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y + 2\right),$$

$$F_4(x,y) = \left(\frac{1}{3}x - 2, \frac{1}{3}y\right), \quad F_5(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y - 2\right),$$

and pre-fractals K_n starting from

$$K_0 = \{(x,0) \mid -3 \le x \le 3\} \cup \{(0,y) \mid -3 \le y \le 3\}.$$

The limit fractal K is called the Vicsek fractal. Each K_n is embedded in K in a distance-preserving manner.

Theorem 4.1 (Approximation of the Vicsek fractal)**.** *The pre-fractal approximation* (K_n, d_n^K) *satisfies condition (H2), i.e., for arbitrary sequences* $(a_n) \in S(a)$ *and* $(b_n) \in S(b)$

under the metric d^K, the convergence

$$d_n^K(a_n, b_n) \to d^K(a, b)$$

holds.

Proof. Since $d_n^K = d^K$ on K_n, we see that

$$|d_n^K(a_n, b_n) - d^K(a, b)| = |d^K(a_n, b_n) - d^K(a, b)| \leq d^K(a, a_n) + d^K(b, b_n) \to 0.$$

This completes the proof. □

4.3 The Sierpiński gasket

Consider the vertices of the unit triangle on two dimensional Euclidean space $(\mathbb{R}^2, d^E)$ given by

$$O_1 = \left(-\frac{1}{2}, 0\right), \quad O_2 = \left(\frac{1}{2}, 0\right), \quad O_3 = \left(0, \frac{\sqrt{3}}{2}\right).$$

The Sierpiński gasket is generated by the family of contraction mappings

$$F_i(z) = \frac{1}{2}(z - O_i) + O_i \quad \text{for } i = 1, 2, 3.$$

Consider the intrinsic metric d^K from the Euclidean metric d^E.

Lemma 4.2 ([3, Lemma 2.12])**.** *The metric d^K is equivalent to d^E. Moreover, the inequality*

$$d^E(a, b) \leq d^K(a, b) \leq 8d^E(a, b) \textit{ for all } a, b \in K$$

holds.

For details on the metric structure on the Sierpiński gasket, we refer the reader to [3, 14, 24, 25].

There are three pre-fractal approximations of the Sierpiński gasket K. The first one is graph approximation V_n starting from the vertices $V_0 = \{O_1, O_2, O_3\}$. The second one is Cantor-type approximation D_n starting from the convex hull D_0 of V_0, which is the unit triangle. The last one is network approximation G_n starting from the boundary G_0 of D_0, which is the boundary of the unit triangle. One can introduce natural geodesic metrics d_n^G and d_n^D into G_n and D_n respectively, and natural graph structure into V_n with the metric d_n^V.

Lemma 4.3. *The restriction of the metrics d^K, d_n^G and d_n^D to the vertices V_n is nothing but d_n^V:*

$$d^K(a, b) = d_n^G(a, b) = d_n^D(a, b) = d_n^V(a, b) \text{ for all } a, b \in V_n$$

holds.

We omit the proof because this is obvious from the explicit formula of the metric of the Sierpiński gasket.

The network and Cantor-type approximations satisfy condition (H2).

Theorem 4.4 (Network approximation of the Sierpiński gasket)**.** *The network approximation (G_n, d_n^G) satisfies condition (H2)*: for arbitrary sequences $(a_n) \in S(a)$ and $(b_n) \in S(b)$ under the metric d^K, the convergence

$$d_n^G(a_n, b_n) \to d^K(a, b)$$

holds.

Proof. For each a_n, b_n take the nearest points $\tilde{a}_n, \tilde{b}_n$ of V_n in the metric d^K. Note that $d^K(a_n, \tilde{a}_n), d^K(b_n, \tilde{b}_n) \leq 2^{-n}$. We then see that

$$\begin{aligned} |d_n^G(a_n, b_n) - d^K(a, b)| &\leq 2 \cdot 2^{-n} + |d_n^G(\tilde{a}_n, \tilde{b}_n) - d^K(a, b)| \\ &= 2 \cdot 2^{-n} + |d^K(\tilde{a}_n, \tilde{b}_n) - d^K(a, b)| \\ &\leq 4 \cdot 2^{-n} + d^K(a, a_n) + d^K(b, b_n) \to 0. \end{aligned}$$

This completes the proof. □

Theorem 4.5 (Cantor-type approximation of the Sierpiński gasket)**.** *The Cantor-type approximation (D_n, d_n^D) satisfies condition (H2)*: for arbitrary sequences $(a_n) \in S(a)$ and $(b_n) \in S(b)$ under the metric d^E, the convergence

$$d_n^D(a_n, b_n) \to d^K(a, b)$$

holds.

Proof. For each a_n, b_n take the nearest points $\tilde{a}_n, \tilde{b}_n$ of V_n in the metric d^E. Note that $d^E(a_n, \tilde{a}_n), d^E(b_n, \tilde{b}_n) \leq 2^{-n}$. We then see that

$$\begin{aligned} |d_n^D(a_n, b_n) - d^K(a, b)| &\leq 2 \cdot 2^{-n} + |d_n^D(\tilde{a}_n, \tilde{b}_n) - d^K(a, b)| \\ &= 2 \cdot 2^{-n} + |d^K(\tilde{a}_n, \tilde{b}_n) - d^K(a, b)| \\ &\leq 4 \cdot 2^{-n} + 8d^E(a, a_n) + 8d^E(b, b_n) \to 0. \end{aligned}$$

This completes the proof. □

4.4 On the Koch curve

We remark that every self-similar fractal and its pre-fractal need not satisfy condition (H2). For instance, consider the Koch curve, which is generated from the iterated function system on the plane

$$F_1(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y\right), \quad F_2(x,y) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y\right),$$

$$F_3(x,y) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2}, \frac{\sqrt{3}}{6}x - \frac{1}{6}y + \frac{\sqrt{3}}{6}\right), \quad F_4(x,y) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right),$$

and its pre-fractal starting from

$$K_0 = \{(x,0) \mid 0 \le x \le 1\}.$$

Then, the geodesic distance on the pre-fractal K_n between $(0,0)$ and $(1,0)$ can be calculated as

$$d_n((0,0),(1,0)) = \left(\frac{4}{3}\right)^n,$$

which diverges as $n \to \infty$.

5 Example: Junction of shrinking tubes

In order to define a shrinking tube in $\mathbb{R}^2$, set

$$e_1 = (1,0), \quad e_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad e_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$y_1 = (0,1), \quad y_2 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \quad y_3 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

For $i = 1,2,3$ define the tube parts R_i by

$$R_i = \{x = se_i + ty_i \in \mathbb{R}^2 \mid s \ge \frac{\sqrt{3}}{3n}, -\frac{1}{n} \le t \le \frac{1}{n}\}.$$

Set

$$u_1 = -\frac{2\sqrt{3}}{3n}e_3, \quad u_2 = -\frac{2\sqrt{3}}{3n}e_1, \quad u_3 = -\frac{2\sqrt{3}}{3n}e_2, \quad u_4 = u_1,$$

and define the center parts S_i by

$$S_i = \{x = su_i + tu_{i+1} \mid 0 \le s,t \le 1, s+t = 1\}.$$

Using the three parts $T_i := R_i \cup S_i$, define the shrinking tube T_n by

$$T_n = T_1 \cup T_2 \cup T_3.$$

The shrinking tube T_n converges to the Y-junction

$$J = [0,\infty)e_1 \cup [0,\infty)e_2 \cup [0,\infty)e_3$$

in the Hausdorff sense where $[0,\infty)e_i = \{se_i \mid s \geq 0\}$. We are now able to prove that the shrinking tube satisfies condition (H2) via a projection-like map.

Theorem 5.1. *The shrinking tube (T_n, d_n^T) satisfies condition (H2)*: for arbitrary sequences $(a_n) \in S(a)$ and $(b_n) \in S(b)$ under the Euclidean metric d^E, the convergence

$$d_n^T(a_n, b_n) \to d^J(a,b)$$

holds.

Proof. Take the joint part P_i of T_i, which is given by

$$P_i = \{su_i \mid 0 < s \leq 1\}.$$

We construct a projection-like map $F_n : T_n \to J$ as follows.

$$F_n(x) = (x, e_i)e_i \quad \text{if } x \in T_i \setminus P_i.$$

By the triangle inequality and the property of F_n, we see that

$$\begin{aligned}
&|d_n^T(a_n, b_n) - d^J(a,b)| \\
&\quad \leq d_n^T(F_n(a_n), a_n) + d_n^T(F_n(b_n), b_n) + |d_n^T(F_n(a_n), F_n(b_n)) - d^J(a,b)| \\
&\quad \leq 2n^{-1} + |d_n^T(F_n(a_n), F_n(b_n)) - d^J(a,b)|.
\end{aligned}$$

We can evaluate the last term as

$$\begin{aligned}
&|d_n^T(F_n(a_n), F_n(b_n)) - d^J(a,b)| \\
&\quad \leq d_n^T(F_n(b_n), b) + d_n^T(F_n(a_n), a) + |d_n^T(a,b) - d^J(a,b)| \\
&\quad = d^J(F_n(b_n), b) + d^J(F_n(a_n), a) + |d_n^T(a,b) - d^J(a,b)|.
\end{aligned}$$

Since one can easily check $d^J(F_n(a_n), a) \to 0$, $d^J(F_n(b_n), b) \to 0$, and $|d_n^T(a,b) - d^J(a,b)| \to 0$ as $n \to \infty$, the proof is now completed. □

Remark 5.2. The situation we considered is a simple case and it must be generalized. For example, the n^{-1}-neighborhood

$$X_n = \{x \in \mathsf{X} \mid \mathsf{d}(X_\infty, x) \leq n^{-1}\}$$

of a given closed set X_∞ should satisfy condition (H2) as long as X_∞ is compact. However, this generalization is left as an open problem.

6 Example: Lattice lines and the Manhattan distance

Consider the lattice lines with shrinking spacing n^{-1} for $n = 1, 2, 3, \cdots$ in the plane:

$$L_n = \left(\bigcup_{i \in \mathbb{Z}} \{n^{-1} i\} \times \mathbb{R} \right) \cup \left(\bigcup_{j \in \mathbb{Z}} \mathbb{R} \times \{n^{-1} j\} \right)$$

and the intrinsic metric d_n^L from the Euclidean metric d^E. L_n converges to the plane $\mathbb{R}^2$ as $n \to \infty$ in Hausdorff sense but its metric structure should not be the usual Euclidean one d^E. We instead introduce the so-called Manhattan distance

$$d^M((a,b),(x,y)) = |x-a| + |y-b| \quad \text{for } (a,b),(x,y) \in \mathbb{R}^2.$$

The lattice lines (L_n, d_n^L) are embedded in $(\mathbb{R}^2, d^M)$ in distance-preserving manner.

Theorem 6.1 (Lattice lines)**.** *The lattice lines (L_n, d_n^L) satisfy condition (H2)*: for arbitrary sequences $a_n \in L_n \to a \in \mathbb{R}^2$ and $b_n \in L_n \to b \in \mathbb{R}^2$ under the metric d^M, the convergence

$$d_n^L(a_n, b_n) \to d^M(a,b)$$

holds.

Proof. Since $d_n^L = d^M$ on L_n, we see that

$$|d_n^L(a_n,b_n) - d^M(a,b)| = |d^M(a_n,b_n) - d^M(a,b)| \leq d^M(a,a_n) + d^M(b,b_n) \to 0.$$

This completes the proof. □

7 Remarks on the time-dependent case

Our stability theory can be applied to the time-dependent equation immediately. Consider

$$\partial_t u^n(t,x) + H_n(t,x,u^n(t,x), |\nabla_{d_n} u^n|(t,x)) = 0 \quad \text{for } t > 0 \text{ and } x \in X_n, \tag{7.1}$$

and

$$\partial_t u^\infty(t,x) + H_\infty(t,x,u^\infty(t,x), |\nabla_{d_\infty} u^\infty|(t,x)) = 0 \quad \text{for } t > 0 \text{ and } x \in X_\infty, \tag{7.2}$$

where $\partial_t u$ denotes the derivative of the unknown function u in the time variable t. Since the time-dependence is added to H_n, condition (H3) should be modified as follows:

(H3') For arbitrary sequences $t_n \to t_\infty$ in $(0,\infty)$, $(x_n) \in S(x_\infty)$, $u_n \to u_\infty$ in $\mathbb{R}$ and $p_n \to p_\infty$ in $\mathbb{R}_+$, the convergence

$$H_n(t_n, x_n, u_n, p_n) \to H_\infty(t_\infty, x_\infty, u_\infty, p_\infty)$$

holds.

Let us review the definition of metric viscosity solutions to the time-dependent equation

$$\partial_t u(t,x) + H(t,x,u(t,x), |\nabla_d u|(t,x)) = 0 \quad \text{for } t>0 \text{ and } x \in X. \tag{7.3}$$

We call a locally Lipschitz function $\psi(t,x) = \psi_1(t,x) + \psi_2(t,x)$ a test function for subsolutions if ψ_1 and ψ_2 are locally Lipschitz and C^1 in the time variable t and ψ_1 is a $C^{1,-}$ function in the spatial variable x. We call a locally Lipschitz function $\psi(t,x) = \psi_1(t,x) + \psi_2(t,x)$ a test function for supersolutions if ψ_1 and ψ_2 are locally Lipschitz and C^1 in the time variable t and ψ_1 is a $C^{1,+}$ function in the spatial variable x.

Definition 7.1 (Metric viscosity solutions to the time-dependent problem)**.** Let $u : (0,\infty)\times X \to \mathbb{R}$.

- u is a metric viscosity subsolution of (7.3) if for every test function $\psi = \psi_1 + \psi_2$ for subsolutions such that $u^* - \psi$ attains a local maximum at $\hat{z} = (\hat{t}, \hat{x}) \in (0,\infty) \times X$ we have
$$\partial_t \psi(\hat{z}) + H_{|\nabla_d \psi_2|^*(\hat{z})}(\hat{z}, u^*(\hat{z}), |\nabla_d \psi_1|(\hat{z})) \le 0.$$
- u is a metric viscosity supersolution of (7.3) if for every test function $\psi = \psi_1 + \psi_2$ for supersolutions such that $u_* - \psi$ attains a local minimum at $\hat{z} = (\hat{t}, \hat{x}) \in (0,\infty) \times X$ we have
$$\partial_t \psi(\hat{z}) + H^{|\nabla_d \psi_2|^*(\hat{z})}(\hat{z}, u_*(\hat{z}), |\nabla_d \psi_1|(\hat{z})) \ge 0.$$
- u is a metric viscosity solution of (7.3) if u is both a metric viscosity subsolution and a metric viscosity supersolution.

The time-dependent version of our key proposition (Proposition 2.6) can be stated as follows.

Proposition 7.2. *Let (X,d) be a locally compact geodesic space. Then, the following conditions are equivalent.*

(i) A function $u : (0,\infty)\times X \to \mathbb{R}$ is a metric viscosity subsolution of (7.3)*.*

(ii) For all $\hat{a}, \hat{x} \in X$, $k \geq 0$ and a C^1 function ϕ, if $u^*(t,x) - \phi(t) - \frac{k}{2} d(\hat{a}, x)^2$ attains a local maximum at $(t,x) = (\hat{t}, \hat{x})$, then the inequality

$$\partial_t \phi(\hat{t}) + H(\hat{t}, \hat{x}, u^*(\hat{t}, \hat{x}), kd(\hat{a}, \hat{x})) \leq 0$$

holds.

Similarly, the following conditions are equivalent.

(i') A function $u : (0, \infty) \times X \to \mathbb{R}$ is a metric viscosity supersolution of (7.3).

(ii') For all $\hat{a}, \hat{x} \in X$, $k \geq 0$ and a C^1 function ϕ, if $u_*(t,x) - \phi(t) + \frac{k}{2} d(\hat{a}, x)^2$ attains a local minimum at $(t,x) = (\hat{t}, \hat{x})$, then the inequality

$$\partial_t \phi(\hat{t}) + H(\hat{t}, \hat{x}, u_*(\hat{t}, \hat{x}), kd(\hat{a}, \hat{x})) \geq 0$$

holds.

The proof is basically the same as in the time-independent case.

Proof. We only show the assertion for subsolutions and show condition (ii) implies condition (i). Fix a test function $\psi = \psi_1 + \psi_2$ for subsolutions such that $u^* - \psi$ attains a local maximum at $\hat{z} = (\hat{t}, \hat{x}) \in (0, \infty) \times X$. Consider

$$(t, x, s, y) \mapsto u^*(t,x) - \psi(s,y) - \frac{1}{2\varepsilon}|t - s|^2 - \frac{\alpha}{2}|\hat{t} - s|^2 - \frac{1}{2\varepsilon} d(x,y)^2 - \frac{\alpha}{2} d(\hat{x}, y)^2$$

with positive ε, α. Since X is locally compact we can take a maximum point $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon)$ on

$$\overline{I} \times \overline{B} \times \overline{I} \times \overline{B}.$$

Here I denotes an open neighborhood of $\hat{t}$ in $\mathbb{R}_+$ and B denotes an open neighborhood of $\hat{x}$ in X. Then,

$$\begin{aligned} &u^*(t_\varepsilon, x_\varepsilon) - \psi(s_\varepsilon, y_\varepsilon) - \frac{1}{2\varepsilon}|t_\varepsilon - s_\varepsilon|^2 - \frac{\alpha}{2}|\hat{t} - s_\varepsilon|^2 - \frac{1}{2\varepsilon} d(x_\varepsilon, y_\varepsilon)^2 - \frac{\alpha}{2} d(\hat{x}, y_\varepsilon)^2 \\ &\quad \geq u^*(\hat{t}, \hat{x}) - \psi(\hat{t}, \hat{x}) \geq u^*(t_\varepsilon, x_\varepsilon) - \psi(t_\varepsilon, x_\varepsilon) \end{aligned}$$

and hence

$$\frac{1}{2\varepsilon}|t_\varepsilon - s_\varepsilon|^2 + \frac{\alpha}{2}|\hat{t} - s_\varepsilon|^2 + \frac{1}{2\varepsilon} d(x_\varepsilon, y_\varepsilon)^2 + \frac{\alpha}{2} d(\hat{x}, y_\varepsilon)^2 \leq L|t_\varepsilon - s_\varepsilon| + L d(x_\varepsilon, y_\varepsilon)$$

for the (local) Lipschitz constant L of ψ. Therefore, we can see that $(t_\varepsilon, x_\varepsilon, s_\varepsilon, y_\varepsilon) \to (\hat{t}, \hat{x}, \hat{t}, \hat{x})$ as $\varepsilon \to 0$. Now, by the assumption of u we have

$$\frac{1}{\varepsilon}(t_\varepsilon - s_\varepsilon) + H(t_\varepsilon, x_\varepsilon, u^*(t_\varepsilon, x_\varepsilon), \frac{1}{\varepsilon} d(x_\varepsilon, y_\varepsilon)) \leq 0.$$

By Lemma 2.1 we obtain

$$||\nabla_d \psi_1|(s_\varepsilon, y_\varepsilon) - \frac{1}{\varepsilon} d(x_\varepsilon, y_\varepsilon)| \leq |\nabla_d \psi_2|(s_\varepsilon, y_\varepsilon) + \alpha d(\hat{x}, y_\varepsilon).$$

Also note that

$$\partial_t \psi(s_\varepsilon, y_\varepsilon) - \frac{1}{\varepsilon}(t_\varepsilon - s_\varepsilon) - \alpha(\hat{t} - s_\varepsilon) = 0.$$

Combining them,

$$\partial_t \psi(s_\varepsilon, y_\varepsilon) - \alpha(\hat{t} - s_\varepsilon) + H_{|\nabla_d \psi_2|(s_\varepsilon, y_\varepsilon) + \alpha d(\hat{x}, y_\varepsilon)}(t_\varepsilon, x_\varepsilon, u^*(t_\varepsilon, x_\varepsilon), |\nabla_d \psi_1|(s_\varepsilon, y_\varepsilon)) \leq 0$$

and therefore

$$\partial_t \psi(\hat{t}, \hat{x}) + H_{|\nabla_d \psi_2|^*(\hat{t}, \hat{x})}(\hat{t}, \hat{x}, u^*(\hat{t}, \hat{x}), |\nabla_d \psi_1|(\hat{t}, \hat{x})) \leq 0.$$

Hence, u is a metric viscosity subsolution. □

This implies the following:

Theorem 7.3 (Stability for the time-dependent problem)**.** *Let* (X, d) *be a proper metric space. Assume (H1), (H2), and (H3'). Let* u^n *be a metric viscosity subsolution of* (7.1) *for* $n \in \mathbb{N}$. *If the upper semilimit*

$$\overline{u}^\infty(t, x) := \limsup_{n \to \infty, t_n \to t, (x_n) \in S(x)} u^n(t_n, x_n) = \sup_{t_n \to t, (x_n) \in S(x)} \limsup_{n \to \infty} u^n(t_n, x_n)$$

is upper semicontinuous in d_∞*, then* $\overline{u}^\infty$ *is a metric viscosity subsolution of* (7.2). *Similarly, let* u^n *be a metric viscosity supersolution of* (7.1). *If the lower semilimit*

$$\underline{u}^\infty(t, x) := \liminf_{n \to \infty, t_n \to t, (x_n) \in S(x)} u^n(t_n, x_n) = \inf_{t_n \to t, (x_n) \in S(x)} \liminf_{n \to \infty} u^n(t_n, x_n)$$

is lower semicontinuous in d_∞*, then* $\underline{u}^\infty$ *is a metric viscosity supersolution of* (7.2).

8 Remarks on the noncompact case

Return the time-independent problem and study the possibility to remove the (locally) compactness assumption of (X, d) in Theorem 3.1. As in [22], a tool to consider the function on noncompact domain is Ekeland's variational principle.

Proposition 8.1 (Ekeland's variational principle)**.** *Let* (X, d) *be a complete metric space and let* $F : X \to \mathbb{R}$ *be an upper (resp. lower) semicontinuous function bounded from above (resp. below). Then, for each* $\hat{x} \in X$*, there exists* $\bar{x} \in X$ *such that* $d(\hat{x}, \bar{x}) \leq 1$*,* $F(\bar{x}) \geq F(\hat{x})$ *(resp.* $F(\bar{x}) \leq F(\hat{x})$*) and* $x \mapsto F(x) - md(\bar{x}, x)$ *attains a strict maximum (resp. minimum) at* $\bar{x}$ *with* $m := \sup F - F(\hat{x})$ *(resp.* $m := \inf F - F(\hat{x})$*).*

For details of Ekeland's variational principle, see [7, Corollary 11].

In noncompact cases, we cannot drop ψ_2 but we can reduce ψ_2 to the distance function in the test functions.

Proposition 8.2. *Let (X,d) be a complete geodesic space not necessarily locally compact. Then, the following conditions are equivalent.*

(i) A function $u : X \to \mathbb{R}$ is a metric viscosity subsolution of (1.1).

(ii) For all $\hat{a}, \hat{b}, \hat{x} \in X$, $k, c \geq 0$, if $u^*(x) - \frac{k}{2} d(\hat{a}, x)^2 - cd(\hat{b}, x)$ attains a local maximum at $x = \hat{x}$, then the inequality

$$H_c(\hat{x}, u^*(\hat{x}), kd(\hat{a}, \hat{x})) \leq 0$$

holds.

(iii) For all $\hat{a}, \hat{b}, \hat{x} \in X$, $k, c \geq 0$, if $u^*(x) - \frac{k}{2} d(\hat{a}, x)^2 - cd(\hat{b}, x)$ attains a strict local maximum at $x = \hat{x}$, then the inequality

$$H_c(\hat{x}, u^*(\hat{x}), kd(\hat{a}, \hat{x})) \leq 0$$

holds.

Similarly, the following conditions are equivalent.

(i') A function $u : X \to \mathbb{R}$ is a metric viscosity supersolution of (1.1).

(ii') For all $\hat{a}, \hat{b}, \hat{x} \in X$, $k, c \geq 0$, if $u_*(x) + \frac{k}{2} d(\hat{a}, x)^2 + cd(\hat{b}, x)$ attains a local minimum at $x = \hat{x}$, then the inequality

$$H^c(\hat{x}, u_*(\hat{x}), kd(\hat{a}, \hat{x})) \geq 0$$

holds.

(iii') For all $\hat{a}, \hat{b}, \hat{x} \in X$, $k, c \geq 0$, if $u_*(x) + \frac{k}{2} d(\hat{a}, x)^2 + cd(\hat{b}, x)$ attains a strict local minimum at $x = \hat{x}$, then the inequality

$$H^c(\hat{x}, u_*(\hat{x}), kd(\hat{a}, \hat{x})) \geq 0$$

holds.

Proof. We only show the assertion for subsolutions and show the condition (iii) implies the condition (i). Fix a test function $\psi = \psi_1 + \psi_2$ for subsolutions such that $u^* - \psi$ attains a local maximum at $\hat{x} \in X$. Note that there is a neighborhood B of $\hat{x}$ such that $u^* - \psi$ attains the maximum on the closure $\overline{B}$ at $\hat{x}$. Consider

$$F(x,y) = u^*(x) - \psi(y) - \frac{1}{2\varepsilon} d(x,y)^2 - \frac{\alpha}{2} d(\hat{x},y)^2$$

for $x, y \in \overline{B}$ with positive ε, α. We then see that

$$\begin{aligned} F(x,y) &\leq (u^* - \psi)(\hat{x}) + \psi(x) - \psi(y) - \frac{1}{2\varepsilon} d(x,y)^2 \\ &\leq (u^* - \psi)(\hat{x}) + Ld(x,y) - \frac{1}{2\varepsilon} d(x,y)^2 \\ &\leq (u^* - \psi)(\hat{x}) + \frac{1}{2}\varepsilon L^2 < +\infty \end{aligned}$$

for all $x, y \in \overline{B}$ with the (local) Lipschitz constant L of ψ. Hence, in view of Ekeland's variational principle, there exist points $x_\varepsilon, y_\varepsilon \in \overline{B}$ such that $d(\hat{x}, x_\varepsilon), d(\hat{y}, y_\varepsilon) \leq 1$ and

$$u^*(x) - \psi(y) - \frac{1}{2\varepsilon} d(x,y)^2 - \frac{\alpha}{2} d(\hat{x},y)^2 - m_\varepsilon d(x_\varepsilon, x) - m_\varepsilon d(y_\varepsilon, y)$$

attains a strict maximum on $\overline{B} \times \overline{B}$ at $(x,y) = (x_\varepsilon, y_\varepsilon)$. Here,

$$m_\varepsilon = \sup_{\overline{B} \times \overline{B}} F - F(\hat{x}, \hat{x}) \leq \frac{1}{2}\varepsilon L^2$$

and so $m_\varepsilon \to 0$ as $\varepsilon \to 0$. Note that

$$\begin{aligned} &u^*(x_\varepsilon) - \psi(y_\varepsilon) - \frac{1}{2\varepsilon} d(x_\varepsilon, y_\varepsilon)^2 - \frac{\alpha}{2} d(\hat{x}, y_\varepsilon)^2 \\ &\quad \geq u^*(\hat{x}) - \psi(\hat{x}) - m_\varepsilon d(x_\varepsilon, \hat{x}) - m_\varepsilon d(y_\varepsilon, \hat{y}) \\ &\quad \geq u^*(x_\varepsilon) - \psi(x_\varepsilon) - 2m_\varepsilon \end{aligned}$$

and hence

$$\frac{1}{2\varepsilon} d(x_\varepsilon, y_\varepsilon)^2 + \frac{\alpha}{2} d(\hat{x}, y_\varepsilon)^2 \leq Ld(x_\varepsilon, y_\varepsilon) + \varepsilon L^2.$$

Therefore, we can see that $(x_\varepsilon, y_\varepsilon) \to (\hat{x}, \hat{x})$ as $\varepsilon \to 0$. Now, by the assumption (iii), we have

$$H_{m_\varepsilon}(x_\varepsilon, u^*(x_\varepsilon), \frac{1}{\varepsilon} d(x_\varepsilon, y_\varepsilon)) \leq 0.$$

By Lemma 2.1, we obtain

$$||\nabla_d \psi_1|(y_\varepsilon) - \frac{1}{\varepsilon} d(x_\varepsilon, y_\varepsilon)| \leq |\nabla_d \psi_2|(y_\varepsilon) + \alpha d(\hat{x}, y_\varepsilon) + m_\varepsilon.$$

Combining them,

$$H_{|\nabla_d \psi_2|(y_\varepsilon) + \alpha d(\hat{x}, y_\varepsilon) + 2m_\varepsilon}(x_\varepsilon, u^*(x_\varepsilon), |\nabla_d \psi_1|(y_\varepsilon)) \leq 0$$

and therefore

$$H_{|\nabla_d\psi_2|^*(\hat{x})}(\hat{x}, u^*(\hat{x}), |\nabla_d\psi_1|(\hat{x})) \le 0.$$

We have shown that u is a metric viscosity subsolution. □

Let $B(a, r; X)$ denote the closed ball in the metric space (X, d), say,

$$B(a, r; X) = \{x \in X \mid d(a, x) \le r\}.$$

We now can prove the following theorem:

Theorem 8.3 (Stability in the noncompact case)**.** *Let* (X, d) *be a proper metric space. Assume (H1), (H2), and (H3).*

- *Assume*

  (H4$^+$) For each sequence $(a_n) \in S(a_\infty)$ *and upper semicontinuous functions* ϕ_n *on* X_n *and* ϕ_∞ *on* X_∞ *such that* $\limsup_n \phi_n(x_n) \le \phi_\infty(x_\infty)$ *for all* $(x_n) \in S(x_\infty)$*, there exists* $\delta > 0$ *such that the inequality*

  $$\limsup_n \sup_{B(a_n,r;X_n)} \phi_n \le \sup_{B(a_\infty,r;X_\infty)} \phi_\infty$$

  holds for all positive $r < \delta$*.*

  Let u^n *be a metric viscosity subsolution of* (1.3) *for* $n \in \mathbb{N}$*. If the upper semilimit*

  $$\overline{u}^\infty(x) := \limsup_{n\to\infty,\,(x_n)\in S(x)} u^n(x_n) = \sup_{(x_n)\in S(x)} \limsup_{n\to\infty} u^n(x_n)$$

  is upper semicontinuous in d_∞*, then* $\overline{u}^\infty$ *is a metric viscosity subsolution of* (1.4)*.*

- *Similarly, assume*

  (H4$^-$) For each sequence $(a_n) \in S(a_\infty)$ *and lower semicontinuous functions* ϕ_n *on* X_n *and* ϕ_∞ *on* X_∞ *such that* $\liminf_n \phi_n(x_n) \ge \phi_\infty(x_\infty)$ *for all* $(x_n) \in S(x_\infty)$*, there exists* $\delta > 0$ *such that the inequality*

  $$\liminf_n \inf_{B(a_n,r;X_n)} \phi_n \ge \inf_{B(a_\infty,r;X_\infty)} \phi_\infty$$

  holds for all positive $r < \delta$*.*

  Let u^n *be a metric viscosity supersolution of* (1.3) *for* $n \in \mathbb{N}$*. If the lower semilimit*

  $$\underline{u}^\infty(x) := \liminf_{n\to\infty,\,(x_n)\in S(x)} u^n(x_n) = \inf_{(x_n)\in S(x)} \liminf_{n\to\infty} u^n(x_n)$$

  is lower semicontinuous in d_∞*, then* $\underline{u}^\infty$ *is a metric viscosity supersolution of* (1.4)*.*

Proof. We only show the theorem for the subsolution since the proof for the supersolution is similar. Fix $\hat{a}, \hat{b}, \hat{x} \in X_\infty$, $k, c \geq 0$ and assume that the function

$$\phi_\infty(x) = \overline{u}^\infty(x) - \frac{k}{2} d_\infty(\hat{a}, x)^2 - c d_\infty(\hat{b}, x) \quad \text{for } x \in X_\infty$$

attains a strict local maximum at $x = \hat{x}$. Let $R > 0$ be a radius such that

$$\sup_{x \in B(\hat{x}, R; X_\infty)} \phi_\infty(x) = \phi_\infty(\hat{x}).$$

By assumption (H1), there exist sequences $a_n, b_n, \tilde{x}_n \in X_n$ satisfying $a_n \to \hat{a}$, $b_n \to \hat{b}$, $\tilde{x}_n \to \hat{x}$ in d and $u^n(\tilde{x}_n) \to \overline{u}^\infty(\hat{x})$. For $n \in \mathbb{N}$ applying Ekeland's variational principle to the function

$$\phi_n(x) = u_n(x) - \frac{k}{2} d_n(a_n, x)^2 - c d_n(b_n, x) \quad \text{for } x \in X_n,$$

we see that there exists a maximum point $x = x_n$ of

$$x \mapsto \phi_n(x) - m_n d_n(x_n, x) = u_n(x) - \frac{k}{2} d_n(a_n, x)^2 - c d_n(b_n, x) - m_n d_n(x_n, x),$$

where

$$m_n = \sup_{x \in B(\tilde{x}_n, R; X_n)} \phi_n(x) - \phi_n(\tilde{x}_n).$$

Assumptions (H4$^+$) and (H2) imply

$$\limsup_n m_n \leq \sup_{x \in B(\hat{x}, R; X_\infty)} \phi_\infty(x) - \phi_\infty(\hat{x}) \leq 0$$

and therefore $m_n \to 0$. Note that

$$\phi_n(x_n) \geq \phi_n(\tilde{x}_n) - 2R m_n$$

and the right-hand side converges to $\phi_\infty(\hat{x})$. Therefore, since $\hat{x}$ is a strict maximum point, by (H2), we see that $x_n \to \hat{x}$ in d and $u^n(x_n) \to \overline{u}^\infty(\hat{x})$. Now, take the limit of

$$(H_n)_{c+m_n}(x_n, u^n(x_n), k d_n(a_n, x_n)) \leq 0.$$

In view of (H2) and (H3), we have

$$(H_\infty)_c(\hat{x}, \overline{u}^\infty(\hat{x}), k d_\infty(\hat{a}, \hat{x})) \leq 0,$$

which completes the proof. □

Remark 8.4. In the noncompact case the original definition of Hausdorff convergence (H1) may be too strong to handle various problems. For example, consider the half-lines

$$X_n = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y = n^{-1}x\}, \quad X_\infty = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y = 0\}$$

in the plane $\mathbb{R}^2$. At the point $P = (x, n^{-1}x) \in X_n$ the distance from X_∞ is given by

$$d(X_\infty, P) = n^{-1}x$$

so that the Hausdorff distance $d_H(X_\infty, X_n)$ is always infinity. In particular, X_n does not converge to X_∞ in the sense of (H1). Meanwhile, assumption (H2) is fulfilled. There are alternative definitions of Hausdorff convergence to relax the concept for noncompact cases [23]. However, we do not touch this subject any more since it is not essential in the present study.

Part II

On the Gromov–Hausdorff stability of metric viscosity solutions

1 Introduction

Hamilton–Jacobi equations play a pivotal role in interfacial science and optimal control. From the perspective of PDE, the well-posedness and qualitative properties of their viscosity solutions have been studied intensively since the 1970s. In recent decades, this line of inquiry has been extended beyond Euclidean spaces to a wide range of non-Euclidean and metric settings [2, 6, 11, 9, 10, 17]. Notably, applications of Hamilton–Jacobi equations on metric spaces have attracted considerable attention; examples include the construction of viscosity solutions via graph approximations of fractals [5], domain-perturbation problems [18, 20], and an application to a large-deviation principle [13].

Our main aim is to study the Gromov–Hausdorff stability of metric viscosity solutions, which is a natural generalization of the aforementioned stability results for domain perturbations [18, 20]. Parallel to these developments within PDE, such stability has also been extensively studied in broader mathematical contexts, motivated in particular by advances in metric geometry. In [12], Gigli, Mondino, and Savaré studied the stability of heat flows under pointed-measured-Gromov–Hausdorff convergence. In [15], Honda proved the stability of solutions to elliptic PDEs under Gromov–Hausdorff convergence. The present work furnishes a first-order analogue, establishing Gromov–Hausdorff stability for metric viscosity solutions of Hamilton–Jacobi equations.

Let $n \in \mathbb{N}$. Suppose that (X_n, d_n) and (X_∞, d_∞) are geodesic spaces. We assume

$$X_n \to X_\infty \quad \text{in Gromov–Hausdorff sense.}$$

We recall that X_n converges to X_∞ in Gromov–Hausdorff sense if there exists an ϵ_n-isometry $f_n : X_n \to X_\infty$ with $\epsilon_n \to 0$ as $n \to \infty$. See Definition 2.4.

We consider the Hamilton–Jacobi equation on X_n

$$\partial_t u^n(t,x) + H_n\big(x, |\nabla_{d_n} u^n|(t,x)\big) = 0, \quad (t,x) \in (0,\infty) \times X_n, \tag{1.1}$$

$$u^n(0,x) = g^n(x), \quad x \in X_n, \tag{1.2}$$

where the Hamiltonian $H_n : X_n \times [0,\infty) \to \mathbb{R}$ is continuous and the initial function $g^n : X_n \to \mathbb{R}$ is continuous. We also consider the Hamilton–Jacobi equation on X_∞

$$\partial_t u^\infty(t,x) + H_\infty\big(x, |\nabla_{d_\infty} u^\infty|(t,x)\big) = 0, \quad (t,x) \in (0,\infty) \times X_\infty, \tag{1.3}$$

$$u^\infty(0,x) = g^\infty(x), \quad x \in X_\infty, \tag{1.4}$$

where the Hamiltonian $H_\infty : X_\infty \times [0,\infty) \to \mathbb{R}$ is continuous and the initial function $g^\infty : X_\infty \to \mathbb{R}$ is continuous.

For the stability, we assume the following conditions. To handle metric viscosity solutions, we extend the domain of Hamiltonian H_n and H_∞ to $X_n \times \mathbb{R}$ and $X_\infty \times \mathbb{R}$, respectively, by constant extension. Precisely, for $x \in X_n$, $y \in X_\infty$, and for $p < 0$, we set $H_n(x,p) := H_n(x,0)$, $H_\infty(y,p) := H_\infty(y,0)$.

(S1) (Gromov–Hausdorff convergence) $X_n \to X_\infty$ in Gromov–Hausdorff sense. In other words, there exists an ϵ_n-isometry $f_n : X_n \to X_\infty$ with $\epsilon_n \to 0$ as $n \to \infty$.

(S2) (Monotonicity of H_n in the second variable) Let $x \in X_n$. For $p, q \in \mathbb{R}$ with $p \leq q$, the inequality

$$H_n(x,p) \leq H_n(x,q)$$

holds.

(S3) (Uniform convergence of H_n) For $x \in X_\infty$, $p \in \mathbb{R}$, and $x_n \in X_n$, $p_n \in \mathbb{R}$ such that $f_n(x_n) \to x$ in X_∞ and $p_n \to p$ in $\mathbb{R}$ as $n \to \infty$, the convergence

$$H_n(x_n, p_n) \to H_\infty(x,p)$$

holds.

(S4) (Uniform boundedness of u^n) For $T_1, T_2 \in (0,\infty)$ with $T_1 < T_2$, there exists a positive constant M independent of n such that, for $s \in [T_1, T_2]$ and $x \in X_n$, the inequality

$$|u^n(s,x)| \leq M$$

holds.

(S5) (Equi-continuity and pointwise convergence of g^n) There exists a modulus of continuity η such that, for $x, y \in X_n$, the inequality

$$|g^n(x) - g^n(y)| \leq \eta\big(d_n(x,y)\big)$$

holds. Furthermore, for $x \in X_\infty$, $g^n\big(f_n'(x)\big)$ converges to $g^\infty(x)$ as $n \to \infty$, where $f_n' : X_\infty \to X_n$ is an ϵ_n-inverse of f_n. See Definition 2.7.

Under these assumptions, we can prove the Gromov–Hausdorff stability of metric viscosity solutions.

Theorem 1.1 (Stability under Gromov–Hausdorff convergence)**.** *Let (X_n, d_n) and (X_∞, d_∞) be locally compact geodesic spaces; that is, geodesic spaces in which every closed ball is compact. Let u^n be a metric viscosity subsolution (resp. supersolution) of (1.1), (1.2). Define the limit function $u^\infty(t,x)$ (resp. $u_\infty(t,x)$) by*

$$u^\infty(t,x) := \sup_{(t_n,x_n)} \limsup_{n\to\infty} (u^n)^*(t_n, x_n), \quad (t,x) \in [0,\infty) \times X_\infty.$$

$$\left(\textit{resp. } u_\infty(t,x) := \inf_{(t_n,x_n)} \liminf_{n\to\infty} (u^n)_*(t_n,x_n), \quad (t,x) \in [0,\infty) \times X_\infty. \right)$$

In the definition of u^∞ (resp. u_∞), the supremum (resp. infimum) is taken over all sequences $t_n \in [0,\infty)$ and $x_n \in X_n$ such that $t_n \to t$ in $[0,\infty)$ and $f_n(x_n) \to x$ in X_∞ as $n \to \infty$. Furthermore, $(u^n)^*$ and $(u^n)_*$ denote the upper and lower semicontinuous envelopes of u^n, respectively. Suppose that (S1)–(S5) hold and, for $x \in X_\infty$, the following inequality (continuity at the initial time of u^n) holds:

$$u^\infty(0,x) \le \sup_{(x_n)} \limsup_{n\to\infty} (u^n)^*(0,x_n).$$

$$\left(\textit{resp. } u_\infty(0,x) \ge \inf_{(x_n)} \liminf_{n\to\infty} (u^n)_*(0,x_n). \right)$$

We also assume that u^∞ (resp. u_∞) is an upper (resp. lower) semicontinuous function on X_∞. Then, the function u^∞ (resp. u_∞) is a metric viscosity subsolution (resp. supersolution) of (1.3), (1.4).

Note that if we assume (S1)–(S4) and the upper (resp. lower) semicontinuity of u^∞ (resp. u_∞), we can prove the stability of metric viscosity subsolutions (resp. supersolutions) to (1.1) itself in essentially the same way as in the case of the Cauchy problem (1.1), (1.2).

We outline the strategy of the proof. First, by using the characterization of metric viscosity solutions via squared distance functions [20], we show that u^∞ is a metric viscosity subsolution of (1.3). See Proposition 3.1. Next, we demonstrate that u^∞ satisfies the initial condition (1.4) using (S5). A similar argument shows that u_∞ is a metric viscosity supersolution of (1.3), (1.4). Our strategy is a natural adaptation of the approach used to prove the stability under Hausdorff convergence [18, 20].

As an application of the stability of metric viscosity solutions, we give a PDE-based proof of the convergence of maximizers of the dual Kantorovich problems under measured-Gromov–Hausdorff convergence. The dual Kantorovich problem includes the Hopf–Lax semigroup, which can be identified as the unique metric viscosity solution of (1.1), (1.2) with a specific Hamiltonian. Therefore, we can apply our stability results to the stability of maximizers.

Let $\mu_\infty, \nu_\infty \in P_2(X_\infty)$ and $\mu_n, \nu_n \in P_2(X_n)$. Here, $P_2(X_\infty)$ denotes the set of probability measures on X_∞ with finite second moments. We consider the dual Kantorovich problem on X_n

$$\max_{\phi \in L^1(\mu_n)} \left(\int_{X_n} Q_1\phi \, d\nu_n - \int_{X_n} \phi \, d\mu_n \right) \tag{KAN$_n$}$$

and the dual Kantorovich problem on X_∞

$$\max_{\phi \in L^1(\mu_\infty)} \left(\int_{X_\infty} Q_1\phi \, d\nu_\infty - \int_{X_\infty} \phi \, d\mu_\infty \right), \tag{KAN$_\infty$}$$

where $Q_1\phi$ denotes the Hopf–Lax semigroup applied to ϕ at time 1. See Subsection 2.3 for a more detailed explanation of the Kantorovich problem.

Proposition 1.2 ([27, Exercise 28.12]; see also the claim in the proof of Theorem 1.1 in [26])**.** *Let (X_n, d_n) and (X_∞, d_∞) be compact geodesic spaces. Assume (S1). In other words, there exists an ϵ_n-isometry f_n with $\epsilon_n \to 0$ as $n \to \infty$. Suppose further that f_n is a Borel map and the pushforwards of measures $(f_n)_\#\mu_n$, $(f_n)_\#\nu_n$ satisfy*

$$(f_n)_\#\mu_n \to \mu_\infty, \quad (f_n)_\#\nu_n \to \nu_\infty$$

in 2-Wasserstein distance W_2 as $n \to \infty$. Let ϕ^n be a maximizer of (KAN$_n$). Then, there exists a constant $c^n \in \mathbb{R}$ such that a maximizer $\phi^n - c^n$ of (KAN$_n$) converges uniformly to some maximizer $\overline{\phi}$ of (KAN$_\infty$) up to a subsequence. Equivalently, for $x \in X_\infty$, we have

$$\lim_{n\to\infty} (\phi^n(x_n) - c^n) = \overline{\phi}(x),$$

where $x_n \in X_n$ satisfies $f_n(x_n) \to x$ in X_∞ as $n \to \infty$.

The proof is carried out according to the following strategy. First, we show that, for each n, there exists a real number c^n such that $\phi^n - c^n$ is equi-Lipschitz in n and uniformly bounded. Then, by applying the Ascoli–Arzelà theorem, we deduce the existence of a uniformly convergent limit $\overline{\phi}$. Up to this step, the approach is the same as that used in previous study [26]. However, instead of applying the Ascoli–Arzelà theorem directly to $Q_1(\phi^n - c^n)$ as in the previous work, we deduce the uniform convergence of $Q_1(\phi^n - c^n)$ from the stability of metric viscosity solutions. Using the uniform convergence of these functions, we verify that the limit function $\overline{\phi}$ is a maximizer of the dual Kantorovich problem on X_∞.

The remainder of this paper is organized as follows. In Section 2, we give a short review of metric viscosity solutions, Gromov–Hausdorff convergence, and the dual Kantorovich problem. In Section 3, we prove the stability of metric viscosity solutions under Gromov–Hausdorff convergence. In Section 4, we prove the stability of maximizers of the dual Kantorovich problems under Gromov–Hausdorff convergence.

2 Preliminaries

In this section, we summarize the minimal knowledge of metric viscosity solutions, Gromov–Hausdorff convergence, and the dual Kantorovich problem for the reader's convenience.

2.1 Metric viscosity solutions

Let (X,d) be a metric space satisfying the following condition. For $x,y\in X$, there exists a continuous curve $\gamma:[0,1]\to X$ such that $\gamma(0)=x$, $\gamma(1)=y$ and

$$d\big(\gamma(s),\gamma(t)\big)=|s-t|d(x,y),\quad s,t\in[0,1].$$

A metric space with this property is called a geodesic space.

In this subsection, we consider the Hamilton–Jacobi equation with a continuous Hamiltonian $H(x,p):X\times[0,\infty)\to\mathbb{R}$

$$\partial_t u(t,x)+H(x,|\nabla_d u|(t,x))=0,\quad (t,x)\in(0,\infty)\times X, \tag{2.1}$$

$$u(0,x)=g(x),\quad x\in X, \tag{2.2}$$

where $g:X\to\mathbb{R}$ is continuous.

We define the local slope for a locally Lipschitz function $\phi:(0,\infty)\times X\to\mathbb{R}$ by

$$|\nabla_d\phi|(t,x):=\limsup_{y\to x}\frac{|\phi(t,y)-\phi(t,x)|}{d(y,x)}.$$

In the same way, we define the lower and upper local slope for a locally Lipschitz function $\phi:(0,\infty)\times X\to\mathbb{R}$ by

$$|\nabla_d^-\phi|(t,x):=\limsup_{y\to x}\frac{\max\{0,-\phi(t,y)+\phi(t,x)\}}{d(y,x)},$$

$$|\nabla_d^+\phi|(t,x):=\limsup_{y\to x}\frac{\max\{0,\phi(t,y)-\phi(t,x)\}}{d(y,x)}.$$

Definition 2.1 ([10, Definition 2.2])**.** Let $\phi_1,\phi_2:(0,\infty)\times X\to\mathbb{R}$ be locally Lipschitz functions. We call (ϕ_1,ϕ_2) a subsolution (resp. supersolution) test function if (ϕ_1,ϕ_2) satisfies the following conditions.

(i) The identity $|\nabla_d\phi_1|(t,x)=|\nabla_d^-\phi_1|(t,x)$ $\big($resp. $|\nabla_d\phi_1|(t,x)=|\nabla_d^+\phi_1|(t,x)\big)$ holds on $(0,\infty)\times X$ and the slope $|\nabla_d\phi_1|(t,x)$ is continuous on $(0,\infty)\times X$.

(ii) The time derivatives $\partial_t\phi_1$ and $\partial_t\phi_2$ are continuous on $(0,\infty)\times X$.

We denote by $\underline{C}$ (resp. $\overline{C}$) the collection of all subsolution (resp. supersolution) test functions.

For example, by [9, Lemma 7.2], the function $(\phi_1,\phi_2)=(d(a,\cdot)^2,0)$, $a\in X$ is a subsolution test function.

For a subsolution test function v and a supersolution test function u, the following inequality holds, which is used in the proof of Proposition 3.1.

Lemma 2.2 ([20, Lemma 2.1]). *Let $u, v, w : (0, \infty) \times X \to \mathbb{R}$ be locally Lipschitz functions. We assume that u satisfies the condition (i) of the definition of supersolution test function (Definition 2.1) and v satisfies the condition (i) of the definition of subsolution test function (Definition 2.1). If $u - v - w$ attains a local maximum at $(\hat{t}, \hat{x}) \in (0, \infty) \times X$, then the inequality*

$$\big||\nabla_d u|(\hat{t}, \hat{x}) - |\nabla_d v|(\hat{t}, \hat{x})\big| \leq |\nabla_d w|(\hat{t}, \hat{x})$$

holds.

To define the notion of metric viscosity solutions, we extend the domain of the Hamiltonian to $X \times \mathbb{R}$ by constant extension. For $f : [0, \infty) \times X \to \mathbb{R}$, we define the upper (resp. lower) semicontinuous envelope by

$$f^*(t, x) := \limsup_{(s,y) \to (t,x)} f(s, y).$$

$$\left(\text{resp. } f_*(t, x) := \liminf_{(s,y) \to (t,x)} f(s, y).\right)$$

Definition 2.3 ([10, Definition 2.5]). Let $u : [0, \infty) \times X \to \mathbb{R}$ be a locally bounded function. We say that u is a metric viscosity subsolution (resp. supersolution) of (2.1) if $u^* - \phi_1 - \phi_2$, $(\phi_1, \phi_2) \in \underline{C}$ (resp. $u_* - \phi_1 - \phi_2$, $(\phi_1, \phi_2) \in \overline{C}$) attains a local maximum (resp. minimum) at (t, x) in $(0, \infty) \times X$, then the inequality

$$\partial_t \phi_1(t, x) + \partial_t \phi_2(t, x) + H_{|\nabla_d \phi_2|^*(t,x)}\big(x, |\nabla_d \phi_1(t, x)|\big) \leq 0$$

$$\big(\text{resp. } \partial_t \phi_1(t, x) + \partial_t \phi_2(t, x) + H^{|\nabla_d \phi_2|^*(t,x)}\big(x, |\nabla_d \phi_1|(t, x)\big) \geq 0\big)$$

holds. Here, for $\eta \geq 0$, we define

$$H_\eta(x, p) := \inf_{|s-p| \leq \eta} H(x, s).$$

$$\left(\text{resp. } H^\eta(x, p) := \sup_{|s-p| \leq \eta} H(x, s).\right)$$

We say that u is a metric viscosity subsolution (resp. supersolution) of (2.1), (2.2) if u is a metric viscosity subsolution (resp. supersolution) of (2.1) and u satisfies the initial condition (2.2):

$$u(0, x) \leq g(x). \quad \big(\text{resp. } u(0, x) \geq g(x).\big)$$

We say that u is a metric viscosity solution of (2.1), (2.2) if u is a metric viscosity subsolution and supersolution of (2.1), (2.2).

2.2 Gromov–Hausdorff convergence

In this paper, we use the definition of Gromov–Hausdorff convergence via an ϵ-isometry. It is known that there are some characterizations of Gromov–Hausdorff convergence. See [27] for more details.

Definition 2.4. Let (X, d_X), (Y, d_Y) be metric spaces. Let $\epsilon > 0$. We call a map $f : X \to Y$ an ϵ-isometry if f satisfies the following conditions.

- For $x, y \in X$, the inequality

$$\left|d_Y\big(f(x), f(y)\big) - d_X(x, y)\right| \leq \epsilon$$

  holds.

- The inequality $d_H\big(f(X), Y\big) \leq \epsilon$ holds, where d_H denotes the Hausdorff distance in Y.

Definition 2.5. Let (X_n, d_n) and (X_∞, d_∞) be metric spaces. We say that X_n converges to X_∞ in Gromov–Hausdorff sense if there exists an ϵ_n-isometry $f_n : X_n \to X_\infty$ with $\epsilon_n \to 0$ as $n \to \infty$.

Remark 2.6. According to [4, Subsection 7.4], the ordinary definition of Gromov–Hausdorff convergence in terms of embedding maps coincides with the definition adopted in this paper. Moreover, Hausdorff convergence implies Gromov–Hausdorff convergence [4, Example 7.4.1]. However, there is an example that converges in Gromov–Hausdorff sense but not in Hausdorff sense: a unit line segment rotating about the origin in $\mathbb{R}^2$.

For an ϵ-isometry f, one can construct a map f' that serves as an approximate inverse to f.

Definition 2.7. Let (X, d_X), (Y, d_Y) be metric spaces. Let $f : X \to Y$ be an ϵ-isometry. We call a map $f' : Y \to X$ an ϵ-inverse of f if f' satisfies the following conditions.

- The map f' is a 4ϵ-isometry.

- For $x \in X$ and $y \in Y$, the inequalities

$$d_X\big(f' \circ f(x), x\big) \leq 3\epsilon, \quad d_Y\big(f \circ f'(y), y\big) \leq \epsilon$$

  hold.

Remark 2.8. Let (X_n, d_n) and (X_∞, d_∞) be metric spaces. Suppose that X_n converges to X_∞ in Gromov–Hausdorff sense. Then, for $x \in X_\infty$, there exists $x_n \in X_n$ such that $f_n(x_n) \to x$ as $n \to \infty$. Indeed, there exists $x_n \in X_n$ such that $d_\infty\big(x, f_n(x_n)\big) \le d_\infty\big(x, f_n(X_n)\big) + \frac{1}{n}$. Recalling the fact that

$$d_H\big(f_n(X_n), X_\infty\big) := \max\left\{\sup_{y \in f_n(X_n)} d_\infty(y, X_\infty), \sup_{y \in X_\infty} d_\infty\big(y, f_n(X_n)\big)\right\},$$

we have

$$d_\infty\big(x, f_n(x_n)\big) \le d_\infty\big(x, f_n(X_n)\big) + \frac{1}{n} \le d_H\big(f_n(X_n), X_\infty\big) + \frac{1}{n} \le \epsilon_n + \frac{1}{n}.$$

Letting $n \to \infty$, we obtain the desired conclusion.

2.3 The dual Kantorovich problem

Let (X, d) be a compact geodesic space. Let μ, ν be probability measures on X and let $\Pi(\mu, \nu)$ be the set of probability measures π on $X \times X$ satisfying $(p_1)_\# \pi = \mu$ and $(p_2)_\# \pi = \nu$. Here, $p_1, p_2 : X \times X \to X$ are projection maps onto the first and second factors, respectively. For μ, ν, we define 2-Wasserstein distance $W_2(\mu, \nu)$ by

$$W_2(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x, y)^2 d\pi(x, y)\right)^{\frac{1}{2}}.$$

It is well known that W_2 satisfies the axioms of a distance on $P_2(X)$ [27].

Furthermore, W_2 admits the following dual formulation [27, Theorem 5.10]:

$$\frac{1}{2} W_2(\mu, \nu)^2 = \max_{\phi \in L^1(\mu)} \left(\int_X \phi^c \, d\nu - \int_X \phi \, d\mu\right). \tag{KAN}$$

The right hand side of (KAN) is called the dual Kantorovich problem.

We now turn to the admissible class of functions ϕ in the dual formulation and the definition of their $\frac{d^2}{2}$-transform ϕ^c. Since (KAN) admits a maximizer that is $\frac{d^2}{2}$-convex, we may, without loss of generality, restrict the admissible class of functions ϕ in (KAN) to those that are $\frac{d^2}{2}$-convex. Recall that $\phi : X \to \mathbb{R} \cup \{\pm\infty\}$ is a $\frac{d^2}{2}$-convex function if there exists a function $\zeta : X \to \mathbb{R} \cup \{\pm\infty\}$ such that

$$\phi(x) = \sup_{y \in X} \left(\zeta(y) - \frac{1}{2} d(x, y)^2\right), \quad x \in X.$$

Then, for such a ϕ, the $\frac{d^2}{2}$-transform ϕ^c is defined by

$$\phi^c(x) = \inf_{y \in X} \left(\phi(y) + \frac{1}{2} d(x, y)^2\right), \quad x \in X.$$

Thus, if we use the Hopf–Lax semigroup

$$Q_t\phi(x) := \inf_{y\in X}\left(\phi(y) + \frac{1}{2t}d(x,y)^2\right),$$

the dual Kantorovich problem can be written as

$$\max_{\phi\in L^1(\mu)}\left(\int_X Q_1\phi\,d\nu - \int_X \phi\,d\mu\right).$$

3 Stability of metric viscosity solutions

We use a characterization of metric viscosity solutions via squared distance functions, which is a slight modification of Proposition 7.2 in Part I.

Proposition 3.1. *We assume that (X,d) is a locally compact geodesic space. Then, the following statements are equivalent.*

(i) A function $u:(0,\infty)\times X\to\mathbb{R}$ is a metric viscosity subsolution (resp. supersolution) of (2.1)*.*

(ii) Let $\hat{a},\hat{x}\in X$, $k\geq 0$, $l\geq 0$, and ϕ be a $C^1((0,\infty))$ function. If $u^*(t,x)-\phi(t)-\frac{k}{2}d(\hat{a},x)^2-\frac{l}{2}d(\hat{x},x)^2$ (resp. $u_*(t,x)-\phi(t)+\frac{k}{2}d(\hat{a},x)^2+\frac{l}{2}d(\hat{x},x)^2$) attains a strict local maximum (resp. minimum) at $(\hat{t},\hat{x})\in(0,\infty)\times X$, then the inequality

$$\partial_t\phi(\hat{t}) + H\big(\hat{x},kd(\hat{a},\hat{x})\big)\leq 0$$

$$\big(\textit{resp. } \partial_t\phi(\hat{t}) + H\big(\hat{x},kd(\hat{a},\hat{x})\big)\geq 0\big)$$

holds.

Proof. We show that (ii) implies (i). The converse follows directly from the definition of metric viscosity solutions.

Let $(\phi_1,\phi_2)\in\underline{C}$ and set $\phi=\phi_1+\phi_2$. We assume that $u^*-\phi$ attains a local maximum at $(\hat{t},\hat{x})$ in $(0,\infty)\times X$. Let r be a positive constant with $0<r<\frac{\hat{t}}{2}$ and let I, B be open neighborhoods of $\hat{t}$, $\hat{x}$ respectively, with compact closures. We consider a function $F_1:\overline{I}\times\overline{B}\times\overline{I}\times\overline{B}\to\mathbb{R}$ defined by

$$F_1(t,x,s,y) := u^*(t,x)-\phi(s,y)-\frac{1}{2\epsilon}|t-s|^2-\frac{\alpha}{2}|\hat{t}-s|^2-\frac{1}{2\epsilon}d(x,y)^2-\frac{\alpha}{2}d(\hat{x},y)^2$$

where $\epsilon>0$ and $\alpha>0$. Since F_1 is upper semicontinuous, the function F_1 attains a local maximum at $(t_\epsilon,x_\epsilon,s_\epsilon,y_\epsilon)$ on $\overline{I}\times\overline{B}\times\overline{I}\times\overline{B}$. Then, the function

$$F_2(t,x,s,y) := F_1(t,x,s,y)-\frac{\epsilon}{2}d(x_\epsilon,x)^2-\frac{\epsilon}{2}d(y_\epsilon,y)^2-\frac{\epsilon}{2}|t-t_\epsilon|^2-\frac{\epsilon}{2}|s-s_\epsilon|^2$$

attains a strict maximum at $(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon)$. By comparing the values of F_2 at $(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon)$ and $(\hat{t}, \hat{x}, \hat{t}, \hat{x})$, we obtain

$$F_1(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon) \geq u^*(\hat{t}, \hat{x}) - \phi(\hat{t}, \hat{x}) - \frac{\epsilon}{2}d(x_\epsilon, \hat{x})^2 - \frac{\epsilon}{2}d(y_\epsilon, \hat{x})^2 - \frac{\epsilon}{2}|\hat{t} - t_\epsilon|^2 - \frac{\epsilon}{2}|\hat{t} - s_\epsilon|^2.$$

Consequently, we deduce

$$\begin{aligned}
&\frac{1}{2\epsilon}|t_\epsilon - s_\epsilon|^2 + \frac{\alpha}{2}|\hat{t} - s_\epsilon| + \frac{1}{2\epsilon}d(x_\epsilon, y_\epsilon)^2 + \frac{\alpha}{2}d(\hat{x}, y_\epsilon)^2 \\
&\leq L|t_\epsilon - s_\epsilon| + Ld(x_\epsilon, y_\epsilon) \\
&+ \frac{\epsilon}{2}\left(d(x_\epsilon, \hat{x})^2 + d(y_\epsilon, \hat{x})^2 + |\hat{t} - t_\epsilon|^2 + |\hat{t} - s_\epsilon|^2\right) \\
&\leq L|t_\epsilon - s_\epsilon| + Ld(x_\epsilon, y_\epsilon) + 2\epsilon r^2,
\end{aligned}$$

where L is a Lipschitz constant of ϕ on $\overline{I} \times \overline{B}$. Therefore, we have $(t_\epsilon, x_\epsilon, s_\epsilon, y_\epsilon) \to (\hat{t}, \hat{x}, \hat{t}, \hat{x})$ as $\epsilon \to 0$. Furthermore, by the condition (ii), we have

$$\frac{t_\epsilon - s_\epsilon}{\epsilon} + H\left(x_\epsilon, \frac{1}{\epsilon}d(x_\epsilon, y_\epsilon)\right) \leq 0.$$

Since $-\partial_s \phi(s_\epsilon, y_\epsilon) + \frac{t_\epsilon - s_\epsilon}{\epsilon} + \alpha(\hat{t} - s_\epsilon) = 0$, we deduce

$$\partial_s \phi(s_\epsilon, y_\epsilon) - \alpha(\hat{t} - s_\epsilon) + H\left(x_\epsilon, \frac{1}{\epsilon}d(x_\epsilon, y_\epsilon)\right) \leq 0. \tag{3.1}$$

On the other hand, by Lemma 2.2, we get

$$\left| |\nabla_d \phi_1|(s_\epsilon, y_\epsilon) - \frac{1}{\epsilon}d(x_\epsilon, y_\epsilon) \right| \leq |\nabla_d \phi_2|(s_\epsilon, y_\epsilon) + \alpha d(\hat{x}, y_\epsilon). \tag{3.2}$$

Combining (3.1) with (3.2), we have

$$\partial_t \phi(s_\epsilon, y_\epsilon) - \alpha(\hat{t} - s_\epsilon) + (H)_{|\nabla_d \phi_2|^*(s_\epsilon, y_\epsilon) + \alpha d(\hat{x}, y_\epsilon)}(x_\epsilon, |\nabla_d \phi_1|\,(s_\epsilon, y_\epsilon)) \leq 0.$$

Taking $\liminf_{\epsilon \to 0}$ in the above inequality, we get the desired inequality. □

Remark 3.2. An analogous equivalence holds if, in (ii), we replace the assumption of a strict local maximum (resp. minimum) by that of a local maximum (resp. minimum). See Proposition 7.2 in Part I.

We are now prepared to present the proof of Theorem 1.1.

Proof of Theorem 1.1. We first prove that u^∞ is a metric viscosity subsolution of (1.3). We apply the implication from (ii) to (i) in Proposition 3.1.

Let $a_\infty \in X_\infty$, $t_\infty \in (0, \infty)$, and $\phi \in C^1((0, \infty))$. Condition (S1) implies that there exists $a_n \in X_n$ such that $f_n(a_n) \to a_\infty$. See Remark 2.8 for the details. We assume

$$F_\infty(t,x) := u^\infty(t,x) - \phi(t) - \frac{k}{2} d_\infty(a_\infty, x)^2 - \frac{l}{2} d_\infty(x_\infty, x)^2$$

has a local strict maximum at $(t_\infty, x_\infty) \in (0,\infty) \times X_\infty$. Let $m > 0$. We consider a function

$$F_n(t,x) := (u^n)^*(t,x) - \phi(t) - \frac{k}{2} d_n(a_n, x)^2 - \frac{l}{2} d_n(x_n, x)^2 - \frac{m}{2} d_n(x_n, x)^2 - \frac{m}{2}|t_n - t|^2.$$

By the definition of u^∞ and the upper semicontinuity of u^∞, there exist $x_n \in X_n$ and $t_n \in (0,\infty)$ such that $f_n(x_n) \to x_\infty$, $t_n \to t_\infty$ and $\lim_{n\to\infty} (u^n)^*(t_n, x_n) = u^\infty(t_\infty, x_\infty)$.

Since F_n is a upper semicontinuous function, there exists a maximizer $(\hat{t}_n, \hat{x}_n)$ of F_n on $[t_n - d, t_n + d] \times \overline{B_d(x_n)}$ with $0 < d < \frac{t_\infty}{2}$. This implies

$$\begin{aligned} &(u^n)^*(t_n, x_n) - \phi(t_n) - \frac{k}{2}\, d_n(a_n, x_n)^2 \\ &\le (u^n)^*(\hat{t}_n, \hat{x}_n) - \phi(\hat{t}_n) - \frac{k}{2}\, d_n(a_n, \hat{x}_n)^2 - \frac{l}{2}\, d_n(x_n, \hat{x}_n)^2 - \frac{m}{2}\, d_n(x_n, \hat{x}_n)^2 - \frac{m}{2}|t_n - \hat{t}_n|^2. \end{aligned} \tag{3.3}$$

We claim that the left hand side of (3.3) converges to $u^\infty(t_\infty, x_\infty) - \phi(t_\infty) - \frac{k}{2} d_\infty(a_\infty, x_\infty)^2$. Indeed, by the definition of f_n, we compute directly

$$d_\infty\big(f_n(a_n), f_n(x_n)\big) - \epsilon_n \le d_n(a_n, x_n) \le d_\infty\big(f_n(a_n), f_n(x_n)\big) + \epsilon_n.$$

Hence, we get $(u^n)^*(t_n, x_n) - \phi(t_n) - \frac{k}{2} d_n(a_n, x_n)^2 \to u^\infty(t_\infty, x_\infty) - \phi(t_\infty) - \frac{k}{2} d_\infty(a_\infty, x_\infty)^2$.

To prove $f_n(\hat{x}_n) \to x_\infty$, we show that $f_n(\hat{x}_n)$ converges to some $c_1 \in X_\infty$ as $n \to \infty$. Indeed, by the triangle inequality and the Cauchy inequality, we have

$$\begin{aligned} &d_\infty\big(a_\infty, f_n(\hat{x}_n)\big)^2 \\ &\le 2\Big(d_\infty\big(a_\infty, f_n(a_n)\big) + \epsilon_n\Big)^2 + 2\Big(d_\infty\big(f_n(a_n), f_n(\hat{x}_n)\big) - \epsilon_n\Big)^2. \end{aligned} \tag{3.4}$$

We evaluate the second term in the right hand side of (3.4). By the definition of f_n, we have

$$\Big| d_n(a_n, \hat{x}_n) - d_\infty\big(f_n(a_n), f_n(\hat{x}_n)\big)\Big| \le \epsilon_n,$$

and

$$\Big| d_n(a_n, x_n) - d_\infty\big(f_n(a_n), f_n(x_n)\big)\Big| \le \epsilon_n.$$

Thus, by (3.3), we deduce

$$
\begin{aligned}
&\frac{k}{2}\Big(d_\infty\big(f_n(a_n), f_n(\hat{x}_n)\big) - \epsilon_n\Big)^2 - \frac{k}{2}\Big(d_\infty\big(f_n(a_n), f_n(x_n)\big) + \epsilon_n\Big)^2 \\
&\le \frac{k}{2}\Big(d_\infty\big(f_n(a_n), f_n(\hat{x}_n)\big) - \epsilon_n\Big)^2 - \frac{k}{2}\Big(d_\infty\big(f_n(a_n), f_n(x_n)\big) + \epsilon_n\Big)^2 \\
&+ \frac{l}{2}d_n(x_n, \hat{x}_n)^2 + \frac{m}{2}d_n(x_n, \hat{x}_n)^2 + \frac{m}{2}|t_n - \hat{t}_n|^2 \\
&\le (u^n)^*(\hat{t}_n, \hat{x}_n) - (u^n)^*(t_n, x_n) + \phi(\hat{t}_n) - \phi(t_n).
\end{aligned}
\tag{3.5}
$$

By (S4), we have

$$
(u^n)^*(\hat{t}_n, \hat{x}_n) - (u^n)^*(t_n, x_n) \le 2M. \tag{3.6}
$$

From (3.4), (3.5), and (3.6), we get

$$
\begin{aligned}
&d_\infty\big(a_\infty, f_n(\hat{x}_n)\big)^2 \\
&\le 2\Big(d_\infty\big(a_\infty, f_n(a_n)\big) + \epsilon_n\Big)^2 + 2\Big(d_\infty\big(f_n(a_n), f_n(x_n)\big) + \epsilon_n\Big)^2 + \frac{8M}{k} + \frac{4L_\phi d}{k},
\end{aligned}
$$

where L_ϕ is a Lipschitz constant of ϕ on some closed set including $[t_n - d, t_n + d]$. Thus, for large n, some closed ball independent of n in X_∞ contains $f_n(\hat{x}_n)$. Consequently, by the local compactness of X_∞, we have $f_n(\hat{x}_n)$ converges to some $c_1 \in X_\infty$ up to subsequences.

We prove that $f_n(\hat{x}_n) \to x_\infty$ and $\hat{t}_n \to t_\infty$. Using the triangle inequality and the definition of f_n, we have

$$
\begin{aligned}
d_\infty\big(x_\infty, f_n(\hat{x}_n)\big) &\le d_\infty\big(x_\infty, f_n(x_n)\big) + d_n(x_n, \hat{x}_n) + \epsilon_n \\
&\le d_\infty\big(x_\infty, f_n(x_n)\big) + d + \epsilon_n.
\end{aligned}
$$

Letting $n \to \infty$, we obtain $c_1 \in \overline{B_d(x_\infty)}$. Since $\hat{t}_n \in [t_n - d, t_n + d]$, by passing to a subsequence (still denoted by the same index), we deduce that $\hat{t}_n$ converges to some $c_2 \in (0, \infty)$. Furthermore, letting $n \to \infty$ in (3.3), we obtain that $f_n(\hat{x}_n) \to c_1 = x_\infty$ and $\hat{t}_n \to c_2 = t_\infty$, independently of the choice of subsequence.

We are now in a position to prove u^∞ is a metric viscosity subsolution of (1.3). Note that, by choosing m sufficiently large, one may ensure that F_n attains its maximum at an interior point of $[t_n - d, t_n + d] \times \overline{B_d(x_n)}$. Indeed, in the middle term of (3.5), we observe that $\frac{k}{2}\Big(d_\infty\big(f_n(a_n), f_n(\hat{x}_n)\big) - \epsilon_n\Big)^2 - \frac{k}{2}\Big(d_\infty\big(f_n(a_n), f_n(x_n)\big) + \epsilon_n\Big)^2 \to 0$. Therefore, by choosing m sufficiently large, we can make both $d_n(x_n, \hat{x}_n)$ and $|t_n - \hat{t}_n|$ arbitrarily small, uniformly in all sufficiently large n.

Consider the subsolution test function

$$
(\phi_1(t,x), \phi_2(t,x)) = \left(\phi(t) + \frac{k}{2}d_n(a_n, x)^2, \frac{l}{2}d_n(x_n, x)^2 + \frac{m}{2}d_n(x_n, x)^2 + \frac{m}{2}|t_n - t|^2\right).
$$

Since u^n is a metric viscosity subsolution of (1.1), it follows that

$$\partial_t \phi(\hat{t}_n) + m(\hat{t}_n - t_n) + (H_n)_{|\nabla_{d_n}\phi_2|^*(\hat{t}_n,\hat{x}_n)}\big(\hat{x}_n, kd_n(a_n,\hat{x}_n)\big) \le 0.$$

Moreover, by (S2), we have $(H_n)_{|\nabla_{d_n}\phi_2|^*(\hat{t}_n,\hat{x}_n)}\big(\hat{x}_n, kd_n(a_n,\hat{x}_n)\big) \ge H_n\big(\hat{x}_n, kd_n(a_n,\hat{x}_n) - ld_n(\hat{x}_n,x_n) - md_n(\hat{x}_n,x_n)\big)$. Consequently, we obtain

$$\partial_t \phi(\hat{t}_n) + m(\hat{t}_n - t_n) + H_n\big(\hat{x}_n, kd_n(a_n,\hat{x}_n) - ld_n(\hat{x}_n,x_n) - md_n(\hat{x}_n,x_n)\big) \le 0. \tag{3.7}$$

We deduce from (S2), the definition of f_n, and (3.7) that

$$\begin{aligned}&\partial_t \phi(\hat{t}_n) + m(\hat{t}_n - t_n)\\ &\quad + H_n\Big(\hat{x}_n,\, k\, d_\infty\big(f_n(a_n), f_n(\hat{x}_n)\big) - (l+m)\, d_\infty\big(f_n(\hat{x}_n), f_n(x_n)\big) - (k+l+m)\epsilon_n\Big) \le 0.\end{aligned}$$

Letting $n \to \infty$ and applying (S3), we have

$$\partial_t \phi(t_\infty) + H_\infty\big(x_\infty, kd_\infty(a_\infty, x_\infty)\big) \le 0.$$

Thus, by Proposition 3.1, u^∞ is a metric viscosity subsolution of (1.3).

Finally, we prove that u^∞ satisfies the initial condition (1.4). Let $x_\infty \in X_\infty$. Take arbitrary $t_n \in (0,\infty)$ and $x_n \in X_n$ such that $t_n \to 0$ and $f_n(x_n) \to x_\infty$ in X_∞. By the continuity at the initial time of u^n, we have

$$u^\infty(0,x_\infty) = \sup_{(t_n,x_n)} \limsup_{n\to\infty} (u^n)^*(t_n,x_n) \le \sup_{(x_n)} \limsup_{n\to\infty} (u^n)^*(0,x_n). \tag{3.8}$$

Since u^n satisfies the initial condition (1.2), we get

$$(u^n)^*(0,x_n) \le g^n(x_n). \tag{3.9}$$

By (S5) and the definition of the ϵ_n-inverse, the right hand side of (3.9) can be estimated from above as

$$g^n(x_n) \le g^\infty(x_\infty) + \eta(3\epsilon_n) + \Big| g^n\big(f_n' \circ f_n(x_n)\big) - g^\infty(x_\infty)\Big|. \tag{3.10}$$

Furthermore, by the triangle inequality, (S5) and the definition of the ϵ_n-inverse, we have

$$\begin{aligned}&\big|g^n\big(f_n' \circ f_n(x_n)\big) - g^\infty(x_\infty)\big|\\ &\le \eta\Big(d_\infty\big(f_n(x_n), x_\infty\big) + 4\epsilon_n\Big) + \big|g^n\big(f_n'(x_\infty)\big) - g^\infty(x_\infty)\big|.\end{aligned} \tag{3.11}$$

Thus, in view of (3.8), (3.9), (3.10), (3.11) and (S5), we conclude that

$$
\begin{aligned}
&u^\infty(0, x_\infty) - g^\infty(x_\infty) \le \sup_{(x_n)} \limsup_{n\to\infty} \{(u^n)^*(0, x_n) - g^\infty(x_\infty)\} \\
&\le \sup_{(x_n)} \limsup_{n\to\infty} \Big\{\eta(3\epsilon_n) + \eta\Big(d_\infty\big(f_n(x_n), x_\infty\big) + 4\epsilon_n\Big) + \big|g^n\big(f_n'(x_\infty)\big) - g^\infty(x_\infty)\big|\Big\} \\
&\le 0
\end{aligned}
$$

In the same way, we can prove that u_∞ is a metric viscosity supersolution of (1.3), (1.4). □

By Theorem 1.1, the convergence of metric viscosity solutions follows immediately.

Corollary 3.3 (Uniform convergence)**.** *We make the same assumptions as in Theorem 1.1. Furthermore, we assume that (X_n, d_n) and (X_∞, d_∞) are compact geodesic spaces. We also assume that the comparison principle holds for the Hamilton–Jacobi equation (1.3), (1.4); that is, for continuous metric viscosity subsolution u and supersolution v of (1.3), (1.4), we have $u \le v$ on $[0,\infty) \times X_\infty$. Then, $u^\infty = u_\infty (=: \overline{u})$ on $[0,\infty) \times X_\infty$ and u^n converges uniformly to $\overline{u}$:*

$$\lim_{n\to\infty} u^n(t_n, x_n) = \overline{u}(t, x), \quad (t, x) \in [0,\infty) \times X_\infty,$$

where $x_n \in X_n$ satisfies $f_n(x_n) \to x$ in X_∞ and $t_n \in [0,\infty)$ satisfies $t_n \to t$ in $[0,\infty)$ as $n \to \infty$.

Proof. By the definition of u^∞ and u_∞, we have $u^\infty \ge u_\infty$ in $[0,\infty) \times X_\infty$. Furthermore, by Theorem 1.1 and the comparison principle, we have $u^\infty \le u_\infty$. Consequently, $u^\infty = u_\infty$ in $[0,\infty) \times X_\infty$.

By a standard argument in the theory of viscosity solutions, we obtain

$$\lim_{n\to\infty} u^n(t_n, x_n) \ = \ \overline{u}(t, x), \quad (t, x) \in [0,\infty) \times X_\infty,$$

where $x_n \in X_n$ satisfies $f_n(x_n) \to x$ and $t_n \in [0,\infty)$ satisfies $t_n \to t$ as $n \to \infty$. Thus, we obtain the conclusion. □

4 Stability of the dual Kantorovich problem

To prove Proposition 1.2, we demonstrate the convergence of the Hopf–Lax semigroups under Gromov–Hausdorff convergence.

Lemma 4.1. *Let (X_n, d_n) and (X_∞, d_∞) be compact geodesic spaces. Assume (S1). We also assume that $g^n : X_n \to \mathbb{R}$ and $g^\infty : X_\infty \to \mathbb{R}$ satisfy the following condition.*

(S5)' The function g^n is equi-Lipschitz in n, that is, there exists a positive constant L such that, for $x, y \in X_n$, the inequality

$$|g^n(x) - g^n(y)| \leq L d_n(x, y)$$

holds. Furthermore, for $x \in X_\infty$, $g^n(f_n'(x))$ converges to $g^\infty(x)$ as $n \to \infty$.

Then, for $t > 0$, the Hopf–Lax semigroup $Q_t g^n$ converges uniformly to $Q_t g^\infty$ as $n \to \infty$. More precisely, for $t > 0$, the convergence

$$\lim_{n\to\infty} Q_t g^n(x_n) = Q_t g^\infty(x), \quad x \in X_\infty \tag{4.1}$$

holds, where $x_n \in X_n$ satisfies $f_n(x_n) \to x$ in X_∞ as $n \to \infty$.

Proof. We note that $Q_t g^n$ is the unique metric viscosity solution of

$$\partial_t u^n(t,x) + \frac{1}{2}|\nabla u^n(t,x)|^2 = 0, \quad (t,x) \in (0,\infty) \times X_n,$$

$$u^n(0,x) = g^n(x), \quad x \in X_n.$$

Likewise, $Q_t g^\infty$ is the unique metric viscosity solution on X_∞; see [9, Theorem 7.7].

We verify in turn the assumptions required for Corollary 3.3. Conditions (S2), (S3) can be checked directly. We can check that u^n is equi-Lipschitz in n by the equi-coercivity of the Hamiltonian and the equi-Lipschitz continuity of g^n in n (which follows by a standard argument from viscosity solution theory). Thus, by the compactness of X_n, we can prove that u^n satisfies (S4). The equi-Lipschitz continuity of u^n in n also implies the continuity at initial time of u^n and the semicontinuity of both u^∞ and u_∞. Moreover, (S5) follows immediately from (S5)'. Thus, by Corollary 3.3, we obtain (4.1). □

Remark 4.2. By the definition of an ϵ_n-inverse f_n' of f_n, the convergence (4.1) corresponds to the standard notion of convergence in optimal transport theory [27]. More precisely, the convergence

$$\lim_{n\to\infty} Q_t g^n \circ f_n'(x) = Q_t g^\infty(x)$$

holds for $t > 0$ and uniformly for $x \in X_\infty$.

We are now in position to prove Proposition 1.2 using Lemma 4.1. To this end, we may assume that f_n' is a Borel map.

Proof of Proposition 1.2. Let ϕ^n be a maximizer of (KAN$_n$). Without loss of generality, we can assume that ϕ^n is $\frac{d_n^2}{2}$-convex [27, Theorem 5.10 (iii)]. Fix a point $z \in X_\infty$. Define $c^n := \phi^n\big(f_n'(z)\big)$. Then, $\big(\phi^n - c^n\big)\big(f_n'(z)\big) = 0$ holds. Since ϕ^n is $\frac{d_n^2}{2}$-convex, $\phi^n - c^n$ is

1-Lipschitz with respect to $\frac{d_n^2}{2}$. Thus, we have

$$\begin{aligned}
&\big|(\phi^n - c^n)(x) - (\phi^n - c^n)(y)\big| \\
&\le \frac{d_n(x,y)^2}{2} \le \frac{\mathrm{diam}(X_n)}{2}\, d_n(x,y), \quad x, y \in X_n,
\end{aligned} \tag{4.2}$$

where $\mathrm{diam}(X_n) := \sup_{x,y\in X_n} d_n(x,y)$. By the compactness of X_∞ and the definition of f_n, we get

$$\mathrm{diam}(X_n) = \sup_{x,y\in X_n} d_n(x,y) \le \sup_{x,y\in X_n} d_\infty\big(f_n(x), f_n(y)\big) + \epsilon_n \le K \tag{4.3}$$

for some positive constant K independent of n. Combining (4.2) and (4.3), $\phi^n - c^n$ is equi-Lipschitz with respect to d_n. Furthermore, by the choice of z, we have

$$|(\phi^n - c^n)(x)| \le \frac{K}{2} d_n(x,z) \le \frac{K^2}{2}, \quad x \in X_n.$$

Thus, $\phi^n - c^n$ is uniformly bounded. By the Ascoli–Arzelà theorem, $(\phi^n - c^n)\circ f_n'$ converges uniformly on X_∞ to some Lipschitz function $\overline{\phi}$ up to subsequences. Applying Lemma 4.1, we obtain the uniform convergence of $Q_1(\phi^n - c^n)\circ f_n'$ to $Q_1(\overline{\phi})$.

Since ϕ^n is a maximizer of (KAN_n), we have

$$\begin{aligned}
&\int_{X_n} Q_1\big(\phi^n - c^n\big)\, d\nu_n - \int_{X_n} \big(\phi^n - c^n\big)\, d\mu_n \\
&= \int_{X_n} Q_1(\phi^n)\, d\nu_n - \int_{X_n} \phi^n\, d\mu_n = \frac{1}{2} W_2(\mu_n, \nu_n)^2.
\end{aligned} \tag{4.4}$$

By the triangle inequality, we observe

$$\begin{aligned}
&\Big|\int_{X_n} Q_1\big(\phi^n - c^n\big)(x)\, d\nu_n - \int_{X_\infty} Q_1(\overline{\phi})(x)\, d\nu_\infty\Big| \\
&\le \int_{X_n} \Big|Q_1\big(\phi^n - c^n\big)(x) - Q_1\big(\phi^n - c^n\big)\big(f_n' \circ f_n(x)\big)\Big|\, d\nu_n \\
&\quad + \Big|\int_{X_\infty} Q_1\big(\phi^n - c^n\big)\big(f_n'(x)\big)\, d\big(f_n\big)_{\#}\nu_n - \int_{X_\infty} Q_1\big(\phi^n - c^n\big)\big(f_n'(x)\big)\, d\nu_\infty\Big| \\
&\quad + \Big|\int_{X_\infty} Q_1(\overline{\phi})(x)\, d\nu_\infty - \int_{X_\infty} Q_1\big(\phi^n - c^n\big)\big(f_n'(x)\big)\, d\nu_\infty\Big|.
\end{aligned} \tag{4.5}$$

Since $Q_1(\phi^n - c^n)$ is equi-Lipschitz, the first term in the right hand side of (4.5) converges to 0 as $n \to \infty$. The second term is estimated above as

$$\begin{aligned}
&\Big|\int_{X_\infty} Q_1\big(\phi^n - c^n\big)\big(f_n'(x)\big)\, d(f_n)_{\#}\nu_n - \int_{X_\infty} Q_1(\overline{\phi})(x)\, d(f_n)_{\#}\nu_n\Big| \\
+&\Big|\int_{X_\infty} Q_1(\overline{\phi})(x)\, d(f_n)_{\#}\nu_n - \int_{X_\infty} Q_1\big(\phi^n - c^n\big)\big(f_n'(x)\big)\, d\nu_\infty\Big|.
\end{aligned}$$

Thus, the second term converges to 0 by the uniform convergence of $Q_1(\phi^n - c^n) \circ f_n'$. The third term also tends to zero, again due to the uniform convergence of $Q_1(\phi^n - c^n) \circ f_n'$. Consequently, the right side of (4.5) vanishes as $n \to \infty$.

A similar argument applies to the term $\int_{X_n} (\phi^n - c^n) d\mu_n$ in (4.4). Then, by the stability of 2-Wasserstein distance under measured Gromov–Hausdorff convergence [27, Theorem 28.6], we conclude

$$\int_{X_\infty} Q_1(\overline{\phi})\, d\nu_\infty - \int_{X_\infty} \overline{\phi}\, d\mu_\infty = \frac{1}{2} W_2(\mu_\infty, \nu_\infty)^2.$$

Therefore, $\overline{\phi}$ is a maximizer of (KAN_∞). □

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