Doctoral Dissertation

Flavor structures in nonfactorizable torus and orbifolds with background magnetic fluxes
（背景磁場入りの分解不可能なトーラスおよびオービフォールドコンパクト化におけるフレーバー構造）

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Abstract

We study the realization of flavor structures in quark and lepton from string compactifications such as six-dimensional (6D) torus T^6, or its twisted orbifolds $T^6/\mathbb{Z}_N$ with background magnetic fluxes. In this paper, we consider models with supersymmetry (SUSY). We study four-dimensional (4D) low energy effective field theory (EFT) at compactification scale.

We study fermion zero-mode wave functions in magnetized T^6. In particular, we focus on two cases. That is, when 6D overall chirality is fixed as positive, we analyze wave functions with positive chirality $(+++)$, and wave functions with various chiralities like $(+--)$, $(-+-)$, and $(--+)$ in magnetized T^6. First, we introduce these wave functions satisfying both the Dirac equation and the boundary conditions on T^6. Second, we introduce $SO(3)$, (or $SO(6)$) rotation and a new complex coordinate when we particularly consider the wave functions with various chiralities. We also derive the wave functions behaviors under the $Sp(6,\mathbb{Z})$ modular transformation in order to count the number of zero modes in magnetized $T^6/\mathbb{Z}_N$ orbifolds. We introduce magnetized $T^6/\mathbb{Z}_N$ models, in particular, $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$. This is because these lattices correspond to $SU(7)$ and E_6 root lattices, respectively. We then analyze the number of zero modes in magnetized $T^6/\mathbb{Z}_N$ by using generators of the $Sp(6,\mathbb{Z})$ modular symmetry.

For $T^6/\mathbb{Z}_7$, we do not find three-generation model to satisfy conditions that can realize 4D $\mathcal{N}=1$ SUSY EFT. For $T^6/\mathbb{Z}_{12}$, on the other hands, we have some three-generation models which can realize 4D $\mathcal{N}=1$ SUSY. We can realize them by not only the wave functions with positive chirality, but the wave functions with various chiralities. Lastly, in order to obtain 4D Yukawa couplings, we need to take chirality and symmetry into account. Then, we calculate overlap integral of left-handed, right-handed fermions, and Higgs sectors. This leads to Yukawa couplings taking chirality into account.

Moreover, we consider 4D $\mathcal{N}=1$ SUSY and $Sp(2g,\mathbb{Z})$, $(g=2,3)$, modular invariant EFT using the Yukawa couplings, the modular transformations, and chirality. The Yukawa couplings in 4D EFT can be derived from the superpotential. We find that the overlap integral leading to the Yukawa couplings in 4D EFT can be holomorphic under the modular symmetry when we take appropriate magnetic fluxes. Next, we study finite non-Abelian discrete groups. They are derived from the modular transformations and magnetic fluxes. We find some non-Abelian discrete symmetries in magnetized T^4 and T^6.

Contents

Chapter 1

Introduction

'Are we able to find the origin of existence and the universe?' Humans have been facing this problem for a long time and this remains one of the most fundamental unsolved problems even today. In current physics, the above question can be rephrased as 'Are we able to clarify the origin of elementary particle and its phenomenology?' Scientists have discovered various particles, for examples, electron, neutrino, quark, and Higgs boson. In recent years, observed particles have been classified as gauge boson, matter fermion, and Higgs boson, which is described in the Standard model (SM) of particle physics. The SM was established by the discovery of Higgs boson, and this is believed to be one of the most successful theory in particle physics. The SM has gauge symmetry $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$ to describe fundamental interactions, namely strong and electro-weak interactions for gauge bosons. Additionally, the SM is chiral theory and we need to regard matter particles as chiral fermions, namely left-handed and right-handed fermions. On the other hand, renormalizable couplings appear as gauge couplings, Yukawa couplings, and Higgs self-couplings. In particular, Yukawa couplings appear among matter left-handed fermions, right-handed fermions, and Higgs boson. They play an important role in generating gauge bosons masses and matter fermions masses. Specifically, both gauge bosons and chiral matter fermions have to be massless because of the gauge symmetries G_{SM} and chiral symmetry before spontaneously symmetry breaking. After the spontaneous gauge symmetry breaking, mass matrix in matter fermions is obtained by Yukawa couplings and vacuum expectation value (VEV) of Higgs boson. Furthermore, through the weak interactions, one can also obtain two kinds of mixing matrices called Cabbibo-Kobayashi-Masukawa (CKM), Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrices in quark, lepton sector, respectively. The mixing matrices are also consistent with the experimental values.

In this way, predictions in the SM quite consistently explains lots experimental facts so far in particle physics. Meanwhile, there are still unsolved mysteries of matter fermions in the SM. The origin of the flavor structure of quarks and leptons such as three-generation structure and mass hierarchy among generations is one of unsolved mysteries. Matter fermions are classified as charged-leptons, neutrinos, up-quarks, and down-quarks sectors, respectively. In these sectors, multiplets of matter fermions have a kind of repreating structure, what is called, three-generation structure. Furthermore, matter sectors except neutrinos have huge mass hierarchies, however, neutrinos have extremely small masses. Moreover, we need 22 real parameters in order to describe experimental facts in the SM. In the mixing matrices, mixing angles and the CP angles appear among the generations and the CP violation. The SM does not explain the origins of their values derived from the experiments. Thus, we need underlying physics beyond the SM to explain the origin of these parameters, in particular, the flavor structure. We can consider two approaches to understand the origin of the flavor structure, namely bottom-up and top-down approaches. For example, introducing non-Abelian discrete symmetries such as S_N, A_N, $\Delta(3N^2)$, $\Delta(6N^2)$ has been proposed to explain mixing angles in mixing matrices. This is one of bottom-up approaches. These groups have been used to realize phenomenological models that can explain the flavor structures such as the values of CKM/PMNS, mixing angles [1–11]. Modular flavor symmetries have been also proposed [12]. The word 'modular' implies the $SL(2,\mathbb{Z}) = Sp(2,\mathbb{Z})$ modular symmetry. In the models, the superpotentials are invariant under the above modular transformation. Furthermore, the finite modular groups Γ_N for $N = 2, 3, 4$ and 5 are isomorphic to the non-Abelian discrete symmetries S_3, A_4, S_4, and A_5, respectively [13]. That is, the modular flavor symmetric models in $\Gamma_2 \simeq S_3$ [14], $\Gamma_3 \simeq A_4$ [12, 15, 16], $\Gamma_4 \simeq S_4$ [17], $\Gamma_5 \simeq A_5$ [18, 19], and $\Gamma_6 \simeq S_3 \times A_4$ [20] have been proposed. Furthermore, when considering their double covering groups [21], for examples, the models with $\Gamma'_3 \simeq T'$ [22, 23], $\Gamma'_4 \simeq S'_4$ [24, 25], $\Gamma'_5 \simeq A'_5$ [26], and Γ'_6 [27] have been proposed so far. Lastly, the models with multi moduli in modular flavor symmetry have been proposed so far in Refs. [28–31]. (See for reviews [32, 33].)

If ideas for these bottom-up approaches are derived from top-down approach, that is very interesting because we can clarify these origins phenomenologically. That is, we can construct a lot of realistic models and obtain background physical laws that we have not found so far from bottom-up approaches. We are interested in such top-down approaches and assume one of them. That is called superstring theory.

Superstring theory is one of higher-dimensional theory and the best candidate for unified theory of particle physics which can describe behavior of fundamental interactions including quantum gravity [34–38]. In the following, we impose the supersymmetry (SUSY) and consider four-dimensional (4D) $\mathcal{N} = 1$ SUSY effective field theory (EFT) to realize chiral theory because the SM is chiral theory. We will mention its details in this paper. In superstring theory, on the other hand, ten-dimensional (10D) space-time background is required to be consistent with various properties such as Lorentz transformation and SUSY in background space-time. However, since we see 4D observed universe, we need to compactify the extra dimension, what is called, the compactification.

We expect that string compactification can solve mysteries in particle physics, e.g. origins of three-generation structure and the mass hierarchy of quarks and leptons. So far, many compactifications such as Calabi-Yau (CY) manifolds have been proposed. The CY manifold is one of the best candidates for compact space. In particular, the Calabi-Yau 3-fold (CY_3) manifolds can realize 4D $\mathcal{N} = 1$ SUSY from 10D $\mathcal{N} = 1$ SUSY because of the $SU(3)$ holonomy. Furthermore, topological properties such as characteristic class have been studied in the CY_3. Then, we can think of generation structure in 4D EFT by the Atitya-Singer index theorem [34,38–40]. However, the analysis of various fields on the CY_3 is very difficult. This is because the metric can be locally written, whereas the global and analytical representation is not straightforward on the CY_3 [40–45]. In other words, it is very difficult to describe the wave functions over the entire CY_3 because of the metric. Thus, it is extremely difficult to calculate the n-point couplings derived from an overlap integral of multiple fields on the CY_3 such as Yukawa couplings. Computing the values of Yukawa couplings and Higgs fields leads to mass matrices of quark and lepton, which is very important to verify whether we can quantitatively realize 4D realistic models or not. Thus, in this paper, we especially focus on flat compact spaces whose metrics are globally defined because one of our purposes is to realize 4D EFT of matter particles phenomenologically.

For examples, six-dimensional torus T^6 is one of the simplest examples of six-dimensional (6D) compactifications. However, 4D $\mathcal{N} = 4$ SUSY remains when we suppose T^6 compactification. This means that it has the $SU(4)_R$ symmetry, which leads to 4D non-chiral theory. On the other hands, orbifolding $T^6/\mathbb{Z}_N$ is one of compactifications to realize 4D chiral theory. We can see that 4D SUSY can be broken to $\mathcal{N} = 1$ or $\mathcal{N} = 0$, where the $SU(4)_R$ symmetry is also broken [46, 47]. However, 4D $\mathcal{N} = 2$ SUSY, that is, 4D non-chiral theory, can be realized by orbifolding.

Here, we can introduce background magnetic fluxes in compact space when we would like to realize 4D chiral theory [48–53]. Then, we can find the number of zero modes by the values of the magnetic fluxes. We can describe their properties in the compact space, however, they are non-trivial [54, 55]. Furthermore, we can calculate overlap integral of wave functions leading to Yukawa couplings in 4D low energy EFT [54–57]. Furthermore, we mention the modular symmetry. The modular symmetry is one of geometrical symmetries in compact space. This appears in 4D low energy EFT and controls allowed couplings. The modular transformations denote geometrical transformations corresponding to basis transformations in compact space. They can be used when counting zero-mode numbers in compact space. The modular transformation behaviors of zero-mode wave functions were also studied [58–67]. In these analyses, we need to introduce the wave functions explicitly.

The orbifold geometry can be related to background magnetic fluxes in compact space. In other words, we can consider toroidal orbifold compactification with background magnetic fluxes. So far, the six-dimensional space which can be factorizable to product of two-dimensional spaces was particularly studied. For examples, $T^2/\mathbb{Z}_N$ compactifications with magnetic fluxes [68, 69] were studied. When T^6 is factorized as $T^6 = T^2 \times T^2 \times T^2$, we can extend to $(T^2 \times T^2 \times T^2)/\mathbb{Z}_N$ and $(T^2 \times T^2 \times T^2)/(\mathbb{Z}_M \times \mathbb{Z}_N)$ compactifications. Then, one can classify three-generation models [70, 71]. Furthermore, realization of masses and mixing angles in quark and lepton was studied [72–77].

We can also extend such a study to nonfactorizable T^4, T^6, and their orbifolds with generic background magnetic fluxes [54, 55, 78]. One of our purposes is to study nonfactorizable cases. These compactifications lead to richer structures, because they include more degree of freedom of moduli. In this paper, we analyze their zero-mode numbers by the modular transformations in compact space. We show that we can realize three-generation models derived from magnetized T^6 or its $T^6/\mathbb{Z}_N$ orbifold compactifications. Here, we study $T^6/\mathbb{Z}_N$ orbifold models with magnetic fluxes, whose T^4 or T^6 parts are nonfactorizable. Since some nonfactorizable $T^4/\mathbb{Z}_N$ orbifolds were already studied [79,80], we analyze zero-mode numbers in magnetized T^6, or its orbifolds $T^6/\mathbb{Z}_N$. (See the details in Refs. [78,81,82].) In particular, we verify whether three-generation models are realized or not in magnetized $T^6/\mathbb{Z}_N$, $(N = 7, 12)$, with $SU(7)$ or E_6 root lattices, respectively. We need to obtain the wave functions and realize 4D chiral theory with three-generation structure. In Refs. [54, 55], the wave functions with $SO(6)$ chiralities $(+, +, +)$ and $(-, -, -)$ on T^6 have been already studied for the zero modes. They include the $SO(4)$ chiralities $(+,+)$ and $(-,-)$ on T^4. The wave functions with generic chirality were also studied in Refs. [54,55]. On the other hands, the index theorem can lead to only the number of zero modes [83–85].

We note that the invariant combinations of $SO(6)$ scalar, spinor, and vector representations appear in allowed couplings of 4D low energy EFT. It is very important to consider Yukawa couplings in 4D low energy EFT. In particular, in 4D $\mathcal{N}=1$ SUSY EFT, the invariant combinations of $SU(3)$ triplets are very important because they appear in 4D Yukawa couplings, where $SU(3) \subset SU(4)_R$. Therefore, we have to describe zero-mode wave functions with various chiralities in order to obtain 4D Yukawa couplings taking chirality and symmetry into account. This is one of our purposes in this paper. So far, since T^6 can be factorized as $T^6 = T_1^2 \times T_2^2 \times T_3^2$ and its background magnetic fluxes are non-vanishing on only each T_i^2, the wave functions on factorizable T^6 are obtained as just products of the wave functions $\psi_\pm^{(i)}$ on T_i^2, that is, $\psi_\pm^{(1)}\psi_\pm^{(2)}\psi_\pm^{(3)}$. In this paper, to consider 4D chiral theory with three-generation structure and 4D Yukawa couplings taking chirality and symmetry into account, we study zero-mode wave functions with various chiralities on generic T^6 with background magnetic fluxes. We note that the generic T^6 cannot be factorized as $T^6 = T_1^2 \times T_2^2 \times T_3^2$ and we can introduce the non-vanishing background magnetic fluxes over the generic T^6.

This paper is organized as follows. In chapter 2, we give a review on 10D $\mathcal{N}=1$ super Yang-Mills theory with magnetized D-brane. We particularly review magnetized $T^2 \times T^2 \times T^2$, T^2 with the $Sp(2,\mathbb{Z})$ modular symmetry, and its $T^2/\mathbb{Z}_N$ orbifold models. We will show that we can construct non-Abelian gauge groups $U(N)$ and its spontaneously symmetry breaking $U(N_1) \times U(N_2) \times U(N_3)$ by background magnetic fluxes for the Cartan directions of their gauge groups, where $N = N_1 + N_2 + N_3$. In this paper, we will consider the $U(1)$ gauge theory for simplicity. When we introduce the $(1,1)$-form magnetic flux on T^2, $T^2/\mathbb{Z}_N$, $(N = 2,3,4,6)$ models, we can realize 4D chiral theory with generation structures taking chirality and SUSY into account. Also, we can derive three-point couplings corresponding to Yukawa couplings in T^2. In chapter 3, we introduce the $(1,1)$-form background magnetic fluxes and boundary conditions on T^6 under the gauge symmetry. Then, we derive the F-term SUSY condition. Then, we introduce the massless Dirac equation and spinor wave functions with positive chirality or various chiralities. The wave functions satisfy both the Dirac equation and the boundary conditions. We also obtain the wave functions when we consider the $(1,1)$-form background magnetic fluxes and the $SO(3)$, (or $SO(6)$) rotations keeping 6D overall chirality invariant. We also analyze the bosonic sector and its mass spectrum.

When analyzing the mass spectrum by using the Laplacian, we derive the D-term SUSY condition to realize stable 4D effective field theory with 4D $\mathcal{N}=1$ SUSY. We also mention the wave functions behaviors under the $Sp(6,\mathbb{Z})$ modular symmetry. By use of the wave functions and its modular transformations, we can analyze the number of zero modes in $T^6/\mathbb{Z}_N$, particularly $N=7,12$. In chapter 4, we introduce $T^6/\mathbb{Z}_N$, $(N=7,12)$, orbifolds whose lattices are $SU(7)$, E_6 root lattices, respectively. Also, we explicitly analyze the number of the wave functions in their $\mathbb{Z}_N$ sectors. In chapter 5, we consider the overlap integral leading to Yukawa couplings taking chirality into account, and realize non-Abelian discrete symmetries from magnetized T^4 or T^6. Chapter 6 is devoted to the conclusion. In Appendix A, we briefly prove the g-dimensional Landsberg-Schaar relation. In Appendix B, we briefly consider magnetized T^6 with non-trivial scale factors. In Appendix C, we explicitly prove that wave functions in the main text satisfy boundary conditions in magnetized T^6. In Appendix D, we explicitly prove that wave functions in the main text satisfy the Dirac equation in magnetized T^6. In Appendix E, we consider wave functions with the non-vanishing Scherk-Schwarz phases and Wilson lines. In Appendix F, we briefly consider relation between complex coordinate z and z' defined in the main text. In Appendix G, we briefly review $Sp(2g,\mathbb{Z})$ symplectic group and prove modular transformations. In Appendix H, we consider the number of zero-mode wave functions with various chiralities in magnetized $T^6/\mathbb{Z}_7$ orbifold. In Appendix I, we explicitly calculate overlap integral leading to Yukawa couplings in magnetized T^6.

Chapter 2

Magnetized D-brane model

In this chapter, we review the magnetized D-brane models of the superstring theory. In particular, we focus on compactifications of the extra dimensions with flat bulk such as torus and its orbifolds to realize flavor structures of both quark and lepton sectors. First, if we consider ten-dimensional (10D) $\mathcal{N}=1$ supersymmetric models and suppose six-dimensional torus T^6 without magnetic fluxes, we obtain four-dimensional (4D) $\mathcal{N}=4$ supersymmetric effective field theory. Since this leads to 4D non-chiral theory, we cannot derive 4D chiral theory like the Standard model (SM). Therefore, we note that the supersymmetry (SUSY) has to be broken to 4D $\mathcal{N}=1$ or $\mathcal{N}=0$ in order to realize a 4D chiral theory. This is because the SM is chiral theory. In this paper, we basically consider supersymmetric models with magnetic fluxes along the compact space direction to keep 4D Lorentz transformation which we can obtain 4D chiral theory, and verify whether their vacua are stable or not. In this chapter, we particularly focus on magnetized $T^2 \times T^2 \times T^2$, T^2, and its orbifolds $T^2/\mathbb{Z}_N$, $(N = 2, 3, 4, 6)$. This chapter is along in Refs. [54, 58, 69, 86].

2.1 Super Yang-Mills theory

2.1.1 Magnetized $T_1^2 \times T_2^2 \times T_3^2$ model

We review 10D $\mathcal{N}=1$ $U(N)$ Super Yang-Mills (SYM) theory with magnetized D-brane as effective field theory (EFT) of Type IIB superstring theory [54]. In this chapter, we focus on three two-dimensional (2D) torus compactifications of the extra six-dimensions with background magnetic fluxes,

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6 = \mathbb{R}^{3,1} \times T_1^2 \times T_2^2 \times T_3^2, \tag{2.1}$$

where T_i^2 denotes the i-th flat 2D torus, $(i = 1, 2, 3)$. When we define basis vectors $e_i = 2\pi r_i$ and $\tau_i e_i$, $(r_i > 0,\ \tau_i \in \mathbb{C}_i)$, we find $T_i^2 \simeq \mathbb{C}_i/\Lambda_i$, where Λ_i denotes a 2D lattice spanned by e_i and $\tau_i e_i$. We suppose four-dimensional space-time $\mathcal{M}_4$ and the extra six-dimensional compact space $\mathcal{M}_6$ as the Minkowski space-time $\mathbb{R}^{3,1}$, and three of two-dimensional tori $T_1^2 \times T_2^2 \times T_3^2$, respectively. Many studies on six-dimensional (6D) compact space including the Calabi-Yau manifold have been conducted, but in this paper, we will particularly focus on studying nonfactorizable T^4, T^6 or their orbifolds with background magnetic fluxes rather than factorizable $T_1^2 \times T_2^2 \times T_3^2$ or its orbifolds. Nonfactorizable T^6 with background magnetic fluxes can be more general than factorizable $T^6 = T_1^2 \times T_2^2 \times T_3^2$ with respect to compact space. Furthermore, nonfactorizable geometries with background magnetic fluxes can lead to richer flavor models. In this chapter, we review magnetized T^2, and $T^2/\mathbb{Z}_N$ orbifolds. Then, we will focus on nonfactorizable T^6 or its orbifolds after this chapter. Additionally, in what follows, the background magnetic fluxes in this paper can be derived from non-dynamical vacuum expectation values of the gauge fields for only the direction of the compact spaces like $T_1^2 \times T_2^2 \times T_3^2$ under the Lorentz transformation on $\mathbb{R}^{3,1}$. We will introduce them specifically.

Let us consider 10D SYM action with $SO(9,1)$ Lorentz transformation

$$S = \int_{\mathbb{R}^{3,1}} d^4x \prod_{i=1,2,3} \int_{T_i^2} dz_i d\bar{z}_i \left[-\frac{1}{4g^2} \mathrm{Tr}(F^{MN} F_{MN}) + \frac{i}{2g} \mathrm{Tr}(\bar{\lambda} \Gamma^M D_M \lambda) \right], \tag{2.2}$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$ and $D_M \lambda = \partial_M \lambda - i[A_M, \lambda]$ for $M, N \in \{0, 1, \cdots, 8, 9\}$. The action S is $U(N)$ gauge invariant

$$\lambda \to U \lambda U^{-1}, \quad A_M \to U A_M U^{-1} - (\partial_M U) U. \tag{2.3}$$

λ represents gaugino fields and F_{MN} is the field strength of the 10D vector potential A_M.

$$\begin{aligned} \frac{i}{2g} \mathrm{Tr}(\bar{\lambda} \Gamma^M D_M \lambda) &= \frac{i}{2g} \mathrm{Tr}(-i \bar{\lambda} \Gamma^M [A_M, \lambda]) + \cdots \\ &= \frac{i}{2g} \mathrm{Tr}(-i \bar{\lambda} \Gamma^\mu [A_\mu, \lambda]) + \frac{i}{2g} \mathrm{Tr}(-i \bar{\lambda} \Gamma^m [A_m, \lambda]) + \cdots, \end{aligned} \tag{2.4}$$

where $\mu \in \{0,1,2,3\}$ and $m \in \{4,5,\cdots,8,9\}$. We note that A_m is the extra 6D vector fields which are regarded as 4D scalar fields. In this paper, we consider Kluza-Klein (KK) decomposition for vector fields A_M and gaugino (spinors) λ. The KK decomposition makes A_M and λ decompose into the 4D space-time parts $\mathcal{M}_4 = \mathbb{R}^{3,1}$ and product of three tori $T_1^2 \times T_2^2 \times T_3^2$ parts, namely

$$\begin{aligned} A_M(x^\mu, z_i) &= \sum_{n_1,n_2,n_3} \phi_{M,n_1,n_2,n_3}(x^\mu) \otimes \phi_{M,n_1}(z_1) \otimes \phi_{M,n_2}(z_2) \otimes \phi_{M,n_3}(z_3), \\ \lambda(x^\mu, z_i) &= \sum_{n_1,n_2,n_3} \psi_{M,n_1,n_2,n_3}(x^\mu) \otimes \psi_{M,n_1}(z_1) \otimes \psi_{M,n_2}(z_2) \otimes \psi_{M,n_3}(z_3), \end{aligned} \tag{2.5}$$

where $\phi_{M,n_i}(z_i)$, $\psi_{M,n_i}(z_i)$, $(i = 1,2,3)$, denote the n_i-th excited modes of 2D vector fields, the 2D Majorana-Weyl spinors, respectively, on the T_i^2. In this paper, we focus on solutions of fermion massless modes (zero modes). On the other hands, the compactification and the KK decomposition can lead to transformation of subgroup of the $SO(9,1)$. That is, we can consider transformation of

$$SO(3,1) \times SO(6) \subset SO(9,1) \tag{2.6}$$

on $\mathbb{R}^{3,1} \times \mathcal{M}_6$. Then, $SO(6)$ rotation transformation can exist on the $\mathcal{M}_6$. We will particularly use this $SO(6)$ rotation when we consider magnetized T^6 model. On magnetized $T_1^2 \times T_2^2 \times T_3^2$, we take $SO(2) \times SO(2) \times SO(2)$ rotation as transformation of a subgroup of the $SO(6)$.

Because of Eqs. (2.2), (2.5), when the Dirac equation on 10D space-time background and the KK decomposition hold in the models, we can introduce 4D bosons $\phi(x^\mu)$, fermions $\psi(x^\mu)$, and Yukawa couplings derived from Eqs. (2.4), (2.5). On the other hands, we take complete system of eigen functions of the Dirac operator $i\not{D}_6$ and the Laplace operator Δ_6 on extra six-dimensional compact space:

$$\begin{aligned} i\not{D}_6 \psi_n(z) &= m_n \psi_n(z), \\ \Delta_6 \phi_{M,n}(z) &= M_{M,n}^2 \phi_{M,n}(z). \end{aligned} \tag{2.7}$$

We will specifically consider these two equations and their solutions in magnetized T^{2g}, $(g = 1,2,3)$, models. In particular, we focus on the massless fermions, that is, $m_n = 0$ for any n, in this paper.

Additionally, we comment a complex structure modulus. Shape of two-dimensional torus is determined by complex structure modulus τ that is regarded as vacuum expectation value of graviton for compact space direction. We will describe the detail when we introduce the modular transformation. In the following, we describe background metric in this model. The ten-dimensional (10D) background metric is given by [95]

$$ds^2 = G_{MN}dX^M dX^N = \eta_{\mu\nu}dx^\mu dx^\nu + g_{ij}dx^i dy^j, \tag{2.8}$$

where $M, N \in \{0, 1, \cdots, 8, 9\}$, and $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, +)$, $\mu, \nu \in \{0, 1, 2, 3\}$, denotes the metric of 4D Minkowski space-time background. On the other hand, $i = 1, 2, 3$ labels compact space T_i^2 directions, and x, y denote real coordinates with $0 \le x, y \le 1$ because of the identifications $x^i \sim x^i + 1$ and $y^i \sim y^i + 1$.

In this chapter, since we suppose a product of factorizable three tori, $T_1^2 \times T_2^2 \times T_3^2$, their complex structure are given by τ_1, τ_2, and τ_3, respectively. We can then take metric of 6D compact space

$$g_{ij} = \delta_{i,j} dZ^i d\bar{Z}^j = \begin{bmatrix} g^{(1)} & 0 & 0 \\ 0 & g^{(2)} & 0 \\ 0 & 0 & g^{(3)} \end{bmatrix}, \tag{2.9}$$

where $g^{(i)}$ denotes the metric of each $\mathbb{C}_i$, $(i = 1, 2, 3)$, and is given by

$$g^{(i)} = (2\pi r_i)^2 \begin{bmatrix} 1 & \mathrm{Re}\tau_i \\ \mathrm{Re}\tau_i & |\tau_i|^2 \end{bmatrix}, \tag{2.10}$$

where we define complex coordinates on $\mathbb{C}_i$-plane, $Z_i \equiv 2\pi r_i(x_i + \tau_i y_i) \equiv 2\pi r_i z_i$ and $\bar{Z}_i \equiv 2\pi r_i(x_i + \bar{\tau}_i y_i) \equiv 2\pi r_i \bar{z}_i$, $(r_i > 0,\ i = 1, 2, 3)$. We use complex coordinates of each i-th torus T_i^2, z_i, with the identification $z^i \sim z^i + e_i \sim z^i + \tau_i e_i$, in what follows. The area of the i-th torus, $\mathcal{A}^{(i)}$, is given by

$$\mathcal{A}^{(i)} = (2\pi r_i)^2 \mathrm{Im}\tau_i, \tag{2.11}$$

where we will make use of the above area when we consider a normalization constant of wave functions in magnetized T^2 model. Also, in the following, we will normalize a scale factor $2\pi r_i = 1$.

Next, we introduce background magnetic flux on each torus T_i^2, F, in order to realize a 4D $\mathcal{N} = 1$ chiral theory,

$$F \equiv F_{z_i \bar{z}_j} \cdot \delta_{i,j}(idz^i \wedge d\bar{z}^j) = \pi(\mathrm{Im}\tau_i)^{-1} \begin{bmatrix} M_a^i \mathbf{1}_{N_a} & & \\ & M_b^i \mathbf{1}_{N_b} & \\ & & M_c^i \mathbf{1}_{N_c} \end{bmatrix} (idz^i \wedge d\bar{z}^i), \tag{2.12}$$

where $i = 1, 2, 3$ and $\mathbf{1}_{N_a}$ denotes $N_a \times N_a$ unit matrix. Since the above F is $(1,1)$-form, the components $F_{z_i \bar{z}_j}$ need to be real symmetric. Then, M_a^i can be taken as real number. Note that $N = N_a + N_b + N_c$. Then, the $U(N)$ gauge symmetry is broken to $U(N_a) \times U(N_b) \times U(N_c)$ gauge symmetry because we introduce background magnetic fluxes along the diagonal (Cartan) direction of $U(N_a) \times U(N_b) \times U(N_c)$ by the real fluxes $M^i_{a,b,c}$.

Similarly to magnetized $T_1^2 \times T_2^2 \times T_3^2$, we can take $U(N)$ gauge theory by magnetic fluxes along the $U(N)$ diagonal direction,

$$F = \pi(\mathrm{Im}\tau)^{-1} M_a \mathbf{1}_{N_a} (idz \wedge d\bar{z}). \tag{2.13}$$

We can break the $U(N)$ gauge symmetry to $U(N_1) \times U(N_2) \times U(N_3)$ gauge symmetry by the following fluxes

$$F = \pi(\mathrm{Im}\tau)^{-1} \begin{bmatrix} M_1 \mathbf{1}_{N_1} & & \\ & M_2 \mathbf{1}_{N_2} & \\ & & M_3 \mathbf{1}_{N_3} \end{bmatrix} (idz \wedge d\bar{z}), \tag{2.14}$$

where $N_a = N_1 + N_2 + N_3$ and $M_i \neq M_j$, $(i, j \in \{1, 2, 3\})$. Then, when we introduce the fluxes, we can realize non-Abelian gauge theory such as $U(N)$, $U(N_1) \times U(N_2) \times U(N_3)$. Since the gauge group of the SM is $SU(3) \times SU(2) \times U(1)$, and we can describe $U(N) \simeq SU(N) \times U(1)$ ignoring the quotient by the center $\mathbb{Z}_N$, we can realize the gauge group of the SM from $U(3) \times U(2) \times U(1)$ derived from the fluxes. In the following, we consider magnetized T^2 model with the $U(1)$ gauge symmetry, for simplicity. Note that the following discussions can hold not only magnetized T^2 models, but magnetized $T_1^2 \times T_2^2 \times T_3^2$ models.

2.1.2 Magnetized T^2 model

We consider T^2 with the basis $e_1 = 2\pi r$, $e_2 = \tau e_1$, the radius of T^2, r, complex structure modulus $\tau \equiv e_2/e_1$, and complex coordinate $z = x + \tau y$, $(0 \leq x, y \leq 1,\ z \sim z+1 \sim z+\tau)$ [54,69]. Here, we have introduced the KK decomposition, the Dirac equation, and the eigen equation of the Laplacian on compact space, that is, Eqs. (2.5), (2.7). Then, we can consider massive modes as well as massless modes. However, since we are interested in Yukawa couplings as three-point couplings among massless modes, we consider not KK massive modes but massless modes, that is, zero modes. Thus, we analyze only zero modes in this paper. We take $2\pi r = 1$ and $\mathrm{Im}\tau > 0$, in the following. We introduce the upper-half plane $\mathcal{H}$ as follows

$$\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\tau > 0\} \tag{2.15}$$

to describe well-defined spinor wave functions in magnetized T^2. We will use this. We introduce $U(1)$ background magnetic flux on T^2,

$$F = F_{z\bar{z}}(idz \wedge d\bar{z}) = \pi M(\mathrm{Im}\tau)^{-1}(idz \wedge d\bar{z}). \tag{2.16}$$

Thus we introduce the corresponding $U(1)$ vector potential, $A = A(z, \bar{z})$, by $F = dA$:

$$\begin{aligned} A(z,\bar{z}) &= \pi \mathrm{Im}[M(\bar{z}+\bar{\eta})(\mathrm{Im}\tau)^{-1}dz] \\ &= -\frac{i\pi}{2}[M(\bar{z}+\bar{\eta})(\mathrm{Im}\tau)^{-1}]dz + \frac{i\pi}{2}[M(z+\eta)(\mathrm{Im}\tau)^{-1}]d\bar{z} \\ &\equiv A_z dz + A_{\bar{z}} d\bar{z}, \end{aligned} \tag{2.17}$$

where we describe the complex Wilson line η under the $U(1)$ gauge symmetry on T^2. We note that we can always make the η absorb into the Scherk-Schwarz (SS) phases by appropriate redefinition of the gauge field and the gauge transformation. We can consider models and their wave functions with the non-vanishing complex Wilson line when we use notations $z_w = z + \eta$, $\bar{z}_w = \bar{z} + \bar{\eta}$ in the $U(1)$ background gauge fields A_z and $A_{\bar{z}}$. The complex Wilson lines are global, $U(1)$ gauge-invariant, and complex constant on T^2, so their derivatives with respect to the T^2 coordinates become zero. We explicitly describe following derivatives with respect to T^2 coordinates z, $\bar{z}$, under the $U(1)$ gauge symmetry:

$$\frac{\partial z_w}{\partial z} = \frac{\partial \bar{z}_w}{\partial \bar{z}} = 1, \quad \frac{\partial \bar{z}_w}{\partial z} = \frac{\partial z_w}{\partial \bar{z}} = 0. \tag{2.18}$$

This means that we take

$$\frac{\partial \eta}{\partial z} = \frac{\partial \bar{\eta}}{\partial z} = \frac{\partial \eta}{\partial \bar{z}} = \frac{\partial \bar{\eta}}{\partial \bar{z}} = 0. \tag{2.19}$$

Under the identification $z \sim z+1 \sim z+\tau$, we calculate the $U(1)$ gauge transformations corresponding to the boundary conditions on T^2:

$$\begin{aligned} A(z+1) &= A(z) + d\zeta_1(z_w), \\ A(z+\tau) &= A(z) + d\zeta_\tau(z_w), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \zeta_1(z_w) &\equiv \pi M(\mathrm{Im}\tau)^{-1}\mathrm{Im}z_w, \\ \zeta_\tau(z_w) &\equiv \pi \mathrm{Im}[M\bar{\tau}(\mathrm{Im}\tau)^{-1}z_w] \\ &= \pi M(\mathrm{Re}\tau(\mathrm{Im}\tau)^{-1}\mathrm{Im}z_w - \mathrm{Re}z_w). \end{aligned} \tag{2.21}$$

We will use the boundary conditions when we introduce spinor wave functions on magnetized T^2 under the $U(1)$ gauge symmetry. We have introduced the metric on $\mathbb{C}$:

$$ds^2 = dZd\bar{Z}. \tag{2.22}$$

Here, when we normalize the scale factor $2\pi r = 1$, the metric simply becomes $dzd\bar{z}$. When we introduce the following Pauli matrices, σ_i, $(i = 1, 2, 3)$,

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{2.23}$$

we define the Gamma matrices $\Gamma^z, \Gamma^{\bar{z}}$ to introduce the Dirac equation on T^6:

$$\Gamma^z \equiv \frac{1}{2\pi r}\Gamma^Z = \frac{1}{2\pi r}\sigma^z = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \ \Gamma^{\bar{z}} \equiv \frac{1}{2\pi r}\Gamma^{\bar{Z}} = \frac{1}{2\pi r}\sigma^{\bar{z}} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \tag{2.24}$$

where $\sigma^z = \sigma^1 + i\sigma^2$, $\sigma^{\bar{z}} = \sigma^1 - i\sigma^2$. By use of them, we also define the chirality Γ^5

$$\Gamma^5 \equiv \sigma^3. \tag{2.25}$$

In order to define the Dirac operator $i\not{D}_2$ on T^2, we need to describe the covariant derivatives D_z, $D_{\bar{z}}$ as follows:

$$D_z = \partial_z - iA_z, \ D_{\bar{z}} = \partial_{\bar{z}} - iA_{\bar{z}}, \tag{2.26}$$

where we take $U(1)$ charge q as $+1$. Therefore, we introduce the Dirac operator

$$i\not{D}_2 \equiv i(\Gamma^z D_z + \Gamma^{\bar{z}} D_{\bar{z}}) = i\begin{bmatrix} 0 & 2D_z \\ 2D_{\bar{z}} & 0 \end{bmatrix}. \tag{2.27}$$

When we describe a 2D massless Majorana-Weyl spinor $\Psi = [\psi_+, \psi_-]^T$, the Dirac equation of the spinor Ψ, in particular zero modes, in magnetized T^2 is given by

$$i\not{D}_2\Psi = i\begin{bmatrix} 0 & 2D_z \\ 2D_{\bar{z}} & 0 \end{bmatrix}\begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = 0. \tag{2.28}$$

From Eq. (2.21), the boundary condition of spinor wave functions can be obtained under the $U(1)$ gauge symmetry:

$$\begin{aligned} \psi_\pm(z+1) &= e^{2\pi i\alpha_1^S} e^{i\zeta_1(z_w)}\psi_\pm(z), \\ \psi_\pm(z+\tau) &= e^{2\pi i\beta_\tau^S} e^{i\zeta_\tau(z_w)}\psi_\pm(z), \end{aligned} \tag{2.29}$$

where α_1^S, β_τ^S, $(0 \leq \alpha_1^S, \beta_\tau^S \leq 1)$, are the so-called Scherk-Schwarz (SS) phases. The boundary conditions Eq. (2.29) are pseudo-periodic and consistent with condition that the action should be single-valued. When we consider a contractible loop $z \to z+1 \to z+1+\tau \to z+\tau \to z$ along the lattice Λ spanned by two basis vectors $e_1 = 2\pi r = 1$, $e_2 = \tau e_1 = \tau$, and use Eq. (2.29), we obtain $\psi(z) = e^{2\pi i M} \cdot \psi(z)$. We therefore find a following condition

$$M \in \mathbb{Z} \tag{2.30}$$

since the wave functions have to be single-valued. This is the Dirac quantization condition, and the flux M must be quantized in T^2. In terms of topology, this corresponds to the first Chern number $c_1 = \frac{1}{2\pi}\int_{T^2} F$, where F denotes the $U(1)$ background magnetic flux on T^2.

2.1.3 Zero-mode wave functions in magnetized T^2 model

So far, we have introduced the Dirac equation (2.28), the boundary conditions Eq. (2.29) and the Dirac quantization condition $M \in \mathbb{Z}$. We particularly focus on the massless Dirac equation,

$$\begin{aligned} D_{\bar{z}}\psi_+ &= \left(\partial_{\bar{z}} + \frac{\pi}{2}[M(z+\eta)(\mathrm{Im}\tau)^{-1}]\right)\psi_+ = 0, \\ D_z\psi_- &= \left(\partial_z - \frac{\pi}{2}[M(\bar{z}+\bar{\eta})(\mathrm{Im}\tau)^{-1}]\right)\psi_- = 0, \end{aligned} \tag{2.31}$$

$$\begin{aligned} \psi_\pm(z+1) &= e^{2\pi i\alpha_1^S} e^{i\zeta_1(z_w)}\psi_\pm(z), \\ \psi_\pm(z+\tau) &= e^{2\pi i\beta_\tau^S} e^{i\zeta_\tau(z_w)}\psi_\pm(z). \end{aligned} \tag{2.32}$$

We can obtain the solutions of fermion zero-mode wave functions ψ_+, ψ_-, with the non-trivial SS phases and complex Wilson lines [69],

$$\begin{aligned} \psi_+ &= \mathcal{N} \cdot e^{-2\pi i(J+\alpha_1^S)^T M^{-1}\beta_\tau^S} \cdot e^{i\pi M(z+\eta)(\mathrm{Im}\tau)^{-1}\mathrm{Im}(z+\eta)} \\ &\quad \cdot \theta\begin{bmatrix}(J+\alpha_1^S)M^{-1} \\ -\beta_\tau^S\end{bmatrix}(M(z+\eta), M\tau) \\ &\equiv \psi_+^{(J+\alpha_1^S,\beta_\tau^S,M)}(z+\eta,\tau), \end{aligned} \tag{2.33}$$

$$\begin{aligned} \psi_- &= \mathcal{N} \cdot e^{-2\pi i(J+\alpha_1^S)^T M^{-1}\beta_\tau^S} \cdot e^{i\pi M(\bar{z}+\bar{\eta})(\mathrm{Im}\bar{\tau})^{-1}\mathrm{Im}(\bar{z}+\bar{\eta})} \\ &\quad \cdot \theta\begin{bmatrix}(J+\alpha_1^S)M^{-1} \\ -\beta_\tau^S\end{bmatrix}(M(\bar{z}+\bar{\eta}), M\bar{\tau}) \\ &\equiv \psi_-^{(J+\alpha_1^S,\beta_\tau^S,M)}(\bar{z}+\bar{\eta},\bar{\tau}), \end{aligned} \tag{2.34}$$

where $J \in \{0, 1, \cdots, |M|-1\}$ labels $|M|$-independent zero modes, and $\mathcal{N}$ denotes normalization constant defined by

$$\int_{T^2} dz d\bar{z}\ (\psi^J)^* \psi^K = (2\mathrm{Im}\tau)^{-\frac{1}{2}} \delta_{J,K}, \quad \mathcal{N} = M^{\frac{1}{4}} \cdot \mathcal{A}^{-\frac{1}{2}}. \tag{2.35}$$

We also define the Jacobi Theta-functions with characters as follows

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) &\equiv \sum_{m \in \mathbb{Z}} e^{i\pi(m+a)\tau(m+a)} e^{2\pi i(m+a)(z+b)}, \\ &\qquad (a, b \in \mathbb{R}, \quad z \in \mathbb{C}, \quad \tau \in \mathcal{H}). \end{aligned} \tag{2.36}$$

We stress that the wave functions with positive chirality are well-defined only if $M\tau \in \mathcal{H}$. Then, the flux M must be positive under $\mathrm{Im}\tau > 0$. Similarly, the flux M must be negative under $\mathrm{Im}\tau > 0$ when we consider the wave functions with negative chirality because $M\bar{\tau}$ has to be an element of $\mathcal{H}$. Therefore, the solutions of the zero-mode wave functions under $\mathrm{Im}\tau > 0$ are well-defined if

$$\begin{aligned} \psi &= \psi_+^{(J+\alpha_1^S, \beta_\tau^S, M)}(z + \eta, \tau) \quad \text{for} \quad M > 0, \\ \psi &= \psi_-^{(J+\alpha_1^S, \beta_\tau^S, M)}(\bar{z} + \bar{\eta}, \bar{\tau}) \quad \text{for} \quad M < 0. \end{aligned} \tag{2.37}$$

That is, if $M > 0$, we can interpret that we realize the wave functions corresponding to 4D left-handed fermion. If $M < 0$, we can interpret that we realize the wave functions corresponding to 4D right-handed fermion. Furthermore, complex conjugates of these wave functions correspond to their 4D anti-particles, according to Ref. [54]. This implies that we can realize chiral wave functions by flux M. Additionally, the number $|M|$ corresponds to the number of independent fermion zero modes in magnetized T^2, so the number $|M|$ can be regarded as 4D $|M|$-generation number, according to the KK decompositions. In other words, we can realize 4D chiral theories with $|M|$-generation structure when we consider T^2 compactification with the non-trivial flux $M \neq 0$. For example, when we take $M = 3$ on T^2, we can realize 4D chiral theory with three-generation structure.

2.1.4 Three-point couplings and mass in bosonic sector

Here, we note that the products of the Jacobi-theta functions with characters are expressed as [54]

$$\begin{aligned}
&\theta\begin{bmatrix}\frac{r}{M_1}\\0\end{bmatrix}(z_1,M_1\tau)\times\theta\begin{bmatrix}\frac{s}{M_2}\\0\end{bmatrix}(z_2,M_2\tau)\\
&=\sum_{m\in\mathbb{Z}_{M_1+M_2}}\theta\begin{bmatrix}\frac{r+s+M_1m}{M_1+M_2}\\0\end{bmatrix}(z_1+z_2,(M_1+M_2)\tau)\\
&\qquad\times\theta\begin{bmatrix}\frac{M_2r-M_1s+M_1M_2m}{M_1M_2(M_1+M_2)}\\0\end{bmatrix}(M_2z_1-M_1z_2,M_1M_2(M_1+M_2)\tau).
\end{aligned}\tag{2.38}$$

By use of Eq. (2.38), when we suppose the vanishing Wilson line and Scherk-Schwarz (SS) phases, that is, $\eta=\alpha_1^S=\beta_\tau^S=0$, we can derive the following product expansions of the zero-mode wave functions corresponding to left-handed and right-handed fermions under the gauge symmetry

$$\psi_+^{I,|M_1|}(z,\tau)\cdot\psi_+^{J,|M_2|}(z,\tau)=\sum_{K\in\mathbb{Z}_{(|M_1|+|M_2|)}}Y_{T^2}^{IJK}(\tau)\cdot\psi^{I+J+K|M_2|,(|M_1|+|M_2|)}(z,\tau),\tag{2.39}$$

where $Y_{T^2}^{IJK}=Y_{T^2}^{IJK}(\tau)$ represents three-point couplings

$$\begin{aligned}
Y_{T^2}^{IJK}&=(2\mathrm{Im}\tau)^{-\frac{1}{2}}\cdot\mathcal{A}^{-\frac{1}{2}}\cdot\left|\frac{M_1M_2}{|M_1|+|M_2|}\right|^{\frac{1}{4}}\\
&\quad\cdot\sum_{m=0}^{|M_2|-1}\theta\begin{bmatrix}\frac{|M_2|I-|M_1|J+|M_1||M_2|m}{|M_1||M_2|(|M_1|+|M_2|)}\\0\end{bmatrix}(0,|M_1||M_2|(|M_1|+|M_2|)\tau)\\
&\quad\cdot\delta_{I+J-K,(|M_1|+|M_2|)n-|M_1|m},\quad(n\in\mathbb{Z}).
\end{aligned}\tag{2.40}$$

The normalization conditions leads to the following overall integral

$$y_{T^2}^{IJK}=gY_{T^2}^{IJK}=g(2\mathrm{Im}\tau)^{\frac{1}{2}}\cdot\int_{T^2}dzd\bar{z}\;\psi^{I,|M_1|}\cdot\psi^{J,|M_2|}\cdot(\psi^{K,(|M_1|+|M_2|)})^*.\tag{2.41}$$

In order to realize SUSY in 4D EFT, we introduce the D-term SUSY condition for the bosonic sectors. When we define the Laplacian Δ on compact space, we can derive the eigen equation of the Laplacian under the 2D massless spinor fields by calculating the squared Dirac operators as follows

$$(i\not{D}_2)^2\Psi=\left[\frac{1}{2}\{D^\dagger,D\}+2i\begin{bmatrix}F_{z\bar{z}}&0\\0&F_{\bar{z}z}\end{bmatrix}\right]\Psi\equiv\left[\Delta+2i\begin{bmatrix}F_{z\bar{z}}&0\\0&F_{\bar{z}z}\end{bmatrix}\right]\Psi=0.\tag{2.42}$$

Here, we normalize the scale factor as $2\pi R = 1$. Then, we derive the eigen equation of the Laplacian Δ

$$\Delta\Psi = -2i \begin{bmatrix} F_{z\bar{z}} & 0 \\ 0 & F_{\bar{z}z} \end{bmatrix} \Psi, \tag{2.43}$$

which leads to the following equation

$$\Delta\psi_+ = \frac{2\pi M}{\mathcal{A}}\psi_+. \tag{2.44}$$

When we consider $T_1^2 \times T_2^2 \times T_3^2$ again, we can expand the above equation and find the eigenvalues

$$m_i^2 = \frac{2\pi M_i}{\mathcal{A}_i}, \quad i = 1, 2, 3. \tag{2.45}$$

We will consider the D-term SUSY condition where 4D scalar becomes massless and realize the D-flat vacuum in chapter 3.

2.1.5 $Sp(2, \mathbb{Z})$ modular symmetry

We review $Sp(2, \mathbb{Z})$ modular symmetry on T^2, and modular form. (See Refs. [23,61,103].) The T^2-torus is described by lattice Λ as $T^2 \simeq \mathbb{C}/\Lambda$. The Λ is spanned by the basis vectors e_1, e_2. When we consider a complex structure modulus τ and complex coordinate z on T^2, they are given by

$$\tau \equiv \frac{e_2}{e_1}, \quad z = \frac{Z}{e_1}, \quad \text{where} \quad Z \in \mathbb{C}. \tag{2.46}$$

We can transform the lattice vectors e_1, e_2, as follows

$$\begin{bmatrix} e_2 \\ e_1 \end{bmatrix} \Rightarrow \begin{bmatrix} e_2' \\ e_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e_2 \\ e_1 \end{bmatrix} \equiv \gamma \begin{bmatrix} e_2 \\ e_1 \end{bmatrix}. \tag{2.47}$$

Let us consider the γ. Since the basis vectors construct the lattice Λ and the T^2-torus, we need to take a, b, c, d as integers when we consider basis transformation. Additionally, when we keep the area (volume) of the lattice, we impose a condition $ad - bc = 1$. Then, we take γ as an element of the $SL(2, \mathbb{Z})$

$$SL(2, \mathbb{Z}) = \left\{ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\} = Sp(2, \mathbb{Z}). \tag{2.48}$$

We find $Sp(2g,\mathbb{Z}) \subseteq SL(2g,\mathbb{Z})$, $(g=1,2,3,\cdots)$. In particular, we obtain $Sp(2,\mathbb{Z}) = SL(2,\mathbb{Z})$, because the matrix $\gamma \in Sp(2,\mathbb{Z})$ has the determinant 1. In the context of $Sp(2,\mathbb{Z})$, we can describe the γ as

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a,b,c,d \in \mathbb{Z}, \quad \gamma J \gamma^T = J, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.49}$$

Then, the complex structure modulus τ and the complex coordinate z on T^2 are transformed under the $Sp(2,\mathbb{Z})$ modular symmetry

$$\begin{aligned} \gamma : \tau = \frac{e_2}{e_1} &\Rightarrow \frac{ae_2 + be_1}{ce_2 + de_1} = \frac{a\tau + b}{c\tau + d}, \\ \gamma : z = \frac{Z}{e_1} &\Rightarrow \frac{Z}{ce_2 + de_1} = \frac{z}{c\tau + d}. \end{aligned} \tag{2.50}$$

The $Sp(2,\mathbb{Z})$ modular symmetry has two generators, S and T. We can represent them as follows

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{2.51}$$

They satisfy the following algebraic structures

$$S^2 = -\mathbf{1}_2, \quad S^4 = (ST)^3 = \mathbf{1}_2. \tag{2.52}$$

Under these two generators, the τ and z are transformed as

$$\begin{aligned} S &: (z,\tau) \Rightarrow \left(-\frac{z}{\tau}, -\frac{1}{\tau}\right), \\ T &: (z,\tau) \Rightarrow (z, \tau+1). \end{aligned} \tag{2.53}$$

When we consider the complex structure modulus behavior under the modular symmetry, we obtain the fixed modular-invariant modulus. For examples, when using the algebraic structures in Eq. (2.52), we find the S-invariant or ST-invariant modulus, τ_S, τ_{ST} as elements of the $\mathcal{H}$, respectively,

$$\begin{aligned} S &: \tau_S = -\frac{1}{\tau_S} \Rightarrow \tau_S = i \in \mathcal{H}, \\ ST &: \tau_{ST} = -\frac{1}{\tau_{ST}+1} \Rightarrow \tau_{ST} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \equiv \omega \in \mathcal{H}. \end{aligned} \tag{2.54}$$

We will make use of the above invariant modulus when we analyze magnetized $T^2/\mathbb{Z}_N$, $(N=3,4,6)$ orbifold models.

Additionally, let us study the modular forms. When we consider complex function $f(\tau)$, we can define the modular form of weight k. The modular form $f(\tau)$ is given as the holomorphic function of the τ [23]

$$f(\gamma\tau) = (c\tau + d)^k f(\tau). \tag{2.55}$$

This is the modular form of singlet. The factor $(c\tau + d)^k$ is called the automorphy factor. Similarly to the singlet modular form, when we construct multiplets of $f_i(\tau)$, the modular forms are given as holomorphic functions of τ

$$f_i(\gamma\tau) = (c\tau + d)^k \rho_{ij}(\gamma) f_j(\tau), \tag{2.56}$$

where $\rho_{ij}(\gamma)$ denotes a complex representation of the γ. Let us take an example of the modular forms. In magnetized T^2, we have found the massless spinor wave function with positive chirality, that is, $\psi_+^{(J+\alpha_1^S,\beta_\tau^S)}(z+\eta,\tau)$. Here, we suppose the vanishing Wilson line and SS phases, namely $\eta = \alpha_1^S = \beta_\tau^S = 0$. Then, we can derive the wave functions with flux $M > 0$ under the modular S, T transformations

$$\begin{aligned}
S : \psi_+^J\left(-\frac{z}{\tau}, -\frac{1}{\tau}\right) &= (-\tau)^{\frac{1}{2}} \cdot \frac{e^{\pi i/4}}{\sqrt{M}} \sum_{K=0}^{M-1} e^{2\pi i J M^{-1} K} \cdot \psi_+^K(z,\tau) \\
&= (-\tau)^{\frac{1}{2}} \cdot \sum_{K=0}^{M-1} \rho_{JK}(S) \cdot \psi_+^K(z,\tau), \\
T : \psi_+^J(z, \tau+1) &= e^{\pi i J M^{-1} K} \cdot \delta_{J,K} \cdot \psi_+^K(z,\tau) \\
&= \rho_{JK}(T) \cdot \psi_+^K(z,\tau),
\end{aligned} \tag{2.57}$$

where $J, K \in \{0, 1, \cdots, M-1\}$, and

$$\rho_{JK}(S) = \frac{e^{\pi i/4}}{\sqrt{M}} \cdot e^{2\pi i J M^{-1} K}, \quad \rho_{JK}(T) = e^{\pi i J M^{-1} K} \cdot \delta_{J,K}. \tag{2.58}$$

Here, the flux M becomes even under the modular T symmetry. In chapter 5, we will extend these representations of the modular symmetry to nonfactorizable T^4, T^6 with background magnetic fluxes.

2.1.6 Magnetized $T^2/\mathbb{Z}_N$ models

In this subsection, we give a review on magnetized $T^2/\mathbb{Z}_N$, $(N = 2, 3, 4, 6)$, orbifold models. This subsection is along in Refs. [58, 69]. So far, we have considered magnetized T^2 model. The T^2-torus is constructed by the identification $z \sim z+1 \sim$

$z+\tau$ derived from the lattice Λ, and has the $SO(2)$ rotation transformation. In addition to the identification, we can impose the $\mathbb{Z}_N$-twist identification, that is,

$$z \sim \theta z, \quad \theta = e^{2\pi i/N}, \quad \theta^N = 1. \tag{2.59}$$

This means that we can divide T^2 by $\mathbb{Z}_N$. This is the $T^2/\mathbb{Z}_N$ orbifold. In general, we define the $T^2/\mathbb{Z}_N$ orbifold as the following identification

$$z \sim \theta z + m\tau + n, \tag{2.60}$$

where m, n are arbitrary integers. It is known that there are four kinds of the $T^2/\mathbb{Z}_N$, namely $T^2/\mathbb{Z}_2$, $T^2/\mathbb{Z}_3$, $T^2/\mathbb{Z}_4$, and $T^2/\mathbb{Z}_6$. For example, we can take arbitrary modulus $\tau \in \mathcal{H}$ in $T^2/\mathbb{Z}_2$, however, we must fix the modulus $\tau = \theta = e^{2\pi i/N}$ because of the crystallography in $T^2/\mathbb{Z}_N$, $(N = 3, 4, 6)$ [87].

We define the fixed points $z_{\rm fp}$ as follows

$$z_{\rm fp} = \theta z_{\rm fp} + m + n\tau, \tag{2.61}$$

where $m, n \in \mathbb{Z}$. By use of the above definition, we can generalize the formula as follows

$$z_{\rm fp} = \theta z_{\rm fp} + (\theta - 1)(m' + n'\tau) + m + n\tau, \tag{2.62}$$

where $m', n' \in \mathbb{Z}$. This means that we introduce the identification $(\theta, m + n\tau) \sim (\theta, (\theta - 1)(m' + n'\tau) + m + n\tau)$. For example, we consider fixed points of $T^2/\mathbb{Z}_3$. The lattice is fixed and the modulus is fixed as $\tau_{ST} = \omega$ because of $(ST)^3 = \mathbf{1}_2$. Then, we can derive

$$\begin{aligned} z_{\rm fp} &= \omega z_{\rm fp} + (\omega - 1)(m' + n'\tau) + m + n\tau, \\ (1-\omega) z_{\rm fp} &= -(1-\omega)(m' + n'\tau) + m + n\tau, \\ z_{\rm fp} &= -(m' + n'\tau) + \frac{m + n\tau}{1 - \omega}. \end{aligned} \tag{2.63}$$

Thus we obtain

$$z_{\rm fp} = 0,\ \frac{2+\tau}{3},\ \frac{1+2\tau}{3}. \tag{2.64}$$

Similarly, we can derive the four fixed points of $T^2/\mathbb{Z}_2$ as follows

$$z_{\rm fp} = 0,\ \frac{1}{2},\ \frac{\tau}{2},\ \frac{1+\tau}{2}, \tag{2.65}$$

where τ is not fixed generally in $T^2/\mathbb{Z}_2$. Two fixed points $z_{\rm fp} = 0, \frac{1+\tau}{2}$ and one fixed point $z_{\rm fp} = 0$ exist in $T^2/\mathbb{Z}_4$ and $T^2/\mathbb{Z}_6$, respectively.

Furthermore, we define a $\mathbb{Z}_N$-twist boundary condition of the 2D spinor $\Phi_{T^2/\mathbb{Z}_N} = [\phi_{(T^2/\mathbb{Z}_N,+)}, \phi_{(T^2/\mathbb{Z}_N,-)}]^T = [\phi_+, \phi_-]^T$ in magnetized $T^2/\mathbb{Z}_N$. First, since we have the $SO(2)$ rotation transformation on T^2, we need to introduce $Spin(2)$ spinor rotation $\mathcal{S}^{spin}$ to describe spinor rotation on T^2. Specifically, the $Spin(2)$ is defined as double covering group of $SO(2)$, which implies that we can describe rotation of 2D spinor $\Phi = [\phi_+, \phi_-]^T$ on the $\mathbb{C}$. Also, when we consider the $(1,1)$-form background magnetic flux $F = F_{z\bar{z}}(idz \wedge d\bar{z})$ under $SO(2)$ rotation keeping 2D chirality σ^3 invariant, we can specifically write rotation of the 2D spinor under following transformation $f = f(\mathrm{Re}z, \mathrm{Im}z)$ in $(\mathrm{Re}z, \mathrm{Im}z)$ basis

$$f : (\mathrm{Re}z, \mathrm{Im}z) \to (\mathrm{Re}(\theta z), \mathrm{Im}(\theta z)), \tag{2.66}$$

where $\theta \in \mathbb{C}$. In particular, we can regard this rotation as the $\mathbb{Z}_N$-twist under the identifications $\mathrm{Re}z \sim \mathrm{Re}(\theta z)$ and $\mathrm{Im}z \sim \mathrm{Im}(\theta z)$ such that $\theta^N = 1$. We thus consider $\mathbb{Z}_N$-twist in the complex coordinates $(z, \bar{z})$, namely

$$\mathbb{Z}_N : (z, \bar{z}) \Rightarrow (\theta z, \bar{\theta}\bar{z}). \tag{2.67}$$

We will also explain the details of magnetized T^6 written in Chapter 3 . Then, we can derive the general element $\mathcal{S}^{spin} \in Spin(2)$ as follows

$$\begin{aligned}
\mathcal{S}^{spin} &\equiv \exp\left(\theta_{1\bar{1}}\Sigma^{1\bar{1}}\right) \\
&= \exp\left(\frac{\theta_{1\bar{1}}}{4i}[\Gamma^z, \Gamma^{\bar{z}}]\right) \\
&= \exp\left(-i\theta_{1\bar{1}}\sigma^3\right) \\
&= \exp\left(\begin{bmatrix} -i\theta_{1\bar{1}} & 0 \\ 0 & i\theta_{1\bar{1}} \end{bmatrix}\right) \\
&= \begin{bmatrix} e^{-i\theta_{1\bar{1}}} & 0 \\ 0 & e^{i\theta_{1\bar{1}}} \end{bmatrix},
\end{aligned} \tag{2.68}$$

where $\theta_{1\bar{1}} \in \mathbb{R}$. Here, we take $spin(2)$-algebra generator $\Sigma^{1\bar{1}} \equiv \frac{1}{4i}[\Gamma^z, \Gamma^{\bar{z}}] = -i\sigma^3$ and define a matrix exponential $\exp(A)$ as follows

$$\exp(A) \equiv \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \quad (\exp(A))_{ab} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} (A^k)_{ab}, \tag{2.69}$$

where A denotes $n \times n$ complex matrix and $a, b \in \{1, 2, \cdots, n\}$. We also define A^0 as $n \times n$ unit matrix $\mathbf{1}_{n\times n}$. From Eq. (2.68), we can introduce the spinor rotation under $Spin(2)$

$$\Phi(\theta z, \bar{\theta}\bar{z}) = \mathcal{S}^{spin}\Phi(z, \bar{z}) = \begin{bmatrix} e^{-i\theta_{1\bar{1}}} & 0 \\ 0 & e^{i\theta_{1\bar{1}}} \end{bmatrix} \begin{bmatrix} \phi_+(z, \bar{z}) \\ \phi_-(z, \bar{z}) \end{bmatrix}. \tag{2.70}$$

To define the $\mathbb{Z}_N$-twist boundary condition of the 2D spinor Φ, we can make use of degree of freedom derived from $U(1)$ global symmetry in the action. That is, we can introduce the boundary conditions taking the $U(1)$ global symmetry as well as the $Spin(2)$ transformation into account. Note that the $U(1)$ written as the global group is different from the $Spin(2)$ written as the spinor rotation. We first define and fix the boundary condition of the spinor ϕ_+ as follows

$$\phi_+(\theta z, \bar{\theta}\bar{z}) \equiv V(z, \bar{z}) \cdot \phi_+(z, \bar{z}) \equiv e^{2\pi i\beta} \cdot \phi_+(z, \bar{z}) \equiv e^0 \cdot e^{-i\theta_{1\bar{1}}} \cdot \phi_+(z, \bar{z}), \tag{2.71}$$

where e^0, $e^{-i\theta_{1\bar{1}}}$ come from the $U(1)$ global transformation, the $Spin(2)$ spinor rotation transformation, respectively. Also, $V(z, \bar{z})$ is defined as transformation function for $\mathbb{Z}_N$-twist, and β is a real number such that $\beta N \equiv 0 \pmod 1$. Then, we can take $\beta = \frac{m}{N}$, where $m \in \{0, 1, \cdots, N-1\}$ denotes integer called $\mathbb{Z}_N$-charge. Then, we can take the $\theta_{1\bar{1}}$ as follows

$$\begin{aligned} e^{2\pi i\frac{m}{N}} &= e^{-i\theta_{1\bar{1}}} \cdot e^{2\pi i\ell}, \quad \ell \in \mathbb{Z}, \\ \theta_{1\bar{1}} &= 2\pi \left[\ell - \frac{m}{N}\right]. \end{aligned} \tag{2.72}$$

Next, we can introduce the boundary condition of the spinor ϕ_- taking the $U(1)$ global symmetry and the $Spin(2)$ rotation transformation into account using a global function χ

$$\phi_-(\theta z, \bar{\theta}\bar{z}) \equiv e^{i\chi} \cdot e^{2\pi i\left[\ell - \frac{m}{N}\right]} \cdot \phi_-(z, \bar{z}) \equiv e^{\frac{2\pi i}{N}[m+1]} \cdot \phi_-(z, \bar{z}), \tag{2.73}$$

where we can assign $\chi = 2\pi\left[\frac{2m+1}{N} + \ell'\right]$, $(\ell' \in \mathbb{Z})$. We conclude that we can define the following boundary condition of the $\Phi_{T^2/\mathbb{Z}_N} = \Phi_{T^2/\mathbb{Z}_N}(z, \bar{z})$ as follows [69, 86]

$$\begin{aligned} \Phi_{T^2/\mathbb{Z}_N}(\theta z, \bar{\theta}\bar{z}) &\equiv \begin{bmatrix} \phi_{(T^2/\mathbb{Z}_N,+)}(\theta z, \bar{\theta}\bar{z}) \\ \phi_{(T^2/\mathbb{Z}_N,-)}(\theta z, \bar{\theta}\bar{z}) \end{bmatrix} \\ &= e^{2\pi i\frac{m}{N}} \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{N}} \end{bmatrix} \begin{bmatrix} \phi_{(T^2/\mathbb{Z}_N,+)}(z, \bar{z}) \\ \phi_{(T^2/\mathbb{Z}_N,-)}(z, \bar{z}) \end{bmatrix} \\ &\equiv V(z, \bar{z})\mathcal{S}^{twist}\Phi_{T^2/\mathbb{Z}_N}(z, \bar{z}), \end{aligned} \tag{2.74}$$

where we introduce 2×2 complex matrix $\mathcal{S}^{twist}$ so that ψ_+ acts trivially under the $\mathbb{Z}_N$-twist,

$$\mathcal{S}^{twist} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{N}} \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & \rho' \end{bmatrix}, \tag{2.75}$$

where $\rho' \equiv e^{\frac{2\pi i}{N}}$. By use of the m written in the $\theta_{1\bar{1}}$, we find the $\mathbb{Z}_N$-twist boundary conditions of ϕ_+ and ϕ_-,

$$\begin{aligned} \phi_{(T^2/\mathbb{Z}_N,+)}(\theta z, \bar{\theta}\bar{z}) &= (\rho')^m \cdot \phi_{(T^2/\mathbb{Z}_N,+)}(z,\bar{z}), \\ \phi_{(T^2/\mathbb{Z}_N,-)}(\theta z, \bar{\theta}\bar{z}) &= (\rho')^{m+1} \cdot \phi_{(T^2/\mathbb{Z}_N,-)}(z,\bar{z}), \end{aligned} \tag{2.76}$$

where $m \in \{0, 1, \cdots N-1\}$ denotes $\mathbb{Z}_N$-twist charges, what is called, $\mathbb{Z}_N$-charges, and leads to $\mathbb{Z}_N$-sectors in magnetized $T^2/\mathbb{Z}_N$ orbifold models. Note that only ϕ_+ is excited and $\phi_- = 0$ when the flux M is fixed as positive. Similarly, $\phi_+ = 0$ and only ϕ_- is excited when the M is fixed as negative. Thus, the wave functions of magnetized $T^2/\mathbb{Z}_N$ orbifold can be described as the linear combinations of the wave functions in magnetized T^2 [69, 86]

$$\begin{aligned} \phi^{(J+\alpha_1^S, \beta_\tau^S, M)}_{(T^2/\mathbb{Z}_N^m,+)}(z) &= \mathcal{N}_{T^2/\mathbb{Z}_N} \cdot \sum_{k=0}^{N-1} (\rho')^{-km} \cdot \psi^{(J+\alpha_1^S, \beta_\tau^S, M)}_{(T^2,+)}((\rho')^k z), \\ \phi^{(J+\alpha_1^S, \beta_\tau^S, M)}_{(T^2/\mathbb{Z}_N^m,-)}(z) &= \mathcal{N}_{T^2/\mathbb{Z}_N} \cdot \sum_{k=0}^{N-1} (\rho')^{-k(m+1)} \cdot \psi^{(J+\alpha_1^S, \beta_\tau^S, M)}_{(T^2,-)}((\rho')^k z). \end{aligned} \tag{2.77}$$

When we determine the value of the flux M on T^2, we find that we can realize a chiral theory with $|M|$-generation structure. On the other hand, when we consider $T^2/\mathbb{Z}_N$, we define D_m as degenerated zero-mode number in a $\mathbb{Z}_N$-sector with the charge m, $(m \in \{0, 1, \cdots, N-1\})$. That is, we can realize N different sectors called $\mathbb{Z}_N$-sectors with $\mathbb{Z}_N$-twist charges, $e^{2\pi i \frac{m}{N}}$, $(m \in \{0, 1, \cdots, N-1\})$, and we obtain degenerated zero modes in each sector. Then, we derive a following relation

$$D_0 + D_1 + \cdots + D_{N-1} = |M|, \tag{2.78}$$

and may find more three-generation models in each sector than the T^2. Furthermore, according to Eq. (2.57), we have the modular transformations of the wave functions on magnetized T^2. According to Ref. [58], since the lattices of $T^2/\mathbb{Z}_N$, $(N = 3, 4)$, can be realized by the algebraic structure $(ST)^3 = S^4 = \mathbf{1}_2$ written in the generators of the $Sp(2,\mathbb{Z})$ modular symmetry, each complex structure modulus $\tau = \tau_\gamma$ is also fixed as $\tau_{ST} = \omega = e^{2\pi i/3}$, $\tau_S = i$ for $N = 3, 4$, respectively.

For example, let us consider $T^2/\mathbb{Z}_3$ with $SU(3)$ root lattice. The modulus can be fixed as $\tau = e^{2\pi i/3} = \omega$. Additionally, since we can introduce this modulus by the algebraic structure $(ST)^3 = \mathbf{1}_2$, that is, $\omega = \tau_{ST}$, the ST transformation leads to $\mathbb{Z}_3$-twist. The modular transformations of the wave functions on T^2 corresponding to their $\mathbb{Z}_3$-twist are expressed as

$$ST : \psi_{(T^2,+)}(\omega z) = \rho(ST) \cdot \psi_{(T^2,+)}(z). \tag{2.79}$$

On the other hands, the eigen states of $T^2/\mathbb{Z}_3$ can be given by

$$\begin{aligned}
\phi_{(T^2/\mathbb{Z}_3^0,+)}(\omega z) &= \phi_{(T^2/\mathbb{Z}_3^0,+)}(z), \\
\phi_{(T^2/\mathbb{Z}_3^1,+)}(\omega z) &= \omega \cdot \phi_{(T^2/\mathbb{Z}_3^1,+)}(z), \\
\phi_{(T^2/\mathbb{Z}_3^2,+)}(\omega z) &= \omega^2 \cdot \phi_{(T^2/\mathbb{Z}_3^2,+)}(z).
\end{aligned} \tag{2.80}$$

Also, the $\rho(ST)$ has $\mathbb{Z}_3$ eigenvalues $1, \omega, \omega^2$ because of $(ST)^3 = \mathbf{1}_2$. From the linear combinations of the zero-mode wave functions in Eq. (2.77) and the eigenvalues of $\rho(ST)$, we can find following relations

$$\begin{aligned}
&D_0 + D_1 + D_2 = |M|, \\
&D_0 + \omega D_1 + \omega^2 D_2 = \mathrm{tr}\rho(ST).
\end{aligned} \tag{2.81}$$

We can generally derive following relations in $T^2/\mathbb{Z}_N$ orbifold

$$\begin{aligned}
&\sum_{k=0}^{N-1} D_k = |M|, \\
&\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}k} \cdot D_k = \mathrm{tr}\rho(\gamma),
\end{aligned} \tag{2.82}$$

where $\gamma \in Sp(2,\mathbb{Z})$ such that $\gamma^N = \mathbf{1}_2$. Since we can find the eigenvalues of the states $\phi_{(T^2/\mathbb{Z}_3^k,+)}$, we classify the number of zero modes in $\mathbb{Z}_N$ sectors with the charge k.

In what follows, let us analyze the number of zero modes with positive chirality, ϕ_+, in magnetized $T^2/\mathbb{Z}_N$, $(N = 2, 3, 4, 6)$, and verify whether three-generation models can be realized or not. Additionally, we will consider the non-trivial SS phases [69]. We define the function $U_{m+n\tau}(z)$ as

$$\begin{aligned}
\Phi_{T^2/\mathbb{Z}_N}(z + m + n\tau) &\equiv U_{m+n\tau}(z) \cdot \Phi_{T^2/\mathbb{Z}_N}(z). \\
\Phi_{T^2/\mathbb{Z}_N}(\theta^k(z + m + n\tau)) &\equiv U_{\theta^k(m+n\tau)}(\theta^k z) \cdot \Phi_{T^2/\mathbb{Z}_N}(\theta^k z),
\end{aligned} \tag{2.83}$$

where $U_1(z) = e^{2\pi i\alpha_1^S} e^{i\zeta_1(z_w)}$ and $U_\tau(z) = e^{2\pi i\beta_\tau^S} e^{i\zeta_\tau(z_w)}$. Here, we suppose the vanishing Wilson line $\eta = 0$. They have the following properties

$$\begin{aligned}
&U_{m+n\tau+1} = U_1(z+n\tau)U_{m+n\tau}(z), \\
&U_{m+(n+1)\tau} = U_\tau(z+m)U_{m+n\tau}(z), \\
&U_m(z) = (U_1(z))^m, \quad U_{n\tau}(z) = (U_\tau(z))^n. \\
&U_{-(m+n\tau)}(z) = U^{-1}_{m+n\tau}(z-m-n\tau)
\end{aligned} \tag{2.84}$$

We have to impose the relation

$$U_{m+n\tau}(z) = U_{\theta^k(m+n\tau)}(\theta^k z) \tag{2.85}$$

$$\begin{aligned}
&U_1(z) = U_{-1}(-z), \quad U_\tau(z) = U_{-\tau}(-z) \quad \text{for } N=2, \\
&U_1(z) = U_\theta(\theta z), \quad U_\tau(z) = U_{\theta\tau}(\theta z), \quad \text{for } N = 3,4,6.
\end{aligned} \tag{2.86}$$

For $N = 2$, we obtain the following conditions

$$e^{2\pi i\alpha_1^S} = e^{-2\pi i\alpha_1^S}, \quad e^{2\pi i\beta_\tau^S} = e^{-2\pi i\beta_\tau^S}. \tag{2.87}$$

Then, we find

$$(\alpha_1^S, \beta_\tau^S) \equiv (0,0),\ (1/2,0),\ (0,1/2),\ (1/2,1/2), \quad (\text{mod } 1). \tag{2.88}$$

For $N = 3$, we obtain

$$e^{2\pi i\alpha_1^S} = e^{2\pi i\beta_\tau^S}, \quad e^{2\pi i\beta_\tau^S} = e^{-2\pi i(\alpha_1^S+\beta_\tau^S)+i\pi M}, \tag{2.89}$$

that is, we find

$$\begin{aligned}
&(\alpha_1^S, \beta_\tau^S) = (0,0),\ (1/3,1/3),\ (2/3,2/3), \quad \text{for } M = 2\ell,\ (\ell \in \mathbb{Z}), \\
&(\alpha_1^S, \beta_\tau^S) = (1/6,1/6),\ (1/2,1/2),\ (5/6,5/6), \quad \text{for } M = 2\ell+1, (\ell \in \mathbb{Z}).
\end{aligned} \tag{2.90}$$

For $N = 4$, we obtain

$$e^{2\pi i\alpha_1^S} = e^{2\pi i\beta_\tau^S}, \quad e^{2\pi i\beta_\tau^S} = e^{-2\pi i\alpha_1^S}, \tag{2.91}$$

that is, we find

$$(\alpha_1^S, \beta_\tau^S) \equiv (0,0),\ (1/2,1/2), \quad (\text{mod } 1). \tag{2.92}$$

For $N = 6$, we obtain

$$e^{2\pi i\alpha_1^S} = e^{2\pi i\beta_\tau^S}, \quad e^{2\pi i\beta_\tau^S} = e^{2\pi i(-\alpha_1^S+\beta_\tau^S)+i\pi M}, \tag{2.93}$$

that is, we find

$$\begin{aligned} &(\alpha_1^S, \beta_\tau^S) \equiv (0,0), \quad \text{for } M = 2\ell,\ \ell \in \mathbb{Z}, \\ &(\alpha_1^S, \beta_\tau^S) \equiv (1/2,1/2), \quad \text{for } M = 2\ell+1,\ \ell \in \mathbb{Z}. \end{aligned} \tag{2.94}$$

We comment the SS phases for magnetized $T^2/\mathbb{Z}_6$ orbifold. Since $\mathbb{Z}_6$ group has $\mathbb{Z}_2$ and $\mathbb{Z}_3$ subgroups, we find the non-trivial SS phases as common SS phases. For examples, when we take an even flux $M = 2\ell$, ($\ell \in \mathbb{Z}$), we find the SS phases $(0,0)$, $(1/3,1/3)$, and $(2/3,2/3)$ in $T^2/\mathbb{Z}_3$ orbifold. Also, we find the SS phases $(0,0)$, $(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$ in $T^2/\mathbb{Z}_2$ orbifold. Then, the common SS phase is only $(0,0)$. For an odd flux $M = 2\ell+1$, ($\ell \in \mathbb{Z}$), we find the SS phases $(1/6,1/6)$, $(1/2,1/2)$, and $(5/6,5/6)$ in $T^2/\mathbb{Z}_3$ orbifold. Also, we find the SS phases $(0,0)$, $(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$ in $T^2/\mathbb{Z}_2$ orbifold. Then, the common SS phase is only $(1/2,1/2)$.

Magnetized $T^2/\mathbb{Z}_2$ model

We consider the number of zero modes with positive chirality in magnetized $T^2/\mathbb{Z}_2$. Suppose a positive flux $M > 0$. The wave functions of $T^2/\mathbb{Z}_N$ are described as

$$\phi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2/\mathbb{Z}_2^m,+)}(z) = \mathcal{N}_{T^2/\mathbb{Z}_N}(\psi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2,+)}(z) \pm \psi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2,+)}(-z)), \tag{2.95}$$

where $m = 0,1$ describe $\mathbb{Z}_2$ even, $\mathbb{Z}_2$ odd zero modes, respectively. We impose the $\mathbb{Z}_2$-twist boundary condition

$$\phi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2/\mathbb{Z}_2^m,+)}(-z) = e^{i\pi m}\phi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2/\mathbb{Z}_2^m,+)}(z), \tag{2.96}$$

Under the $\mathbb{Z}_2$-twist $z \Rightarrow -z$, we find

$$\psi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2,+)}(-z) = \psi^{(M-(J+\alpha_1^S),-\beta_\tau^S)}_{(T^2,+)}(z) = e^{-4\pi i(J+\alpha_1^S)\beta_\tau^S/M} \cdot \psi^{(M-(J+\alpha_1^S),\beta_\tau^S)}_{(T^2,+)}(z). \tag{2.97}$$

Then, the zero modes on magnetized $T^2/\mathbb{Z}_2$ orbifold are given by

$$\phi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2/\mathbb{Z}_2,+)}(z) = \mathcal{N}_{T^2/\mathbb{Z}_2} \cdot \left(\psi^{(J+\alpha_1^S,\beta_\tau^S)}_{(T^2,+)}(z) \pm e^{-4\pi i(J+\alpha_1^S)\beta_\tau^S/M} \cdot \psi^{(M-(J+\alpha_1^S),\beta_\tau^S)}_{(T^2,+)}(z)\right). \tag{2.98}$$

When we take $(\alpha_1^S, \beta_\tau^S) = (0,0)$, the wave functions are given by

$$\phi^J_{(T^2/\mathbb{Z}_2,+)}(z) = \mathcal{N}_{T^2/\mathbb{Z}_2} \cdot \left(\psi^J_{(T^2,+)}(z) \pm \psi^{M-J}_{(T^2,+)}(z)\right). \tag{2.99}$$

They correspond to the $\mathbb{Z}_2$ even and odd modes. For $M = 2\ell$, $(\ell \in \mathbb{Z}, \ell > 0)$, we find the following relation for the $\mathbb{Z}_2$ even modes

$$\phi^J_{(T^2/\mathbb{Z}_2^0,+)}(z) = \phi^{M-J}_{(T^2/\mathbb{Z}_2^0,+)}(z) \tag{2.100}$$

for $J = 0, 1, 2, \cdots \frac{M}{2}$. This means that the $\mathbb{Z}_2$ even modes $\phi^J_{(T^2/\mathbb{Z}_2^0,+)}(z)$, $(J = 0, 1, \cdots, \frac{M}{2})$ correspond to $(\ell + 1)$ linearly independent zero-mode eigen states. For the $\mathbb{Z}_2$ odd modes, $\phi^J_{(T^2/\mathbb{Z}_2^1,+)}(z)$ with $\mathbb{Z}_2$ charge 1, since they satisfy following condition

$$\phi^J_{(T^2/\mathbb{Z}_2^1,+)}(-z) = -\phi^{M-J}_{(T^2/\mathbb{Z}_2^1,+)}(z), \tag{2.101}$$

we find linearly independent zero-mode solutions $J = 1, 2, \cdots, \frac{M}{2} - 1$. This implies that the number of linearly independent zero-mode wave functions on magnetized $T^2/\mathbb{Z}_2$ is $(\ell - 1)$. The summation of zero modes is

$$(\ell + 1) + (\ell - 1) = 2\ell = M. \tag{2.102}$$

Similarly to the even flux, we can also count the zero-mode numbers for the $\mathbb{Z}_2$ even and odd modes with an odd flux M as $\frac{M+1}{2}$, $\frac{M-1}{2}$, respectively. We can construct the $\mathbb{Z}_2$ even and odd modes with the non-trivial SS phases, and count these zero-mode numbers. We show the zero-mode numbers in Table 2.1 [69].

$(\alpha_1^S, \beta_\tau^S)$	flux $M > 0$	$\mathbb{Z}_2$ even	$\mathbb{Z}_2$ odd
$(0,0)$	even	$\frac{M}{2} + 1$	$\frac{M}{2} - 1$
$(0,0)$	odd	$\frac{M+1}{2}$	$\frac{M-1}{2}$
$(\frac{1}{2},0)$	even	$\frac{M}{2}$	$\frac{M}{2}$
$(\frac{1}{2},0)$	odd	$\frac{M+1}{2}$	$\frac{M-1}{2}$
$(0,\frac{1}{2})$	even	$\frac{M}{2}$	$\frac{M}{2}$
$(0,\frac{1}{2})$	odd	$\frac{M+1}{2}$	$\frac{M-1}{2}$
$(\frac{1}{2},\frac{1}{2})$	even	$\frac{M}{2}$	$\frac{M}{2}$
$(\frac{1}{2},\frac{1}{2})$	odd	$\frac{M-1}{2}$	$\frac{M+1}{2}$

Table 2.1: Zero-mode numbers in magnetized $T^2/\mathbb{Z}_2$.

Magnetized $T^2/\mathbb{Z}_3$ model

We consider $T^2/\mathbb{Z}_3$ with positive flux $M > 0$. Since the lattice corresponds to $SU(3)$ root lattice and $(ST)^3 = \mathbf{1}_2$, we can obtain the fixed complex structure modulus $\tau_{ST} = e^{2\pi i/3} = \omega$. We will describe the reason why we obtain the value of the modulus τ_{ST}.

When we perform the ST modular transformation of τ, we obtain

$$ST : \tau \Rightarrow \tau + 1 \Rightarrow -\frac{1}{\tau + 1}. \tag{2.103}$$

Then, we find ST-invariant modulus τ_{ST}

$$\tau_{ST} = -\frac{1}{\tau_{ST} + 1} \Rightarrow \tau_{ST} = \omega, \tag{2.104}$$

where we take $\tau \in \mathcal{H}$. Then, under the modular transformation, we see that ωz denotes $\mathbb{Z}_3$-twist in the complex coordinate z. That is, the modular ST transformation corresponds to $\mathbb{Z}_3$-twist in the z. Furthermore, since we impose the modular ST symmetry and construct $T^2/\mathbb{Z}_3$ orbifold, we should introduce an even flux $M = 2\ell$, $(\ell \in \mathbb{Z}, \ell > 0)$. Lastly, in the following, we suppose the vanishing SS phases, for simplicity.

To count zero-mode numbers in $\mathbb{Z}_3$ sectors, we need to derive the wave functions behaviors under the $\mathbb{Z}_3$-twist. The $\mathbb{Z}_3$-twist corresponds to the modular ST transformation of the complex coordinates $(z, \bar{z}) \to (\omega z, \bar{\omega}\bar{z})$. Then, the following modular ST transformation of the wave functions in magnetized T^2 leads to the wave functions behaviors under the $\mathbb{Z}_3$-twist,

$$\begin{aligned}\psi^J_{(T^2,+)}(\omega z, \omega) &= (-(\omega+1))^{\frac{1}{2}} \cdot \frac{e^{\pi i/4}}{\sqrt{M}} \cdot e^{2\pi i J M^{-1} K} \cdot e^{\pi i K M^{-1} K} \cdot \psi^K_{(T^2,+)}(z, \omega) \\ &\equiv \rho_{JK}(ST) \cdot \psi^K_{(T^2,+)}(z, \omega),\end{aligned} \tag{2.105}$$

where $\rho(ST)$ corresponds to $M \times M$ complex representation under the $\mathbb{Z}_3$ twist. Since the eigenvalues of the $\rho_{IJ}(ST)$, $(I, J \in \{0, 1, \cdots, M-1\}$ are $\mathbb{Z}_3$ eigenvalues, that is, $1, \omega, \omega^2$, the number of the $\mathbb{Z}_3$ eigenvalues corresponds to the number of zero modes in $\mathbb{Z}_3$ sectors with the $\mathbb{Z}_3$ charge. Thus, in order to count zero-mode numbers in $\mathbb{Z}_3$ sectors, we need to clarify the number of $\mathbb{Z}_3$ eigenvalues in the $\rho(ST)$. One of how to clarify the number is to classify the values of $\mathrm{tr}\rho(ST)$. We can utilize the two-dimensional Landsberg-Schaar relation [1] given by [58]

$$\frac{1}{\sqrt{p}}\sum_{n=0}^{p-1} e^{2\pi i n^2 \frac{q}{p}} = \frac{e^{\frac{\pi}{4}i}}{\sqrt{2q}}\sum_{n=0}^{2q-1} e^{-\pi i n^2 \frac{p}{2q}}, \quad p, q \in \mathbb{N}. \tag{2.106}$$

We show the zero-mode numbers and three-generation models in $\mathbb{Z}_3$ sectors in the Table 2.2 and Table 2.3 [58, 69].

even flux $M > 0$	D_0	D_1	D_2
2	1	0	1
4	1	2	1
6	$\boxed{3}$	2	1
8	$\boxed{3}$	2	$\boxed{3}$
10	$\boxed{3}$	4	$\boxed{3}$
12	5	4	$\boxed{3}$

Table 2.2: Zero-mode numbers and three-generation models in magnetized $T^2/\mathbb{Z}_3$.

even flux $M > 0$	D_0	D_1	D_2
$6n$	$2n+1$	$2n$	$2n-1$
$6n+2$	$2n+1$	$2n$	$2n+1$
$6n+4$	$2n+1$	$2n+2$	$2n+1$

Table 2.3: General zero-mode numbers in magnetized $T^2/\mathbb{Z}_3$.

Magnetized $T^2/\mathbb{Z}_4$ model

We consider magnetized $T^2/\mathbb{Z}_4$ with positive flux $M > 0$ and the vanishing SS phases $(\alpha_1^S, \beta_\tau^S) = (0, 0)$. The algebraic structure is given by

$$S^4 = \mathbf{1}_2. \tag{2.107}$$

Then, we introduce the S-invariant modulus $\tau = \tau_S$ such that

$$\begin{aligned} \tau_S &= -\frac{1}{\tau_S}, \\ \tau_S &= i, \end{aligned} \tag{2.108}$$

[1] The g-dimensional Landsberg-Scharr relation is given in the Appendix A . It is also used when we consider the number of zero modes in magnetized $T^4/\mathbb{Z}_N$. (See Refs. [81, 82] .)

where $\tau \in \mathcal{H}$. We therefore derive the $\mathbb{Z}_4$-twist in the complex coordinates $(z, \bar{z})$ through the modular S transformation $S : (z, \bar{z}) \to (iz, -i\bar{z})$. We introduce the $\mathbb{Z}_4$-twist of the wave functions on T^2 as follows

$$\psi^J_{(T^2,+)}(iz, i) = \frac{1}{\sqrt{M}} \sum_{K=0}^{M-1} e^{2\pi i J M^{-1} K} \cdot \psi^K_{(T^2,+)}(z, i) \equiv \rho_{JK}(S) \cdot \psi^K_{(T^2,+)}(z, i). \quad (2.109)$$

Similarly to $T^4/\mathbb{Z}_3$ orbifold, by use of the two-dimensional Landsberg-Schaar relation Eq. (2.106), we can calculate the $\mathrm{tr}\rho(S)$ as follows

$$\mathrm{tr}\rho(S) = \frac{1}{\sqrt{M}} \sum_{K=0}^{M-1} e^{2\pi i M^{-1} K^2} = \frac{e^{\frac{\pi}{4}i}}{\sqrt{2}} \sum_{n=0}^{1} e^{-\pi i n^2 \frac{M}{2}} = \frac{e^{\frac{\pi}{4}i}}{\sqrt{2}}(1 + e^{-\pi i \frac{M}{2}}). \quad (2.110)$$

Then, $\mathrm{tr}\rho(S)$ is classified by the flux M

$$\mathrm{tr}\rho(S) = \begin{cases} \sqrt{2}e^{\frac{\pi}{4}i} = 1 + i, & \text{for } M \equiv 0 \pmod 4, \\ \frac{e^{\frac{\pi}{4}i}}{\sqrt{2}}(1 - i) = 1, & \text{for } M \equiv 1 \pmod 4, \\ 0, & \text{for } M \equiv 2 \pmod 4, \\ \frac{e^{\frac{\pi}{4}i}}{\sqrt{2}}(1 + i) = i, & \text{for } M \equiv 3 \pmod 4, \end{cases} \quad (2.111)$$

We show the zero-mode numbers and three-generation models in $\mathbb{Z}_4$ sectors, and general zero-mode counting in the Table 2.4 and Table 2.5 [58, 69].

Magnetized $T^2/\mathbb{Z}_6$ model

We consider magnetized $T^2/\mathbb{Z}_6$ orbifold. Since the $\mathbb{Z}_6$-twist can be introduced as the product of the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ twists. From the $\mathbb{Z}_3$ twist, we should impose $M = 2n > 0$, $(n \in \mathbb{Z})$ because of the modular T symmetry. We show the zero-mode numbers, three-generation models, and the general zero-mode numbers in Table 2.6 and Table 2.7 [58, 69].

In conclusion, we have considered 10D $\mathcal{N} = 1$ SYM with the $U(N)$ gauge symmetry and the $SO(9, 1)$ Lorentz transformation. We have introduced the compactification $\mathcal{R}^{3,1} \times \mathcal{M}_6$ with $SO(3, 1) \times SO(6)$ rotation transformation and the KK decomposition for 10D boson and fermion. By use of background magnetic fluxes on $\mathcal{M}_6$, we can realize $U(N)$ gauge theory or its spontaneously symmetry breaking. When we suppose 10D $\mathcal{N} = 1$ supersymmetric $U(1)$ gauge theory compactified on T^2 with quantized flux $|M|$ and $SO(2)$ rotation, we can realize supersymmetric

Flux $M > 0$	D_0	D_1	D_2	D_3
1	1	0	0	0
2	1	0	1	0
3	1	1	1	0
4	2	1	1	0
5	2	1	1	1
6	2	1	2	1
7	2	2	2	1
8	$\boxed{3}$	2	2	1
9	$\boxed{3}$	2	2	2
10	$\boxed{3}$	2	$\boxed{3}$	2
11	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	2
12	4	$\boxed{3}$	$\boxed{3}$	2

Table 2.4: Zero-mode numbers and three-generation models in magnetized $T^2/\mathbb{Z}_4$.

Flux $M > 0$	D_0	D_1	D_2	D_3
$4n$	$n+1$	n	n	$n-1$
$4n+1$	$n+1$	n	n	n
$4n+2$	$n+1$	n	$n+1$	n
$4n+3$	$n+1$	$n+1$	$n+1$	n

Table 2.5: General zero-mode counting in magnetized $T^2/\mathbb{Z}_4$.

chiral theory with $|M|$-generation structure under the Dirac equation of massless fermion on T^2. This can lead to three-generation structure in 4D chiral theory. We can also derive three-point couplings and mass spectrum of 2D vector on the T^2. The T^2 has geometrical symmetry called $Sp(2,\mathbb{Z})$ modular symmetry and this can restrict the values of the flux $|M|$.

Furthermore, when we consider spinor rotation described by $Spin(2)$ on T^2, we can take $SO(2)$ rotation in $(\mathrm{Re}z, \mathrm{Im}z)$ basis keeping chirality of the wave functions on T^2 and SUSY. Under the $SO(2)$ transformation, we can define the identification of $\mathbb{Z}_N$-twist and the boundary conditions of the spinor on $T^2/\mathbb{Z}_N$ twisted orbifold taking the $Spin(2)$ rotation and the $U(1)$ global symmetry into account. We can represent the wave functions of $T^2/\mathbb{Z}_N$ as linear combinations of the wave functions on T^2. Then, we can count degenerated fermion zero modes on $T^2/\mathbb{Z}_N$, $(N = 2, 3, 4, 6)$, with magnetic flux. By use of algebraic structure constructed by

Flux $M > 0$	D_0	D_1	D_2	D_3	D_4	D_5
2	1	0	1	0	0	0
4	1	1	1	0	1	0
6	2	1	1	1	1	0
8	2	1	2	1	1	1
10	2	2	2	1	2	1
12	$\boxed{3}$	2	2	2	2	1
14	$\boxed{3}$	2	$\boxed{3}$	2	2	2
16	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	2	$\boxed{3}$	2
18	4	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$	2
20	4	$\boxed{3}$	4	$\boxed{3}$	$\boxed{3}$	$\boxed{3}$
22	4	4	4	$\boxed{3}$	4	$\boxed{3}$
24	5	4	4	4	4	$\boxed{3}$

Table 2.6: Zero-mode counting and three-generation models in magnetized $T^2/\mathbb{Z}_6$.

Flux $M > 0$	D_0	D_1	D_2	D_3	D_4	D_5
$6n$	$n+1$	n	n	n	n	$n-1$
$6n+2$	$n+1$	n	$n+1$	n	n	n
$6n+4$	$n+1$	$n+1$	$n+1$	n	$n+1$	n

Table 2.7: General zero-mode counting in magnetized $T^2/\mathbb{Z}_6$.

the generators of the $Sp(2,\mathbb{Z})$ modular symmetry, since the transformation in the complex coordinates $(z,\bar{z})$ by the generator γ satisfying $\gamma^N = \mathbf{1}_2$ corresponds to the $\mathbb{Z}_N$-twist, we can classify the number of the $\mathbb{Z}_N$ eigenvalues of the $\rho(\gamma)$ by $\mathrm{tr}\rho(\gamma)$ and the Landsberg-Schaar relation.

So far, we have considered factorizable torus and its orbifolds with background magnetic fluxes, that is, $T^6 = T^2 \times T^2 \times T^2$, T^2, and $T^2/\mathbb{Z}_N$, $(N = 2,3,4,6)$ with background magnetic fluxes. We can find the zero-mode wave functions and their zero-mode numbers in T^2 or its orbifolds $T^2/\mathbb{Z}_N$.

In chapter 3, we will consider nonfactorizable six-dimensional torus T^6 with background magnetic fluxes. In magnetized T^6, we can find zero-mode wave functions.

Chapter 3

Magnetized T^6 model

In this chapter, we study fermion zero-mode wave functions with various chiralities in magnetized T^6 torus. This chapter is along in Refs. [54, 55, 78, 81, 82]. As we have mentioned in the chapter 2, T^6 compactification without magnetic flux can lead to 4D non-chiral theory if we consider 10D $\mathcal{N} = 1$ SYM. We mention this again. T^6 is six-dimensional torus with trivial Calabi-Yau metric. However, since T^6 does not have the $SU(3)$ holonomy structure, 4D $\mathcal{N} = 4$ SUSY remains if we consider T^6 compactification without magnetic fluxes in 10D $\mathcal{N} = 1$ supersymmetric gauge theory. Then, 4D $\mathcal{N} = 4$ SUSY EFT derived from T^6 compactification without background magnetic fluxes leads to non-chiral theory, however, we need 4D $\mathcal{N} = 1$ or $\mathcal{N} = 0$ EFT which realizes chiral theory since the Standard model is chiral theory. In the chapter 2, we have derived 4D chiral theory with generation structure under the T^2 compactfication with magnetic flux. Then, let us consider T^6 compactification with magnetic fluxes, what is called, magnetized T^6 model, and verify whether we can realize 4D chiral EFT with $\mathcal{N} = 1$ SUSY or not. First, we consider the wave functions satisfying the Dirac equation and the boundary conditions on the magnetized T^6. Second, we introduce the $SO(3)$ (or $SO(6)$) transformations and derive the wave functions under the modular transformation.

Let us consider magnetized D-brane models with T^6 compactification. In what follows, we study $U(1)$ gauge theory on T^6 with magnetic flux background. We give a review on the magnetic flux background and the Dirac equation. Similarly to magnetized T^2, we can extend this model to $U(N)$ gauge theory with background magnetic fluxes along the diagonal direction of $U(N)$.

3.1 Introduction of magnetized T^6

3.1.1 Gamma matrices and Dirac operator

First, we review how to introduce magnetized T^6 models [78,81,82]. By use of the 6D lattice Λ, we can introduce the 6D torus $T^6 \simeq \mathbb{C}^3/\Lambda$. (See Refs. [88–93] for various 6D Lie lattices to construct T^6 and its orbifolds.) The 6D lattice Λ can be defined by six basis vectors e'_i, $(i = 1, \cdots, 6)$. These vectors are given by

$$e'_1 = 2\pi r_1 e_1 = 2\pi r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e'_2 = 2\pi r_2 e_2 = 2\pi r_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e'_3 = 2\pi r_3 e_3 = 2\pi r_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$e'_4 = 2\pi r_i \Omega_{i1} e_1 = 2\pi \begin{bmatrix} r_1\omega_{11} \\ r_2\omega_{21} \\ r_3\omega_{31} \end{bmatrix}, e'_5 = 2\pi r_i \Omega_{i2} e_2 = 2\pi \begin{bmatrix} r_1\omega_{12} \\ r_2\omega_{22} \\ r_3\omega_{32} \end{bmatrix},$$

$$e'_6 = 2\pi r_i \Omega_{i3} e_3 = 2\pi \begin{bmatrix} r_1\omega_{13} \\ r_2\omega_{23} \\ r_3\omega_{33} \end{bmatrix},$$

where $2\pi r_i$, $(i = 1, 2, 3)$, denote the scale factors. The e_i represents the Euclidean standard basis. Then, we introduce the complex structure moduli $\Omega = \Omega_{ij}$ on T^6 as follows

$$\Omega = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix}. \tag{3.1}$$

Here, we can consider $\Omega^T = \Omega$ when we introduce lattice of $T^{2g}/\mathbb{Z}_N$, $(g = 2, 3)$. We also introduce complex coordinates on T^6, that is, $z^i = x^i + \Omega_{ij} y^j$, $(\vec{x}, \vec{y} \in \mathbb{R}^3)$, by using complex coordinates $\vec{Z}$ on $\mathbb{C}^3$:

$$Z^i \equiv 2\pi r_i (x^i + \Omega_{ij} y^j) = 2\pi r_i z^i, \tag{3.2}$$

where we take the identifications $z^i \sim z^i + e_i \sim z^i + \Omega_{ij} e_j$ on T^6, namely $x^i \sim x^i + e_i$ and $y^i \sim y^i + e_i$. Here, we can write vector $\vec{v}$ as v. In this paper, we will consider $T^6/\mathbb{Z}_N$ orbifolds with simply-laced Lie Lattices. We then need to normalize the scale factors $2\pi r_i = 1$.[1] We also define the Siegel upper-half plane $\mathcal{H}^3$ as follows [94],

$$\mathcal{H}^3 = \{\Omega \in GL(3, \mathbb{C}) | \Omega^T = \Omega, \mathrm{Im}\Omega > 0\}. \tag{3.3}$$

[1]Magnetized T^6 model with the non-trivial scale factors $2\pi r_i$ is discussed in Appendix B.

By use of Eq. (3.3), we will consider well-defined wave functions in magnetized T^6.

We introduce the following flat metric,

$$ds^2 = \delta_{i,j} dZ^i d\bar{Z}^{\bar{j}} \quad \text{for} \quad i,j = 1,2,3. \tag{3.4}$$

We also define the Gamma matrices Γ^{z^i}, $\Gamma^{\bar{z}^i}$. We use them when we consider the Dirac equation and spinor rotation on T^6:

$$\begin{aligned}
\Gamma^{z^1} &\equiv \frac{1}{2\pi r_1}\Gamma^{Z^1} = \frac{1}{2\pi r_1}(\sigma^z \otimes \sigma^3 \otimes \sigma^3), \quad \Gamma^{\bar{z}^{\bar{1}}} \equiv \frac{1}{2\pi r_1}\Gamma^{\bar{Z}^{\bar{1}}} = \frac{1}{2\pi r_1}(\sigma^{\bar{z}} \otimes \sigma^3 \otimes \sigma^3), \\
\Gamma^{z^2} &\equiv \frac{1}{2\pi r_2}\Gamma^{Z^2} = \frac{1}{2\pi r_2}(\mathbf{1}_2 \otimes \sigma^z \otimes \sigma^3), \quad \Gamma^{\bar{z}^{\bar{2}}} \equiv \frac{1}{2\pi r_2}\Gamma^{\bar{Z}^{\bar{2}}} = \frac{1}{2\pi r_2}(\mathbf{1}_2 \otimes \sigma^{\bar{z}} \otimes \sigma^3), \\
\Gamma^{z^3} &\equiv \frac{1}{2\pi r_3}\Gamma^{Z^3} = \frac{1}{2\pi r_3}(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^z), \quad \Gamma^{\bar{z}^{\bar{3}}} \equiv \frac{1}{2\pi r_3}\Gamma^{\bar{Z}^{\bar{3}}} = \frac{1}{2\pi r_3}(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^{\bar{z}}),
\end{aligned} \tag{3.5}$$

where $\mathbf{1}_n$ represents $n \times n$ unit matrix. We also describe the Pauli matrices σ^i,

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.6}$$

We can define $\sigma^z = \sigma^1 + i\sigma^2$, $\sigma^{\bar{z}} = \sigma^1 - i\sigma^2$ by using the Pauli matrices. One can check that the Gamma matrices satisfy the Clifford algebra,

$$\begin{aligned}
&\{\Gamma^{Z^i}, \Gamma^{Z^j}\} = \{\Gamma^{\bar{Z}^{\bar{i}}}, \Gamma^{\bar{Z}^{\bar{j}}}\} = 0, \\
&\{\Gamma^{Z^i}, \Gamma^{\bar{Z}^{\bar{j}}}\} = 4\delta^{i,j}.
\end{aligned} \tag{3.7}$$

In particular, the chirality of wave functions on T^6 can be determined by the following matrix Γ^7:

$$\Gamma^7 = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 = \mathrm{diag}[+,-,-,+,-,+,+,-]. \tag{3.8}$$

We can define the Dirac operator $\not{D}$ on T^6,

$$\begin{aligned}
i\not{D} &\equiv i(\Gamma^{z^j} D_{z^j} + \Gamma^{\bar{z}^{\bar{j}}} \bar{D}_{\bar{z}^{\bar{j}}}) \\
&= 2i \begin{bmatrix} D_{2,3} & D_1 \\ \bar{D}_{\bar{1}} & D_{2,3} \end{bmatrix},
\end{aligned} \tag{3.9}$$

where we define following covariant derivatives D_{z^j}, $\bar{D}_{\bar{z}^{\bar{j}}}$ and we restrict $2\pi r_i = 1$ for $i = 1,2,3$. Here, we consider $U(1)$ gauge field theory, where the gauge fields

A_z, $A_{\bar{z}}$ are coupled to fermions with $U(1)$ unit charge $q = +1$,

$$\begin{aligned} D_{z^j} &\equiv D_j = \partial_{z^j} - iA_{z^j}, \\ \bar{D}_{\bar{z}^{\bar{j}}} &\equiv \bar{D}_j = \partial_{\bar{z}^{\bar{j}}} - iA_{\bar{z}^{\bar{j}}}. \end{aligned} \tag{3.10}$$

We will introduce background magnetic fluxes in the next subsection 3.1.2 because we would like to describe explicitly both the gauge fields in Eq. (3.10) and boundary conditions of spinor wave functions on T^6.

3.1.2 Background magnetic fluxes and F-term SUSY condition

Here, we introduce $U(1)$ background magnetic fluxes F on T^6,

$$\begin{aligned} F &\equiv \frac{1}{2}(p_{xx})_{ij} dx^i \wedge dx^j + \frac{1}{2}(p_{yy})_{ij} dy^i \wedge dy^j + (p_{xy})_{ij} dx^i \wedge dy^j \\ &= \frac{1}{2}(F_{zz})_{ij} dz^i \wedge dz^j + \frac{1}{2}(F_{\bar{z}\bar{z}})_{ij} d\bar{z}^i \wedge d\bar{z}^j + (F_{z\bar{z}})_{ij}(idz^i \wedge d\bar{z}^j), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} (F_{zz})_{ij} &= \left([(\bar{\Omega} - \Omega)^{-1}]^T (\bar{\Omega}^T p_{xx} \bar{\Omega} + p_{yy} + p_{xy}^T \bar{\Omega} - \bar{\Omega}^T p_{xy})(\bar{\Omega} - \Omega)^{-1}\right)_{ij}, \\ (F_{\bar{z}\bar{z}})_{ij} &= \left([(\bar{\Omega} - \Omega)^{-1}]^T (\Omega^T p_{xx} \Omega + p_{yy} + p_{xy}^T \Omega - \Omega^T p_{xy})(\bar{\Omega} - \Omega)^{-1}\right)_{ij}, \\ (F_{z\bar{z}})_{ij} &= i\left([(\bar{\Omega} - \Omega)^{-1}]^T (\bar{\Omega}^T p_{xx} \Omega + p_{yy} + p_{xy}^T \Omega - \bar{\Omega}^T p_{xy})(\bar{\Omega} - \Omega)^{-1}\right)_{ij}. \end{aligned} \tag{3.12}$$

In what follows, for simplicity, we suppose $p_{xx} = p_{yy} = 0$. The Hermitian Yang-Mills equation keeps SUSY in 10D $\mathcal{N} = 1$ super Yang-Mills (SYM) theory. (See Refs. [54, 95]). That is, we need to define the $(1,1)$-form magnetic fluxes F. We then obtain

$$p_{xy}^T \Omega = \Omega^T p_{xy}. \tag{3.13}$$

Here, we can define the integer fluxes N as $p_{xy} = 2\pi N^T$ through the Dirac quantization condition,

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}, \tag{3.14}$$

where $n_{ij} \in \mathbb{Z}$ denote an integer flux. We therefore obtain

$$(N\Omega)^T = N\Omega. \tag{3.15}$$

This is called the F-term SUSY condition, and we can realize F-flat vacua satisfying the Hermitian Yang-Mills equation. In particular, by use of Eq. (3.15), we find $[N,\Omega]=0$ when we take both $N^T=N$ and $\Omega^T=\Omega$. This means that we can diagonalize N and Ω simultaneously. Then, we consider the case where we take $N^T=N$ and $\Omega^T=\Omega$. [2] We take the following background magnetic fluxes,

$$F=\pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}(idz^i\wedge d\bar{z}^j)\equiv F_{i\bar{j}}(idz^i\wedge d\bar{z}^j). \tag{3.16}$$

We find the gauge potential A,

$$\begin{aligned}A(z,\bar{z})&=\pi\mathrm{Im}[(N(\bar{z}+\bar{\eta}))^T(\mathrm{Im}\Omega)^{-1}dz]\\&=-\frac{i\pi}{2}[(N(\bar{z}+\bar{\eta}))^T(\mathrm{Im}\Omega)^{-1}]_idz^i+\frac{i\pi}{2}[(N(z+\eta))^T(\mathrm{Im}\Omega)^{-1}]_id\bar{z}^i\\&\equiv A_{z^i}dz^i+A_{\bar{z}^{\bar{i}}}d\bar{z}^i,\end{aligned} \tag{3.17}$$

where $F=dA$ and η denotes the Wilson line. In the main text, we take the vanishing Wilson line, namely $\eta=\vec{0}$. Then, we can obtain the following commutation relations by using the values of the gauge fields A_{z^i}, $A_{\bar{z}^{\bar{i}}}$ and Eq. (3.10)

$$\begin{aligned}[][D_i,\bar{D}_j]&=F_{i\bar{j}},\\ [D_i,D_j]&=F_{ij}=0,\\ [\bar{D}_i,\bar{D}_j]&=F_{\bar{i}\bar{j}}=0.\end{aligned} \tag{3.18}$$

From the $U(1)$ gauge symmetry, the gauge potential $A(z,\bar{z})=A(z)$ need to satisfy the following boundary conditions,

$$\begin{aligned}A(z+e_k)&=A(z)+d\zeta_{e_k}(z),\\ A(z+\Omega e_k)&=A(z)+d\zeta_{\Omega e_k}(z),\end{aligned} \tag{3.19}$$

where $k=1,2,3$. Thus, we obtain

$$\begin{aligned}\zeta_{e_k}(z)&=\pi[N^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z]_k,\\ \zeta_{\Omega e_k}(z)&=\pi\mathrm{Im}[(N\bar{\Omega})^T(\mathrm{Im}\Omega)^{-1}z]_k\\&=\pi[N(\mathrm{Re}\Omega(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z-\mathrm{Re}z)]_k,\end{aligned} \tag{3.20}$$

and boundary conditions of spinor wave functions in magnetized T^6 under $(N\Omega)^T=N\Omega$

$$\begin{aligned}\psi(z+e_k)&=e^{i\zeta_{e_k}(z)}\psi(z)=e^{i\pi[N^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z]_k}\psi(z),\\ \psi(z+\Omega e_k)&=e^{i\zeta_{\Omega e_k}(z)}\psi(z)=e^{i\pi[N(\mathrm{Re}\Omega(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z-\mathrm{Re}z)]_k}\psi(z).\end{aligned} \tag{3.21}$$

[2] The discussion can be consistent even if the asymmetric N and Ω are introduced under the magnetic fluxes and the F-term SUSY condition $(N\Omega)^T=N\Omega$.

3.2 Introduction of wave functions in magnetized T^6

3.2.1 Dirac equation and rotation in magnetized T^6

Here, we study zero-mode wave functions on generic T^6 with background magnetic fluxes. Components of the spinor field Ψ can be represented by

$$\begin{aligned}\Psi \equiv \begin{bmatrix}+\\-\end{bmatrix} \otimes \begin{bmatrix}+\\-\end{bmatrix} \otimes \begin{bmatrix}+\\-\end{bmatrix} &= [\psi_{+++}, \psi_{++-}, \psi_{+-+}, \psi_{+--}, \psi_{-++}, \psi_{-+-}, \psi_{--+}, \psi_{---}]^T \\ &\equiv [\psi'_{1+}, \psi_{3-}, \psi_{2-}, \psi_{1+}, \psi_{1-}, \psi_{2+}, \psi_{3+}, \psi'_{1-}]^T.\end{aligned} \tag{3.22}$$

We introduce ψ_{j+} and ψ_{j-}, $(j = 1, 2, 3)$, as the positive and negative components in terms of 6D chirality, respectively. Similarly, we introduce $\psi'_{j+} = \psi_{+++}$ and $\psi'_{j-} = \psi_{---}$. We find that the Dirac equation, $i\not{D}_6\Psi = 0$, is explicitly given by

$$\begin{cases} D_3\psi_{3-} + D_2\psi_{2-} + D_1\psi_{1-} = 0, \\ \bar{D}_3\psi'_{1+} - D_2\psi_{1+} - D_1\psi_{2+} = 0, \\ \bar{D}_2\psi'_{1+} + D_3\psi_{1+} - D_1\psi_{3+} = 0, \\ -\bar{D}_2\psi_{3-} + \bar{D}_3\psi_{2-} + D_1\psi'_{1-} = 0, \\ \bar{D}_1\psi'_{1+} + D_3\psi_{2+} + D_2\psi_{3+} = 0, \\ -\bar{D}_1\psi_{3-} + \bar{D}_3\psi_{1-} - D_2\psi'_{1-} = 0, \\ -\bar{D}_1\psi_{2-} + \bar{D}_2\psi_{1-} + D_3\psi'_{1-} = 0, \\ \bar{D}_1\psi_{1+} - \bar{D}_2\psi_{2+} + \bar{D}_3\psi_{3+} = 0. \end{cases} \tag{3.23}$$

First, we consider a solution that only $\psi_{+++} = \psi'_{1+}$ is excited and the others are vanishing. We focus on the solution corresponding to wave functions with various chiralities after we discuss only $\psi_{+++} = \psi'_{1+}$. For the solution $\psi_{+++} = \psi'_{1+}$, we can describe the following Dirac equation

$$\bar{D}_1\psi'_{1+} = \bar{D}_2\psi'_{1+} = \bar{D}_3\psi'_{1+} = 0. \tag{3.24}$$

Then we can describe its solution as follows [54, 55, 78]

$$\psi^J_{+++} = \mathcal{N} \cdot e^{i\pi[Nz]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z} \cdot \theta \begin{bmatrix} J^TN^{-1} \\ 0 \end{bmatrix} (Nz, N\Omega), \tag{3.25}$$

where $J \in \Lambda_{N^T}$ denotes lattice vectors in Λ_{N^T}. The lattice Λ_{N^T} is spanned by $N^T e_i$, $(i = 1, 2, 3)$. We note that e_i, $(i = 1, 2, 3)$, denote three-dimensional standard unit vectors. The periodicity in the index $\vec{J}$ can be written by

$$\psi_{+++}^{J+N^T e_i} = \psi_{+++}^{J}. \tag{3.26}$$

We can immediately verify Eq. (3.26) as follows

$$\begin{aligned}
\psi_{+++}^{J+N^T e_i} &= \mathcal{N} \cdot e^{i\pi [Nz]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z} \cdot \theta \begin{bmatrix} (J + N^T e_i)^T N^{-1} \\ 0 \end{bmatrix} (Nz, N\Omega) \\
&= \mathcal{N} \cdot e^{i\pi [Nz]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z} \cdot \theta \begin{bmatrix} J^T N^{-1} + e_i^T \\ 0 \end{bmatrix} (Nz, N\Omega) \\
&= \mathcal{N} \cdot e^{i\pi [Nz]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z} \\
&\quad \cdot \sum_{m \in \mathbb{Z}^3} e^{i\pi (m + J^T N^{-1} + e_i^T)^T N\Omega (m + J^T N^{-1} + e_i^T)} \cdot e^{2\pi i (m + J^T N^{-1} + e_i^T)^T Nz} \\
&= \mathcal{N} \cdot e^{i\pi [Nz]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z} \\
&\quad \cdot \sum_{m' = m + e_i^T \in \mathbb{Z}^3} e^{i\pi (m' + J^T N^{-1})^T N\Omega (m' + J^T N^{-1})} \cdot e^{2\pi i (m' + J^T N^{-1})^T Nz} \\
&= \mathcal{N} \cdot e^{i\pi [Nz]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (Nz, N\Omega) \\
&= \psi_{+++}^{J}.
\end{aligned} \tag{3.27}$$

Thus, we obtain $|\det N|$ of independent degenerated zero modes. This means that we can realize $|\det N|$-generation models from magnetized T^6. For $|\det N| = +3$, we can derive three-generation models from magnetized T^6. We define the Riemann-Theta function with characters

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) \equiv \sum_{m \in \mathbb{Z}^3} e^{i\pi (m+a)^T \Omega (m+a)} e^{2\pi i (m+a)^T (z+b)}, \quad a, b \in \mathbb{R}^3, \quad \Omega \in \mathcal{H}^3. \tag{3.28}$$

We define the factor $\mathcal{N}$ as normalization constant,

$$\mathcal{N} = (\det N)^{\frac{1}{4}} \cdot [\mathrm{Vol}(T^6)]^{-\frac{1}{2}} = (\det N)^{\frac{1}{4}} \cdot [\det (\mathrm{Im}\Omega)]^{-\frac{1}{2}}, \tag{3.29}$$

if $\det \mathrm{Im}\Omega > 0$. Here, the wave function is normalized as

$$\int_{T^6} d^3 z d^3 \bar{z} (\psi^J)^* \psi^K = [\det (2\mathrm{Im}\Omega)]^{-1/2} \delta_{J,K}, \tag{3.30}$$

following Ref. [66]. We note that we can describe only ψ_{+++} when $F_{i\bar{j}} = \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}$ has positive-definite, that is, $F_{i\bar{j}} > 0$. This is because ψ_{+++} is well-defined only if $F_{i\bar{j}} > 0$. Then, the other spinor components are vanishing under $F_{i\bar{j}} > 0$. We will use the wave functions with positive chirality, ψ_{+++}, written in Eq. (3.25) when we analyze the number of zero modes in magnetized $T^6/\mathbb{Z}_N$.

Furthermore, the solution for only the component ψ_{---} can be obtained by flipping all chiralities in ψ_{+++}. That is, the sign of $\det N$ is flipped. Then, we can describe ψ_{---} as follows,

$$\psi^J_{---} = \mathcal{N} \cdot e^{i\pi[N\bar{z}]^T(\mathrm{Im}\bar{\Omega})^{-1}\mathrm{Im}\bar{z}} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (N\bar{z}, N\bar{\Omega}). \tag{3.31}$$

We note that we take the following replacement,

$$z \to \bar{z}, \quad \Omega \to \bar{\Omega}, \quad N \to -N \equiv N. \tag{3.32}$$

Next, we consider the solutions that other components are excited. For other $F_{i\bar{j}} = \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}$, we obtain not zero mode ψ_{+++} but other zero modes like $\psi_{\underline{+--}}$, where we describe the underline which represents all of the possible permutations. In what follows, we focus on zero-mode wave functions of other spinor components, that is, $\psi_{1+} = \psi_{+--}$, $\psi_{2+} = \psi_{-+-}$, and $\psi_{3+} = \psi_{--+}$. By use of the replacement Eq. (3.32), we can obtain the others components, that is, $\psi_{1-} = \psi_{-++}$, $\psi_{2-} = \psi_{+-+}$, and $\psi_{3-} = \psi_{++-}$. Here, let us consider the case where only ψ_{1+} is excited and the others are vanishing. Then, the Dirac equation is given by

$$D_2\psi_{1+} = D_3\psi_{1+} = \bar{D}_1\psi_{1+} = 0. \tag{3.33}$$

By use of Eq. (3.18), D_i, and $\bar{D}_i$, we obtain $F_{1\bar{2}}\psi_{1+} = F_{1\bar{3}}\psi_{1+} = 0$. This implies that we cannot derive non-trivial solution if we take $F_{1\bar{2}}, F_{1\bar{3}} \neq 0$. If we take $F_{1\bar{2}}, F_{1\bar{3}} = 0$, we can consider the factorizable torus, that is, $T^2 \times T^4$. Then, we study the case where spinor components $\psi_{1+} = \psi_{+--}$, $\psi_{2+} = \psi_{-+-}$, and $\psi_{3+} = \psi_{--+}$ are excited simultaneously because we would like to obtain the wave functions in nonfactorizable T^6 with magnetic fluxes. According to Ref. [55], we can introduce Ansatz for solutions. In Ansatz, we can describe these components by using a complex function ψ. That is, we suppose $\psi_{i+} = \alpha_i\psi$, $(i = 1, 2, 3)$, where α_i are coefficients [55]. We will see the details of ψ and α_i. By use of the Ansatz, we find the corresponding Dirac equation

$$\begin{cases} (\alpha_1 D_2 + \alpha_2 D_1)\psi = 0, \\ (\alpha_1 D_3 - \alpha_3 D_1)\psi = 0, \\ (\alpha_2 D_3 + \alpha_3 D_2)\psi = 0, \\ (\alpha_1 \bar{D}_1 - \alpha_2 \bar{D}_2 + \alpha_3 \bar{D}_3)\psi = 0. \end{cases} \tag{3.34}$$

Here, we find

$$\alpha_3(\alpha_1 D_2 + \alpha_2 D_1)\psi + \alpha_2(\alpha_1 D_3 - \alpha_3 D_1)\psi = \alpha_1(\alpha_2 D_3 + \alpha_3 D_2)\psi = 0. \tag{3.35}$$

Then, we find that the first three equations depend on each other.

In what follows, we suppose $\alpha_1 \neq 0$. We also introduce the ratios $q_1 = \frac{\alpha_2}{\alpha_1}$, $q_2 = -\frac{\alpha_3}{\alpha_1}$ and $\vec{q} = (q_1, q_2)^T$.[3] We can derive the following integrable conditions when we consider Eq. (3.34),

$$\begin{cases} F_{2\bar{1}} - q_1^2 F_{1\bar{2}} + q_1(F_{1\bar{1}} - F_{2\bar{2}}) - q_2 F_{2\bar{3}} - q_1 q_2 F_{1\bar{3}} = 0, \\ F_{3\bar{1}} - q_1 F_{3\bar{2}} - q_2(F_{3\bar{3}} - F_{1\bar{1}}) - q_1 q_2 F_{1\bar{2}} - q_2^2 F_{1\bar{3}} = 0. \end{cases} \tag{3.36}$$

Under the integrable conditions, we can obtain a non-trivial solution. Here, we consider the magnetic fluxes F. Since the F is defined as the $(1,1)$-form, $F_{i\bar{j}} = \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}$ is a real symmetric matrix. This implies that the F has six degrees of freedom. On the other hands, since we can obtain two constraints from Eq. (3.36), we can consider four degrees of freedom in the F. By using this, we can represent $F_{i\bar{j}}$ by the coefficients q_i, a real number $\hat{N}'$, and 2×2 real symmetric matrix $\tilde{N}'$, that is,

$$\begin{aligned} F_{i\bar{j}} &\equiv \pi[\hat{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij} + \pi[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij}, \\ \pi[\hat{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij} &= \hat{N}' \begin{bmatrix} 1 & -\vec{q}^T \\ -\vec{q} & \vec{q}(\vec{q})^T \end{bmatrix} = \hat{N}' \begin{bmatrix} 1 & -q_1 & -q_2 \\ -q_1 & q_1^2 & q_1 q_2 \\ -q_2 & q_1 q_2 & q_2^2 \end{bmatrix}, \\ \pi[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij} &= \begin{bmatrix} \vec{q}^T \tilde{N}' \vec{q} & \vec{q}^T \tilde{N}' \\ \tilde{N}' \vec{q} & \tilde{N}' \end{bmatrix} \\ &= \begin{bmatrix} \tilde{N}'_{11} q_1^2 + 2\tilde{N}'_{12} q_1 q_2 + \tilde{N}'_{22} q_2^2 & \tilde{N}'_{11} q_1 + \tilde{N}'_{12} q_2 & \tilde{N}'_{12} q_1 + \tilde{N}'_{22} q_2 \\ \tilde{N}'_{11} q_1 + \tilde{N}'_{12} q_2 & \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{12} q_1 + \tilde{N}'_{22} q_2 & \tilde{N}'_{12} & \tilde{N}'_{22} \end{bmatrix}. \end{aligned} \tag{3.37}$$

[3] If $\alpha_1 = 0$, we can use ratio $q = \frac{\alpha_3}{\alpha_2}$. Then, we can consider a similar discussion.

We find that they satisfy Eq. (3.36), where $N \equiv \hat{N} + \tilde{N}$. By use of the above background fluxes, we can explictly describe the solutions by [81, 82] [4]

$$\psi^{J}_{M,N} = \mathcal{N} \cdot f(z,\bar{z}) \cdot \hat{\Theta}(z,\bar{z}), \tag{3.38}$$

where

$$\begin{aligned} f(z,\bar{z}) &\equiv \exp\left[i\pi((\hat{N}z)^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z - (\tilde{N}\bar{z})^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\bar{z})\right] \\ &= \exp\left[i\pi(\hat{N}z + \tilde{N}\bar{z})^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z\right] \\ &= \exp\left[i\pi(N\mathrm{Re}z + iM\mathrm{Im}z)^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z\right], \end{aligned} \tag{3.39}$$

$$\begin{aligned} \hat{\Theta}(z,\bar{z}) \equiv & \sum_{m\in\mathbb{Z}^3} e^{\pi i(m+J^TN^{-1})^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})(m+J^TN^{-1})} e^{2\pi i(m+J^TN^{-1})^T(\hat{N}z+\tilde{N}\bar{z})} \\ = & \sum_{m\in\mathbb{Z}^3} e^{\pi i(m+J^TN^{-1})^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+J^TN^{-1})} e^{2\pi i(m+J^TN^{-1})^T(N\mathrm{Re}z+iM\mathrm{Im}z)} \\ = & \, \theta\begin{bmatrix} J^TN^{-1} \\ 0 \end{bmatrix}(N\mathrm{Re}z + iM\mathrm{Im}z, N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega). \end{aligned} \tag{3.40}$$

We define a following M

$$M \equiv \hat{N} - \tilde{N}. \tag{3.41}$$

Similarly to ψ_{+++}, we can describe the normalization constant $\mathcal{N}$ in Eq. (3.38)

$$\mathcal{N} = |\det N|^{\frac{1}{4}} \cdot [\mathrm{Vol}(T^6)]^{-\frac{1}{2}} = |\det N|^{\frac{1}{4}} \cdot [|\det(\mathrm{Im}\Omega)|]^{-\frac{1}{2}}. \tag{3.42}$$

In addition, we can rewrite $|\det N|$ as $|\det M|$ in Eq. (3.42), because $|\det N| = |\det M|$.

We consider the F-term SUSY condition $(N\Omega)^T = N\Omega$ in Eq. (3.15). Additionally, from Eq. (3.3), we consider the Riemann condition $(N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega)^T = N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega$ and $M\mathrm{Im}\Omega > 0$, namely $N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega \in \mathcal{H}^3$. Then, the spinor wave functions Eq. (3.38) are well-defined and satisfy both Dirac equation and the boundary conditions. Thus, we can obtain the wave functions with chiralities $(\underline{+--})$. In addition, the wave functions Eq. (3.38) become ψ_{+++} in Eq.

[4] For $\Omega = i\mathbf{1}_n$, the wave function with various chiralities was studied in Ref. [55]. We prove that the wave functions Eq. (3.38) satisfy both the boundary conditions Eq. (3.21) and the Dirac equation Eq. (3.34) in Appendices C and D. The wave functions with SS phases and Wilson lines are described in Appendix E.

(3.25) when $N = M$. We can therefore introduce all spinor wave functions in nonfactorizable T^6 with background magnetic fluxes Eq. (3.37).

On the other hands, since the $F_{i\bar{j}}$ is a real symmetric matrix, we can rotate $F_{i\bar{j}}$ by rotation matrix, that is,

$$F^{\rm part}_{i\bar{j}} \equiv P_{12}^{-1} F_{i\bar{j}} P_{12} = \begin{bmatrix} F^{\rm part}_{1\bar{1}} & 0 & 0 \\ 0 & F^{\rm part}_{2\bar{2}} & F^{\rm part}_{2\bar{3}} \\ 0 & F^{\rm part}_{2\bar{3}} & F^{\rm part}_{3\bar{3}} \end{bmatrix}, \tag{3.43}$$

where P_{12} can be element of $SO(3)$ rotation group

$$P_{12} = \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.44}$$

Then, we can represent q_1 and q_2 by θ_1 and θ_2

$$q_1 = -\frac{\tan\theta_1}{\cos\theta_2}, \quad q_2 = \tan\theta_2. \tag{3.45}$$

Also, we find the explicit entries in $F^{\rm part}_{i\bar{j}}$ as follows

$$\begin{aligned}
F^{\rm part}_{1\bar{1}} &= [F_{1\bar{1}} + F_{2\bar{2}} q_1^2 + F_{3\bar{3}} q_2^2 - 2(F_{1\bar{2}} q_1 - F_{2\bar{3}} q_1 q_2 + F_{1\bar{3}} q_2)] \cos^2\theta_1 \cos^2\theta_2, \quad (3.46)\\
F^{\rm part}_{2\bar{2}} &= [[2(F_{1\bar{2}} - F_{2\bar{3}} q_2) q_1 + (-2F_{1\bar{3}} q_2 + F_{1\bar{1}} + F_{3\bar{3}} q_2^2) q_1^2 \cos^2\theta_2] \cos^2\theta_2 + F_{2\bar{2}}] \cos^2\theta_1,\\
F^{\rm part}_{3\bar{3}} &= [F_{1\bar{1}} q_2^2 + F_{3\bar{3}} + 2F_{1\bar{3}} q_2] \cos^2\theta_2,\\
F^{\rm part}_{2\bar{3}} &= [[2F_{1\bar{3}} + (F_{1\bar{1}} - F_{3\bar{3}}) q_2] q_1 \cos^2\theta_2 + (F_{1\bar{2}} q_2 + F_{2\bar{3}} - F_{1\bar{3}} q_1)] \cos\theta_1 \cos\theta_2.
\end{aligned}$$

Therefore, the q_1 and q_2 correspond to parameters of $SO(3)$ (or $SO(6)$) rotation. We will describe the details of α_i. In this basis, ψ_{+--} can be excited if $F^{\rm part}_{1\bar{1}} > 0$. If $F^{\rm part}_{1\bar{1}} < 0$, the solution can be represented as the combination of ψ_{-+-} and ψ_{--+}. This is because the rotation keeps 6D overall chirality invariant. Furthermore, we can diagonalize the bottom right 2×2 matrix of $F^{\rm part}_{i\bar{j}}$ by the rotation P_3

$$P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix}. \tag{3.47}$$

Thus, we can specifically describe the background magnetic fluxes after the diagonalization, that is, $(F_{1\bar{1}}^{\rm diag}, F_{2\bar{2}}^{\rm diag}, F_{3\bar{3}}^{\rm diag})$. Here, we take

$$\begin{aligned}
F_{1\bar{1}}^{\rm diag} &= F_{1\bar{1}}^{\rm part}, \\
F_{2\bar{2}}^{\rm diag} &= [[\sin\theta_3 \tan 2\theta_3 + \cos\theta_3]\cos\theta_3] F_{2\bar{2}}^{\rm part} + [[-\cos\theta_3 \tan 2\theta_3 + \sin\theta_3]\sin\theta_3] F_{3\bar{3}}^{\rm part}, \\
F_{3\bar{3}}^{\rm diag} &= [[-\cos\theta_3 \tan 2\theta_3 + \sin\theta_3]\sin\theta_3] F_{2\bar{2}}^{\rm part} + [[\sin\theta_3 \tan 2\theta_3 + \cos\theta_3]\cos\theta_3] F_{3\bar{3}}^{\rm part},
\end{aligned} \tag{3.48}$$

and

$$\tan 2\theta_3 = \frac{2F_{2\bar{3}}^{\rm part}}{F_{2\bar{2}}^{\rm part} - F_{3\bar{3}}^{\rm part}}, \tag{3.49}$$

if $F_{2\bar{2}}^{\rm part} \neq F_{3\bar{3}}^{\rm part}$. If $F_{2\bar{2}}^{\rm part} = F_{3\bar{3}}^{\rm part}$, we find

$$F_{2\bar{3}}^{\rm part}(\cos^2\theta_3 - \sin^2\theta_3) = 0. \tag{3.50}$$

Since we would like to consider $F_{2\bar{3}}^{\rm part} \neq 0$, we can obtain $\tan^2\theta_3 = 1$. In conclusion, the flux diagonal basis can be defined by

$$w^i = (P^{-1})^{ij} z^j, \tag{3.51}$$

where $P = P_{12}P_3 \in SO(3)$. In the w basis, the solution ψ becomes a single component of $\psi_{+--}, \psi_{-+-}, \psi_{--+}$.

We note that we can relate the wave functions with chiralities $(\underline{+--})$ to the wave functions ψ_{+++} by the rotation. Here, we take $F_{1\bar{1}}^{\rm diag} > 0$, $F_{2\bar{2}}^{\rm diag} < 0$, and $F_{3\bar{3}}^{\rm diag} < 0$. Then, the corresponding solution becomes ψ_{+--} in the w^i basis. Furthermore, we introduce $w'^i = {\rm Re}w^i + i(U{\rm Im}w)^i$, where

$$U \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.52}$$

The fluxes in the w'^i basis have positive-definite, so the corresponding solution becomes ψ_{+++}. This means that we can consider the solution ψ_{+--} for $F_{1\bar{1}}^{\rm diag} > 0$, $F_{2\bar{2}}^{\rm diag} < 0$, $F_{3\bar{3}}^{\rm diag} < 0$, and the solution ψ_{+++} for $F_{1\bar{1}}^{\rm diag} > 0$, $F_{2\bar{2}}^{\rm diag} > 0$, and $F_{3\bar{3}}^{\rm diag} > 0$ by these rotations. In general, we use the following U matrix

$$U \equiv \begin{bmatrix} {\rm sgn}(F_{1\bar{1}}^{\rm diag}) & 0 & 0 \\ 0 & {\rm sgn}(F_{2\bar{2}}^{\rm diag}) & 0 \\ 0 & 0 & {\rm sgn}(F_{3\bar{3}}^{\rm diag}) \end{bmatrix}, \tag{3.53}$$

where $\mathrm{sgn}(x) = 1$ and -1 for $x > 0$ and $x < 0$, respectively. After we rotate the fluxes in the w' basis by the rotation matrix P, we can introduce new complex coordinate $z' = Pw'$ written by the original coordinate z,

$$z' \equiv \mathrm{Re}z + iR\mathrm{Im}z, \tag{3.54}$$

where $R \equiv PUP^{-1} \in SO(3)$ with $R^2 = \mathbf{1}_3$. In addition, we define corresponding complex structure moduli

$$\Omega' \equiv \mathrm{Re}\Omega + iR\mathrm{Im}\Omega, \tag{3.55}$$

where $z' = x+\Omega' y$. Thus, we can find the relation between the solution $\psi'^{J}_{+++}(z', \Omega')$ with the chirality $(+++)$ in z' system and the solution $\psi^J_{M,N}(z, \Omega)$ with chiralities $(\underline{+--})$ as

$$\begin{aligned}
&\psi'^{J}_{+++}(z', \Omega') \\
&= \mathcal{N} \cdot e^{\pi i [Nz']^T (\mathrm{Im}\Omega')^{-1} \mathrm{Im}z'} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (Nz', N\Omega') \\
&= \mathcal{N} \cdot e^{\pi i [N\mathrm{Re}z + iM\mathrm{Im}z]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im}z} \cdot \theta \begin{bmatrix} J^T N^{-1} \\ 0 \end{bmatrix} (N\mathrm{Re}z + iM\mathrm{Im}z, N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega) \\
&= \psi^J_{M,N}(z, \Omega).
\end{aligned} \tag{3.56}$$

We note that $R = N^{-1}M = PUP^{-1} \in SO(3)$. [5]

We then obtain

$$\begin{aligned}
\psi_{+--} &= \alpha_1 \psi = \alpha_1 \psi'_{+++}, \\
\psi_{-+-} &= \alpha_2 \psi = \frac{\alpha_2}{\alpha_1} \cdot \alpha_1 \psi = q_1 \psi_{+--}, \\
\psi_{--+} &= \alpha_3 \psi = \frac{\alpha_3}{\alpha_1} \cdot \alpha_1 \psi = -q_2 \psi_{+--},
\end{aligned} \tag{3.57}$$

where $q_1, q_2 \in \mathbb{R}$. Here, we revisit q_1 and q_2 written in Eq. (3.37). When we suppose $\alpha_1 = \mathrm{Re}\alpha_1 + i\mathrm{Im}\alpha_1 \in \mathbb{C}$, that is, $\mathrm{Im}\alpha_1 \neq 0$, we expect that we derive

$$\begin{aligned}
\alpha_2 &= \mathrm{Re}\alpha_2 + i\mathrm{Im}\alpha_2 \in \mathbb{C}, \\
\alpha_3 &= \mathrm{Re}\alpha_3 + i\mathrm{Im}\alpha_3 \in \mathbb{C},
\end{aligned} \tag{3.58}$$

[5] We can find the relation between $(\mathrm{Re}z, \mathrm{Im}z)$ and $(\mathrm{Re}z', \mathrm{Im}z')$ by the $SO(6)$ rotation. Also, if the fluxes F take of the $(1,1)$-form in the z system, they can be also the $(1,1)$-form in the z' system. See the details in Appendix F.

namely $\mathrm{Im}\alpha_2, \mathrm{Im}\alpha_3 \neq 0$. Since $q_1, q_2 \in \mathbb{R}$, we should derive $q_1^2, q_2^2, q_1 q_2 \in \mathbb{R}$ from Eq. (3.37). This means that we can evaluate the values of α_i, $(i = 1, 2, 3)$, as follows

$$
\begin{aligned}
q_1 &= \frac{\alpha_2}{\alpha_1} = \frac{\alpha_1^* \alpha_2}{|\alpha_1|^2} \in \mathbb{R}, \\
q_2 &= -\frac{\alpha_3}{\alpha_1} = -\frac{\alpha_1^* \alpha_3}{|\alpha_1|^2} \in \mathbb{R}, \\
q_1^2 &= \frac{(\alpha_1^* \alpha_2)^2}{|\alpha_1|^4} \in \mathbb{R}, \\
q_2^2 &= \frac{(\alpha_1^* \alpha_3)^2}{|\alpha_1|^4} \in \mathbb{R}, \\
q_1 q_2 &= -\frac{(\alpha_1^*)^2 \alpha_2 \alpha_3}{|\alpha_1|^4} \in \mathbb{R}.
\end{aligned}
\tag{3.59}
$$

In other words, we should take

$$
\begin{aligned}
&\cdot\ \mathrm{Re}\alpha_1 \mathrm{Im}\alpha_2 = \mathrm{Im}\alpha_1 \mathrm{Re}\alpha_2, \\
&\cdot\ \mathrm{Re}\alpha_1 \mathrm{Im}\alpha_3 = \mathrm{Im}\alpha_1 \mathrm{Re}\alpha_3, \\
&\cdot\ (\mathrm{Re}\alpha_1)^2 (\mathrm{Re}\alpha_2 \mathrm{Im}\alpha_3 + \mathrm{Re}\alpha_3 \mathrm{Im}\alpha_2) \\
&\quad = \mathrm{Im}\alpha_1 [\mathrm{Im}\alpha_1 \mathrm{Im}\alpha_2 \mathrm{Re}\alpha_3 + \mathrm{Im}\alpha_1 \mathrm{Im}\alpha_3 \mathrm{Re}\alpha_2 \\
&\qquad\qquad - 2\mathrm{Re}\alpha_1 [\mathrm{Re}\alpha_2 \mathrm{Re}\alpha_3 - \mathrm{Im}\alpha_2 \mathrm{Im}\alpha_3]].
\end{aligned}
\tag{3.60}
$$

Since we have supposed $\alpha_1 \neq 0$, we do not consider $\mathrm{Re}\alpha_1 = \mathrm{Im}\alpha_1 = 0$. If we suppose $\mathrm{Re}\alpha_1 = 0$ and $\mathrm{Im}\alpha_1 \neq 0$, we find $\mathrm{Re}\alpha_2 = \mathrm{Re}\alpha_3 = 0$. If we suppose $\mathrm{Re}\alpha_1 \neq 0$ and $\mathrm{Im}\alpha_1 = 0$, we find $\mathrm{Im}\alpha_2 = \mathrm{Im}\alpha_3 = 0$. Additionally, under $\mathrm{Re}\alpha_1 \neq 0$ and $\mathrm{Im}\alpha_1 \neq 0$ in nonfactorizable magnetic fluxes, where $q_1 \neq 0$ and $q_2 \neq 0$, we also obtain

$$\mathrm{Re}\alpha_2 \mathrm{Im}\alpha_3 = \mathrm{Re}\alpha_3 \mathrm{Im}\alpha_2, \tag{3.61}$$

which leads to a following formula

$$[(\mathrm{Re}\alpha_1)^2 - (\mathrm{Im}\alpha_1)^2] \mathrm{Re}\alpha_2 \mathrm{Im}\alpha_3 = 0. \tag{3.62}$$

If $\mathrm{Im}\alpha_3 = 0$, we find $\mathrm{Re}\alpha_3 = 0$, which leads to $\alpha_3 = 0$, namely $q_2 = 0$. If $\mathrm{Re}\alpha_2 = 0$, we find $\mathrm{Im}\alpha_2 = 0$, which leads to $\alpha_2 = 0$, namely $q_1 = 0$. If $\mathrm{Re}\alpha_1 = \pm \mathrm{Im}\alpha_1$, we find $\mathrm{Re}\alpha_2 = \pm \mathrm{Im}\alpha_2$ and $\mathrm{Re}\alpha_3 = \pm \mathrm{Im}\alpha_3$, which leads to $\alpha_k = (\pm 1 + i)\mathrm{Im}\alpha_k = \pm(1 \pm i)\mathrm{Im}\alpha_k$, $(k = 1, 2, 3)$. They are consistent with the condition $q_1, q_2 \in \mathbb{R}$. Then, we find that the coefficients α_k can be real or imaginary or complex numbers.

Furthermore, we consider the coefficients α_i in more detail. We have introduced z system with the background magnetic fluxes on nonfactorizable T^6. In the z system, the off-diagonal components of the fluxes lead to various spinor components excited simultaneously. We have also supposed that 6D overall chirality is fixed as positive. Then, we have introduced the Ansatz $\psi_{i+} = \alpha_i \psi$, $(i = 1, 2, 3)$, where ψ denotes a complex function satisfying the boundary conditions on magnetized T^6. We will mention these α_i and ψ.

Here, we mention the $SO(6)$ rotation we have considered so far in detail. Since 10D $\mathcal{N} = 1$ SYM has $SO(9, 1)$ Lorentz transformation, we can derive transformation of its subgroup $SO(3, 1) \times SO(6)$ on $\mathbb{R}^{3,1} \times T^6$ when we consider the T^6 compactification and the KK decomposition. Thus, we can introduce the $SO(6)$ rotation transformation on T^6 from 10D $\mathcal{N} = 1$ SYM.

Furthermore, we need the Hermitian Yang-Mills equation to keep SUSY in 10D $\mathcal{N} = 1$ SYM. This means that the background magnetic fluxes F must be $(1, 1)$-form, that is,

$$F = F_{i\bar{j}}(idz^i \wedge d\bar{z}^j), \quad F_{ij} = F_{\bar{i}\bar{j}} = 0. \tag{3.63}$$

Therefore, we should keep the $(1, 1)$-form under the $SO(6)$ rotation.

Lastly, since 6D overall chirality defined by the Γ^7 in Eq. (3.8) is invariant under the $SO(6)$ rotation, when we fix the 6D overall chirality, we can consider the $SO(6)$ rotation in $(\mathrm{Re}z, \mathrm{Im}z)$ basis, that is, we can define a following function $f = f(\mathrm{Re}z, \mathrm{Im}z)$ under the $(1, 1)$-form magnetic fluxes F

$$f : (\mathrm{Re}z, \mathrm{Im}z) \to (\mathrm{Re}(az), \mathrm{Im}(bz)), \quad a, b \in GL(3, \mathbb{C}). \tag{3.64}$$

This keeps chirality of the wave functions on magnetized T^6 invariant. In conclusion, when we consider magnetized T^6 model, we have the $SO(6)$ rotation transformation and the $(1, 1)$-form magnetic fluxes from 10D $\mathcal{N} = 1$ SYM. This leads to the F and the real coordinates $(\mathrm{Re}z, \mathrm{Im}z)$ taking chirality into account. In this paper, we consider the $SO(6)$ rotation in not only the diagonalization

$$(\mathrm{Re}w, \mathrm{Im}w) = (\mathrm{Re}(P^{-1}z), \mathrm{Im}(P^{-1}z)), \quad P \in SO(3), \tag{3.65}$$

but the $\mathbb{Z}_N$-twist under identification $z \sim \Omega_{twist} z$ such that $\Omega_{twist}^N = \mathbf{1}_3$. We will discuss them.

Then, we consider the fluxes diagonal basis given by w system. In what follows, we take the following signs of the diagonal fluxes in the w system

$$F_{1\bar{1}}^{\mathrm{diag}} > 0, \; F_{2\bar{2}}^{\mathrm{diag}} < 0, \; F_{3\bar{3}}^{\mathrm{diag}} < 0. \tag{3.66}$$

We then find only one spinor component with $(+--)$ chirality corresponding to ψ_{1+} in the w system. Note that the other components become zero. Then, we need to consider double covering group of $SO(6)$, that is, $Spin(6)(\simeq SU(4))$ when we take rotations $w = P^{-1}z$.

Here, we analyze $Spin(6)$ spinor rotations in complex basis $(z, \bar{z})$, and apply them to not only the transformations from the z system to the w system, but also boundary conditions of spinor in magnetized $T^6/\mathbb{Z}_N$ orbifold models. First, we generally analyze generators of $spin(6)$-algebra to introduce general elements of $Spin(6)$. We will describe the elements as $\mathcal{S}^{spin} = \{\mathcal{S}_{ab}\}, (a, b = 1, 2, \cdots, 7, 8)$. When we describe a spinor written in the w system as Φ^w, we can represent the spinor rotation by the $Spin(6)$ spinor rotation $\mathcal{S}^{spin}$ as follows

$$\Phi^w = \mathcal{S}^{spin}\Psi^z, \tag{3.67}$$

where Ψ^z denotes the spinor written in the z system, namely the spinor in Eq. (3.22). Since we consider the $(1,1)$-form of the fluxes $F = F_{i\bar{j}}(idz^i \wedge d\bar{z}^{\bar{j}})$, we can take 9 non-trivial $Spin(6)$ generators. Specifically, according to Refs. [55, 81], we can define following $Spin(6)$ spinor rotation as the matrix exponential,

$$\mathcal{S}^{spin} = \exp\left[\sum_{i,j} \theta_{i\bar{j}} \Sigma^{i\bar{j}}\right] \equiv \exp(A') \equiv \sum_{k=0}^{\infty} \frac{1}{k!}(A')^k, \tag{3.68}$$

where $\theta_{i\bar{j}} \in \mathbb{R}$ and

$$\Sigma^{i\bar{j}} \equiv \frac{1}{4i}[\Gamma^{z^i}, \Gamma^{\bar{z}^{\bar{j}}}]. \tag{3.69}$$

Then, let us derive the spinor representation $\mathcal{S}^{spin} \in Spin(6)$ specifically. From

Eqs. (3.5) and (3.69), we obtain all the generators $\Sigma^{i\bar{j}}$ as follows

$$\begin{aligned}
\Sigma^{1\bar{1}} &= \frac{1}{4i}(4\sigma^3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2) = -i(\sigma^3 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2),\\
\Sigma^{2\bar{2}} &= \frac{1}{4i}(\mathbf{1}_2 \otimes (4\sigma^3) \otimes \mathbf{1}_2) = -i(\mathbf{1}_2 \otimes \sigma^3 \otimes \mathbf{1}_2),\\
\Sigma^{3\bar{3}} &= \frac{1}{4i}(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes (4\sigma^3)) = -i(\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^3),\\
\Sigma^{1\bar{2}} &= \frac{1}{4i}(\sigma^z \otimes (-2\sigma^{\bar{z}}) \otimes \mathbf{1}_2) = \frac{i}{2}(\sigma^z \otimes \sigma^{\bar{z}} \otimes \mathbf{1}_2),\\
\Sigma^{1\bar{3}} &= \frac{1}{4i}(\sigma^z \otimes \sigma^3 \otimes (-2\sigma^{\bar{z}})) = \frac{i}{2}(\sigma^z \otimes \sigma^3 \otimes \sigma^{\bar{z}}),\\
\Sigma^{2\bar{1}} &= \frac{1}{4i}(\sigma^{\bar{z}} \otimes (-2\sigma^{z}) \otimes \mathbf{1}_2) = \frac{i}{2}(\sigma^{\bar{z}} \otimes \sigma^{z} \otimes \mathbf{1}_2),\\
\Sigma^{2\bar{3}} &= \frac{1}{4i}(\mathbf{1}_2 \otimes \sigma^z \otimes (-2\sigma^{\bar{z}})) = \frac{i}{2}(\mathbf{1}_2 \otimes \sigma^z \otimes \sigma^{\bar{z}}),\\
\Sigma^{3\bar{1}} &= \frac{1}{4i}(\sigma^{\bar{z}} \otimes \sigma^3 \otimes (-2\sigma^{z})) = \frac{i}{2}(\sigma^{\bar{z}} \otimes \sigma^3 \otimes \sigma^{z}),\\
\Sigma^{3\bar{2}} &= \frac{1}{4i}(\mathbf{1}_2 \otimes \sigma^{\bar{z}} \otimes (-2\sigma^{z})) = \frac{i}{2}(\mathbf{1}_2 \otimes \sigma^{\bar{z}} \otimes \sigma^{z}),
\end{aligned} \tag{3.70}$$

where $\Sigma^{i\bar{i}}$ will correspond to the diagonal components of the $\mathcal{S}^{spin}$. We use following relations derived from Eq. (3.5)

$$\begin{aligned}
&[\sigma^z, \sigma^{\bar{z}}] = 4\sigma^3,\ [\sigma^3, \sigma^{\bar{z}}] = -2\sigma^{\bar{z}},\ [\sigma^z, \sigma^3] = -2\sigma^z,\\
&(\sigma^z)^2 = (\sigma^{\bar{z}})^2 = 0,\ (\sigma^3)^2 = \mathbf{1}_2.
\end{aligned} \tag{3.71}$$

Then, we find $(\Sigma^{i\bar{j}})^2 = 0$ for $i \neq j$. When we consider $i = j$, we obtain the diagonal components of the $\mathcal{S}^{spin}$

$$\begin{aligned}
\theta_{1\bar{1}}\Sigma^{1\bar{1}} &= \mathrm{diag}[-i\theta_{1\bar{1}}, -i\theta_{1\bar{1}}, -i\theta_{1\bar{1}}, -i\theta_{1\bar{1}}, +i\theta_{1\bar{1}}, +i\theta_{1\bar{1}}, +i\theta_{1\bar{1}}, +i\theta_{1\bar{1}}],\\
\theta_{2\bar{2}}\Sigma^{2\bar{2}} &= \mathrm{diag}[-i\theta_{2\bar{2}}, -i\theta_{2\bar{2}}, +i\theta_{2\bar{2}}, +i\theta_{2\bar{2}}, -i\theta_{2\bar{2}}, -i\theta_{2\bar{2}}, +i\theta_{2\bar{2}}, +i\theta_{2\bar{2}}],\\
\theta_{3\bar{3}}\Sigma^{3\bar{3}} &= \mathrm{diag}[-i\theta_{3\bar{3}}, +i\theta_{3\bar{3}}, -i\theta_{3\bar{3}}, +i\theta_{3\bar{3}}, -i\theta_{3\bar{3}}, +i\theta_{3\bar{3}}, -i\theta_{3\bar{3}}, +i\theta_{3\bar{3}}].
\end{aligned} \tag{3.72}$$

On the other hands, when we consider off-diagonal components of the A', we obtain the explicit forms as follows

$$\theta_{1\bar{2}}\Sigma^{1\bar{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2i\theta_{1\bar{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i\theta_{1\bar{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\theta_{1\bar{3}}\Sigma^{1\bar{3}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2i\theta_{1\bar{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2i\theta_{1\bar{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.73}$$

$$\theta_{2\bar{1}}\Sigma^{2\bar{1}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2i\theta_{2\bar{1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i\theta_{2\bar{1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\theta_{2\bar{3}}\Sigma^{2\bar{3}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2i\theta_{2\bar{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2i\theta_{2\bar{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\theta_{3\bar{1}}\Sigma^{3\bar{1}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2i\theta_{3\bar{1}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2i\theta_{3\bar{1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.74}$$

$$\theta_{3\bar{2}}\Sigma^{3\bar{2}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2i\theta_{3\bar{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i\theta_{3\bar{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we can obtain the A'

$$A' = \sum_{i,j} \theta_{i\bar{j}}\Sigma^{i\bar{j}} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.75}$$

where A, B, C, and D denote 4×4 complex matrix, respectively

$$A=\begin{bmatrix} -i(\theta_{1\bar{1}}+\theta_{2\bar{2}}+\theta_{3\bar{3}}) & 0 & 0 & 0 \\ 0 & -i(\theta_{1\bar{1}}+\theta_{2\bar{2}}-\theta_{3\bar{3}}) & 2i\theta_{2\bar{3}} & 0 \\ 0 & 2i\theta_{3\bar{2}} & -i(\theta_{1\bar{1}}-\theta_{2\bar{2}}+\theta_{3\bar{3}}) & 0 \\ 0 & 0 & 0 & -i(\theta_{1\bar{1}}-\theta_{2\bar{2}}-\theta_{3\bar{3}}) \end{bmatrix},$$
$$B=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2i\theta_{1\bar{3}} & 0 & 0 & 0 \\ 2i\theta_{1\bar{2}} & 0 & 0 & 0 \\ 0 & 2i\theta_{1\bar{2}} & -2i\theta_{1\bar{3}} & 0 \end{bmatrix}, \quad C=\begin{bmatrix} 0 & 2i\theta_{3\bar{1}} & 2i\theta_{2\bar{1}} & 0 \\ 0 & 0 & 0 & 2i\theta_{2\bar{1}} \\ 0 & 0 & 0 & -2i\theta_{3\bar{1}} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$D=\begin{bmatrix} i(\theta_{1\bar{1}}-\theta_{2\bar{2}}-\theta_{3\bar{3}}) & 0 & 0 & 0 \\ 0 & i(\theta_{1\bar{1}}-\theta_{2\bar{2}}+\theta_{3\bar{3}}) & 2i\theta_{2\bar{3}} & 0 \\ 0 & 2i\theta_{3\bar{2}} & i(\theta_{1\bar{1}}+\theta_{2\bar{2}}-\theta_{3\bar{3}}) & 0 \\ 0 & 0 & 0 & i(\theta_{1\bar{1}}+\theta_{2\bar{2}}+\theta_{3\bar{3}}) \end{bmatrix}. \tag{3.76}$$

Here, we can generally prove that when we consider 8×8 complex matrices X' and Y' with shapes like the above A', that is,

$$X'=\begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & x_{23} & 0 & x_{25} & 0 & 0 & 0 \\ 0 & x_{32} & x_{33} & 0 & x_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & x_{46} & x_{47} & 0 \\ 0 & x_{52} & x_{53} & 0 & x_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{64} & 0 & x_{66} & x_{67} & 0 \\ 0 & 0 & 0 & x_{74} & 0 & x_{76} & x_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{88} \end{bmatrix},$$
$$Y'=\begin{bmatrix} y_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{22} & y_{23} & 0 & y_{25} & 0 & 0 & 0 \\ 0 & y_{32} & y_{33} & 0 & y_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{44} & 0 & y_{46} & y_{47} & 0 \\ 0 & y_{52} & y_{53} & 0 & y_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & y_{64} & 0 & y_{66} & y_{67} & 0 \\ 0 & 0 & 0 & y_{74} & 0 & y_{76} & y_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{88} \end{bmatrix}, \tag{3.77}$$

we derive products $Z'=X'Y'$ with the same shape. That is, all the positions of zero and non-zero components are invariant if we consider products

$$Z' = X'Y' = \begin{bmatrix} z_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_{22} & z_{23} & 0 & z_{25} & 0 & 0 & 0 \\ 0 & z_{32} & z_{33} & 0 & z_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{44} & 0 & z_{46} & z_{47} & 0 \\ 0 & z_{52} & z_{53} & 0 & z_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & z_{64} & 0 & z_{66} & z_{67} & 0 \\ 0 & 0 & 0 & z_{74} & 0 & z_{76} & z_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_{88} \end{bmatrix}, \tag{3.78}$$

where all the components of the Z' are written as follows

$$\begin{aligned}
&z_{11} = x_{11}y_{11}, \\
&z_{22} = x_{22}y_{22} + x_{23}y_{32} + x_{25}y_{52}, \quad z_{23} = x_{22}y_{23} + x_{23}y_{33} + x_{25}y_{53}, \\
&z_{25} = x_{22}y_{25} + x_{23}y_{35} + x_{25}y_{55}, \quad z_{32} = x_{32}y_{22} + x_{33}y_{32} + x_{35}y_{52}, \\
&z_{33} = x_{32}y_{23} + x_{33}y_{33} + x_{35}y_{53}, \quad z_{35} = x_{32}y_{25} + x_{33}y_{35} + x_{35}y_{55}, \\
&z_{44} = x_{44}y_{44} + x_{46}y_{64} + x_{47}y_{74}, \quad z_{46} = x_{44}y_{46} + x_{46}y_{66} + x_{47}y_{76}, \\
&z_{47} = x_{44}y_{47} + x_{46}y_{67} + x_{47}y_{77}, \quad z_{52} = x_{52}y_{22} + x_{53}y_{32} + x_{55}y_{52}, \\
&z_{53} = x_{52}y_{23} + x_{53}y_{33} + x_{55}y_{53}, \quad z_{55} = x_{52}y_{25} + x_{53}y_{35} + x_{55}y_{55}, \\
&z_{64} = x_{64}y_{44} + x_{66}y_{64} + x_{67}y_{74}, \quad z_{66} = x_{64}y_{46} + x_{66}y_{66} + x_{67}y_{76}, \\
&z_{67} = x_{64}y_{47} + x_{66}y_{67} + x_{67}y_{77}, \quad z_{74} = x_{74}y_{44} + x_{76}y_{64} + x_{77}y_{74}, \\
&z_{76} = x_{74}y_{46} + x_{76}y_{66} + x_{77}y_{76}, \quad z_{77} = x_{74}y_{47} + x_{76}y_{67} + x_{77}y_{77}, \\
&z_{88} = x_{88}y_{88}.
\end{aligned} \tag{3.79}$$

Thus, when we consider the values of the $\mathcal{S}_{ab}$ derived from

$$\mathcal{S}_{ab} = [\exp(A')]_{ab} = \sum_{k=0}^{\infty} \frac{1}{k!}(A'^k)_{ab}, \tag{3.80}$$

the $(A')^k_{ab}$, $(k \in \mathbb{Z},\ k > 0,\ a \neq b)$, are zero if $A'_{ab} = 0$. Similarly, the $(A')^k_{ab}$ are non-zero if $A'_{ab} \neq 0$. This means that $\mathcal{S}_{ab} = 0$, $(a \neq b)$, if $A'_{ab} = 0$. In conclusion, we finally obtain the $Spin(6)$ rotation matrix $\mathcal{S}^{spin}$ as follows

$$\mathcal{S}^{spin} = \begin{bmatrix} \mathcal{S}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{S}_{22} & \mathcal{S}_{23} & 0 & \mathcal{S}_{25} & 0 & 0 & 0 \\ 0 & \mathcal{S}_{32} & \mathcal{S}_{33} & 0 & \mathcal{S}_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{S}_{44} & 0 & \mathcal{S}_{46} & \mathcal{S}_{47} & 0 \\ 0 & \mathcal{S}_{52} & \mathcal{S}_{53} & 0 & \mathcal{S}_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{S}_{64} & 0 & \mathcal{S}_{66} & \mathcal{S}_{67} & 0 \\ 0 & 0 & 0 & \mathcal{S}_{74} & 0 & \mathcal{S}_{76} & \mathcal{S}_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{S}_{88} \end{bmatrix}. \tag{3.81}$$

Here, we can take $\mathcal{S}_{11} = e^{-i(\theta_{1\bar{1}}+\theta_{2\bar{2}}+\theta_{3\bar{3}})}$ because of the definition Eq. (3.68). This is the specific $Spin(6)$ spinor representation. By use of the $\mathcal{S}^{spin} = \{\mathcal{S}_{ab}\}$, we obtain following spinor rotations

$$\begin{aligned} \psi_{+--} &\Rightarrow \mathcal{S}_{44}\psi_{+--} + \mathcal{S}_{46}\psi_{-+-} + \mathcal{S}_{47}\psi_{--+} = (\mathcal{S}_{44} + \mathcal{S}_{46}q_1 - \mathcal{S}_{47}q_2)\psi_{+--}, \\ \psi_{-+-} &\Rightarrow \mathcal{S}_{64}\psi_{+--} + \mathcal{S}_{66}\psi_{-+-} + \mathcal{S}_{67}\psi_{--+} = (\mathcal{S}_{64} + \mathcal{S}_{66}q_1 - \mathcal{S}_{67}q_2)\psi_{+--}, \\ \psi_{--+} &\Rightarrow \mathcal{S}_{74}\psi_{+--} + \mathcal{S}_{76}\psi_{-+-} + \mathcal{S}_{77}\psi_{--+} = (\mathcal{S}_{74} + \mathcal{S}_{76}q_1 - \mathcal{S}_{77}q_2)\psi_{+--}. \end{aligned} \tag{3.82}$$

Similarly, we can also derive the spinor rotations as linear combinations of the ψ_{1-}, ψ_{2-}, and ψ_{3-} when 6D overall chirality is fixed as negative. Therefore, the $SO(6)$ rotation leads to $Spin(6)$ spinor rotations properly which keep 6D overall chirality invariant.

We note that $\mathcal{S}_{aa}$, $\mathcal{S}_{ab}$, $(a \neq b)$, are generally complex numbers. Since Eq. (3.66) implies that we have only one spinor component corresponding to spinor with $(+--)$ chirality, we can take Φ^w as follows

$$\Phi^w = [0, 0, 0, \phi_{+--}, 0, 0, 0, 0]^T, \tag{3.83}$$

where ϕ_{+--} denotes the corresponding spinor component in the w system. From Eq. (3.82), we can derive

$$\begin{aligned} \psi_{+--} &\Rightarrow (\mathcal{S}_{44} + \mathcal{S}_{46}q_1 - \mathcal{S}_{47}q_2)\psi_{+--} = \phi_{+--}, \\ \psi_{-+-} &\Rightarrow (\mathcal{S}_{64} + \mathcal{S}_{66}q_1 - \mathcal{S}_{67}q_2)\psi_{+--} = 0, \\ \psi_{--+} &\Rightarrow (\mathcal{S}_{74} + \mathcal{S}_{76}q_1 - \mathcal{S}_{77}q_2)\psi_{+--} = 0, \end{aligned} \tag{3.84}$$

under the $Spin(6)$ spinor rotation from z system to w system. Then we obtain the following simultaneous equation

$$\begin{cases} \mathcal{S}_{64} + \mathcal{S}_{66}q_1 - \mathcal{S}_{67}q_2 = 0, \\ \mathcal{S}_{74} + \mathcal{S}_{76}q_1 - \mathcal{S}_{77}q_2 = 0. \end{cases} \tag{3.85}$$

We then obtain the following non-trivial solutions of q_1 and q_2,

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \equiv \frac{1}{\det \mathcal{S}'} \begin{bmatrix} \mathcal{S}_{64}\mathcal{S}_{77} - \mathcal{S}_{74}\mathcal{S}_{67} \\ \mathcal{S}_{64}\mathcal{S}_{76} - \mathcal{S}_{66}\mathcal{S}_{74} \end{bmatrix} = \frac{1}{\alpha_1} \begin{bmatrix} \alpha_2 \\ -\alpha_3 \end{bmatrix}, \tag{3.86}$$

where

$$\mathcal{S}' = \begin{bmatrix} \mathcal{S}_{66} & -\mathcal{S}_{67} \\ \mathcal{S}_{76} & -\mathcal{S}_{77} \end{bmatrix}. \tag{3.87}$$

Under the nonfactorizable fluxes, since we can take $\theta_{i\bar{j}} \neq 0$, this implies that we can assign the values of α_i

$$\begin{aligned} \alpha_1 &= \det \mathcal{S}' \in \mathbb{C}, \\ \alpha_2 &= \mathcal{S}_{64}\mathcal{S}_{77} - \mathcal{S}_{74}\mathcal{S}_{67} \in \mathbb{C}, \\ \alpha_3 &= \mathcal{S}_{66}\mathcal{S}_{74} - \mathcal{S}_{64}\mathcal{S}_{76} \in \mathbb{C}. \end{aligned} \tag{3.88}$$

From Eqs. (3.44), (3.45), and (3.88), we can show following relations between $SO(6)$ and $Spin(6)$ rotations

$$q_1 = \frac{\alpha_2}{\alpha_1} = -\frac{\tan\theta_1}{\cos\theta_2}, \quad q_2 = -\frac{\alpha_3}{\alpha_1} = \tan\theta_2. \tag{3.89}$$

Therefore, we find that the coefficients α_i are generally complex numbers by the $Spin(6)$ spinor rotation derived from $SO(6)$ rotation of the $(1,1)$-form nonfactorizable fluxes. This is because the real rotation parameters $\theta_{i\bar{j}}$ can be non-zero, that is, $\theta_{i\bar{j}} \neq 0$ under the nonfactorizable fluxes.

Then, what are the α_i and ψ in the end? Let us consider spinor wave functions Ψ^{general} of various spinor components in magnetized T^6. They can be written by linear combinations of the basis complex functions $\{\psi_k\}$ in magnetized T^6. Then, we can take Ψ^{general} as follows

$$\Psi^{\text{general}} = \sum_k c_k \psi_k, \tag{3.90}$$

where c_k denotes complex coefficients. Here, for simplicity, we suppose $\psi_k = \psi$ for all k. Also, we suppose $c_k = \alpha_k$, $(k = 1, 2, 3)$, and $c_k = 0$, $(k = 4, \cdots)$, for simplicity. We impose the boundary conditions in magnetized T^6 for the ψ. Thus, we can represent the spinor wave functions Ψ^{general} by the linear combinations of the basis ψ with complex coefficients α_i. In conclusion, when we suppose the Ansatz $\psi_{i+} = \alpha_i \psi$ for simplicity, we find that the coefficients α_i are generally complex numbers, according to the $Spin(6)$ spinor rotation.

Lastly, we mention properties of the metric in z' system. From Eq. (3.4), in real basis, the metric in z system, g, is given by

$$g = dz^i d\bar{z}^{\bar{i}} = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \mathbf{1}_3 & \mathrm{Re}\Omega \\ \mathrm{Re}\Omega & \Omega\bar{\Omega} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{3.91}$$

On the other hands, we can also write the metric in z' system, g', as

$$g' = dz'^i d\bar{z}'^{\bar{i}} = \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} \mathbf{1}_3 & \mathrm{Re}\Omega' \\ \mathrm{Re}\Omega' & \Omega'\bar{\Omega}' \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \tag{3.92}$$

We can find $\mathrm{Re}\Omega' = \mathrm{Re}\Omega$ under the $SO(6)$ rotation keeping the metric invariant. Similarly, we obtain $\Omega\bar{\Omega} = \Omega'\bar{\Omega}'$. In what follows, we impose $[\Omega, \bar{\Omega}] = 0$ to find $[R, \Omega] = 0$ under $[\Omega, \Omega'] = [\Omega, \bar{\Omega}] = 0$. This leads to $\Omega\bar{\Omega} = \Omega'\bar{\Omega}'$. By use of the z' system, we can introduce the chirality operator $\Gamma^{7\prime}$ as follows

$$\Gamma^{7\prime} \equiv \sigma^3 \otimes (-\sigma^3) \otimes (-\sigma^3) = \mathrm{diag}[+,-,-,+,-,+,+,-], \tag{3.93}$$

where $U = U_1 \equiv \mathrm{diag}(+,-,-)$. The $\Gamma^{7\prime}$ is the same as Γ^7 in terms of theory in magnetized T^6. Then, we can regard the wave functions with positive chirality $(+++)$ in z' system, that is, ψ'_{+++}, as solutions corresponding to wave functions with various chiralities in the z system. Furthermore, similarly to ψ_{+++} in the z system, the eigen equation of the Laplacian and the D-term SUSY condition in z' system can be derived by using the magnetic fluxes and the ψ'_{+++} in the z' system. In order to introduce them, we consider mass spectrum on bosonic sector.

3.2.2 Mass spectrum on bosonic sector

We analyze the mass spectrum of the 4D scalar fields. To obtain the mass spectrum, we consider dimensional reduction of 10D gauge boson. We would like to derive the mass spectrum since we particularly focus on the D-term SUSY condition.

Dimensional reduction of 10D $\mathcal{N} = 1$ super Yang-Mills theory

Here, we briefly review the dimensional reduction of 10D super Yang-Mills theory (SYM) based on Refs. [42,78] and Appendix of Ref. [54]. For simplicity, we consider $U(2)$ gauge group. We can extend the discussion to $U(N)$ gauge theory. We focus on T^6 compactification with background magnetic fluxes.

First, the bosonic part of the SYM action is given by

$$S_{YM} = -\frac{1}{4g^2}\int d^{10}w \ \mathrm{Tr}\{F^{MN}F_{MN}\}, \tag{3.94}$$

where $M, N \in \{0, 1, \cdots, 9\}$. We take real orthogonal coordinate system w^M by using the following metric

$$\eta_{MN} = \mathrm{diag}(-, +, +, \cdots, +, +). \tag{3.95}$$

Field strength F_{MN} in the model is given by

$$F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]. \tag{3.96}$$

Similarly, we can write gauge boson A_M as

$$A_M = B_M + W_M = B^a_M U_a + W^{ab}_M e_{ab}, \tag{3.97}$$

where we denote the elements of the $U(2)$ Lie algebra as $(U_a)_{ij} = \delta_{ai}\delta_{aj}$ and $(e_{ab})_{ij} = \delta_{ai}\delta_{bj}$ $(a \neq b)$. By use of $A^\dagger_M = A_M$, B^a_M is real and $(W^{ab}_M)^* = W^{ba}_M$. We finally obtain

$$\mathcal{L}_{YM} = -\frac{1}{2g^2}\mathrm{Tr}\left(D_M W_N D^M W^N - D_M W_N D^N W^M - iG_{MN}[W^M, W^N]\right) + \cdots. \tag{3.98}$$

Here, since we would like to derive mass terms, we take only quadratic terms of W^{ab}_M explicitly. That is, couplings such as 3- and 4-point interactions are irrelevant in this paper. Here, the field strength for B_N is given by

$$G_{MN} = \partial_M B_N - \partial_N B_M. \tag{3.99}$$

We also define a following covariant derivative

$$D_M W_N = \partial_M W_N - i[B_M, W_N]. \tag{3.100}$$

Then, we can analyze VEVs of the B_i which can be regarded as magnetic fluxes,

$$\begin{aligned} B^a_i(w) &= \langle B^a_i\rangle(\eta) + C^a_i(w), \\ W^{ab}_i(w) &= 0 + \Phi^{ab}_i(w), \end{aligned} \tag{3.101}$$

where $\langle B^1_i\rangle \neq \langle B^2_i\rangle$. Then, we find $U(1)\times U(1)$ gauge symmetry. Here, ξ, η denote the real orthogonal coordinates of 4D space-time and the T^6, respectively. we can then introduce a basis $w = (\xi, \eta)$. Also, space-time indices can be written by $M = (\mu, i)$, where $\mu = 0, ..., 3$ and $i = 4, ..., 9$.

Then, we get

$$\begin{aligned}\mathcal{L}_{YM} =& \frac{i}{4g^2}(G^a_{ij} - G^b_{ij})(\Phi^{i,ab}\Phi^{j,ba} - \Phi^{j,ab}\Phi^{i,ba}) \\ &- \frac{1}{2g^2}[(D_\mu \Phi^{ba}_i D^\mu \Phi^{i,ab}) + (\tilde{D}_i \Phi^{ba}_j \tilde{D}^i \Phi^{j,ab}) \\ &\qquad - 2(\tilde{D}_i W^{ba}_\mu)(D^\mu \Phi^{i,ab}) - (\tilde{D}_i \Phi^{ba}_j)(\tilde{D}^j \Phi^{i,ab})] + \cdots ,\end{aligned} \tag{3.102}$$

where

$$\tilde{D}_i W^{ab}_j = \partial_i W^{ab}_j - i(B^a_i - B^b_j)W^{ab}_j. \tag{3.103}$$

If we introduce the gauge fixing condition $\tilde{D}^i \Phi^{ab}_i = 0$, Eq. (3.102) can be given by

$$\mathcal{L}_{YM} = \frac{2i}{2g^2}\Phi^{j,ba}\langle G\rangle^{ab}_{ij}\Phi^{i,ab} + \frac{1}{2g^2}[\Phi^{ba}_i(D_\mu D^\mu \Phi^{i,ab}) + \Phi^{ba}_j(\tilde{D}_i \tilde{D}^i \Phi^{j,ab})] + \cdots , \tag{3.104}$$

where $\langle G\rangle^{ab}_{ij} = \langle G\rangle^a_{ij} - \langle G\rangle^b_{ij}$.

4D scalar mass

To obtain 4D scalar mass spectrum, we introduce the Kaluza-Klein decomposition,

$$\Phi^{ab}_i(w) = \sum_n \varphi^{ab}_{n,i}(\xi) \otimes \phi^{ab}_{n,i}(\eta), \tag{3.105}$$

where the wave functions in T^6 satisfy the following eigen equation of the Laplacian $\Delta = -\tilde{D}_i \tilde{D}^i$

$$\Delta \phi^{ab}_{n,i}(\eta) = \kappa^2_n \phi^{ab}_{n,i}(\eta). \tag{3.106}$$

Here, $\kappa^2_n \geq 0$ denotes the eigenvalue of the Δ. Then, the index $n \in \{0, 1, \cdots\}$ denotes the excitation number, where $\kappa^2_{n+1} > \kappa^2_n$. We substitute Eq. (3.105) into Eq. (3.104) and consider integration with respect to the η

$$S_{4D} = S_{kinetic} + S_{mass} + \cdots , \tag{3.107}$$

where

$$S_{kinetic} = \sum_n \int d^4\xi \ \delta_{ij}\varphi^{i,ba}_n (D_\mu D^\mu)\varphi^{j,ab}_n, \quad (a > b), \tag{3.108}$$

$$S_{mass} = \sum_n \int d^4\xi \ \varphi^{j,ba}_n (2i\langle G\rangle^{ab}_{ij} - \kappa^2_n \delta_{ij})\varphi^{i,ab}_n, \quad (a > b). \tag{3.109}$$

Here, we introduce the normalization condition,

$$g^{-2}\int d^6\eta\ \phi_m^{i,ab}(\eta)^*\phi_n^{i,cd}(\eta)=\delta_{mn}\delta_{ac}\delta_{bd}. \tag{3.110}$$

Next, we consider the coordinate transformation. We define complex coordinates $\vec{z}$ as $z^j=(2\pi r)^{-1}(\eta^{2j+2}+i\eta^{2j+3})$, where $j=1,2,3$ and $r=r_1=r_2=r_3$. This comes from Eq. (3.2). Then, we find

$$\begin{aligned}\varphi^{z^j,ab}(\vec{z})&=\frac{1}{2\pi r}(\varphi^{2j+2,ab}(\eta)+i\varphi^{2j+3,ab}(\eta)),\\ \varphi^{\bar{z}^{\bar{j}},ab}(\vec{z})&=\frac{1}{2\pi r}(\varphi^{2j+2,ab}(\eta)-i\varphi^{2j+3,ab}(\eta)).\end{aligned} \tag{3.111}$$

When representing Eq. (3.108) in the new basis, we get

$$\begin{aligned}S_{kinetic}&=\sum_n\int d^4\xi\ [h_{i\bar{j}}\varphi_n^{z^i,ba}(D_\mu D^\mu)\varphi_n^{\bar{z}^{\bar{j}},ab}+h_{\bar{i}j}\varphi_n^{\bar{z}^{\bar{i}},ba}(D_\mu D^\mu)\varphi_n^{z^j,ab}]\\ &=\frac{(2\pi r)^2}{2}\sum_n\int d^4\xi\ [\varphi_n^{z^j,ba}(D_\mu D^\mu)\varphi_n^{\bar{z}^{\bar{j}},ab}+\varphi_n^{\bar{z}^{\bar{j}},ba}(D_\mu D^\mu)\varphi_n^{z^j,ab}],\end{aligned} \tag{3.112}$$

where the metric is written in Eq. (3.4). We rewrite the 4D scalar fields as

$$\hat{\varphi}^{z^j,ab}=\frac{2\pi r}{\sqrt{2}}\varphi^{z^j,ab},\quad \hat{\varphi}^{\bar{z}^{\bar{j}},ab}=\frac{2\pi r}{\sqrt{2}}\varphi^{\bar{z}^{\bar{j}},ab}, \tag{3.113}$$

in order to consider canonical kinetic terms. We thus derive following 4D scalar mass term

$$\begin{aligned}S_{mass}=\sum_n\int d^4\xi\ &\left[\hat{\varphi}_n^{z^j,ba}\left(\frac{4i}{(2\pi r)^2}\langle G\rangle^{ab}_{\bar{z}^{\bar{j}}z^j}-\kappa_n^2\right)\hat{\varphi}_n^{\bar{z}^{\bar{j}},ab}\right.\\ &\left.+\hat{\varphi}_n^{\bar{z}^{\bar{j}},ba}\left(\frac{4i}{(2\pi r)^2}\langle G\rangle^{ab}_{z^j\bar{z}^{\bar{j}}}-\kappa_n^2\right)\hat{\varphi}_n^{z^j,ab}\right].\end{aligned} \tag{3.114}$$

In particular, when we suppose $\langle G^{b=2}\rangle=0$, the 4D scalar can obtain only $U(1)_{a=1}$ charge. Then, we can see 4D scalar mass spectrum in the $U(1)$ gauge theory. When we consider T^2 compactification and the dimensional reduction of SYM in Ref. [54], we find that their results are realized by Eq. (3.114).

3.2.3 D-term SUSY condition

We can define the Laplacian Δ in z system [78, 81, 82]

$$\Delta\equiv-2\sum_{j=1}^{3}\{D_{z^j},\bar{D}_{\bar{z}^{\bar{j}}}\}. \tag{3.115}$$

We can verify that the solution ψ_{+++} satisfies the eigen equation of the Laplacian Δ:

$$\Delta\psi_{+++} = 2\left[\sum_{j=1}^{3} F_{z^j\bar{z}^{\bar{j}}}\right]\psi_{+++} = 2\cdot(\mathrm{tr}F_{z^i\bar{z}^{\bar{j}}})\psi_{+++}. \tag{3.116}$$

where the eigenvalue must be non-negative. In other words, we need to the condition $\mathrm{tr}F_{z^i\bar{z}^{\bar{j}}} \geq 0$, that is, the positive semi-definiteness of the Δ on T^6. We will use the Δ to derive mass spectrum in bosonic sectors.

The eigenvalue corresponds to the mass squared in 6D scalar field. Also, the 6D vector modes $(A_{z^i}, A_{\bar{z}^i})$ have additional effects. From the discussion so far, we can derive their squared mass matrix $\mathcal{M}^2$ by using $F_{z^i\bar{z}^{\bar{j}}} = F_{i\bar{j}}$ and Δ

$$\mathcal{M}^2 = \Delta + 4\begin{bmatrix} -F_{i\bar{j}} & O \\ O & F_{i\bar{j}} \end{bmatrix}. \tag{3.117}$$

Since we obtain the eigenvalue of the Δ, $\mathcal{M}^2$ can be rewritten in the diagonal basis

$$\mathrm{diag}\mathcal{M}^2 = 2(\lambda_1+\lambda_2+\lambda_3) + 4\cdot\mathrm{diag}[-\lambda_1, -\lambda_2, -\lambda_3, +\lambda_1, +\lambda_2, +\lambda_3], \tag{3.118}$$

where we denote λ_i as the eigenvalues of $F_{i\bar{j}} = \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}$. Then, $\mathrm{tr}F_{i\bar{j}}$ is the lowest eigenvalue of Δ. We note that the largest eigenvalue λ_I need to be equal to summation of the remaining two eigenvalues λ_J, λ_K to realize the massless spectrum, that is,

$$\lambda_I = \lambda_J + \lambda_K. \tag{3.119}$$

This corresponds to the D-term SUSY condition, that is, the condition to realize the D-flat solution [54,78,95]. Similarly, we can obtain the D-term SUSY condition from the the eigen equation of ψ_{---} for Δ given by

$$\Delta\psi_{---} = -2\left[\sum_{j=1}^{3} F_{z^j\bar{z}^{\bar{j}}}\right]\psi_{---}. \tag{3.120}$$

Next, in the z' system, we consider the mass spectrum from magnetic fluxes on T^6 including two negative eigenvalues. We note that 6D overall chirality is fixed. Similarly to the z system so far, for 4D scalar, the eigenvalues of $\mathcal{M}^2$ are invariant under the $SO(6)$ rotation. That is, we can describe the squared 4D scalar masses $m^2_{4D,I}$, $(I = 1, 2, 3)$,

$$\mathrm{diag}(m^2_{4D,1}, m^2_{4D.2}, m^2_{4D,3}) = D^{-1}\mathcal{M}^2_z D = D'^{-1}\mathcal{M}'^2_{z'} D', \tag{3.121}$$

where we denote the squared mass matrices in z, z' basis as $\mathcal{M}_z^2$, $\mathcal{M}_{z'}'^2$, respectively. D, D' represent the digonalization matrices. We will obtain $\mathcal{M}_{z'}'^2$. We can write the spinor in z', Ψ', as

$$\Psi' = [0, 0, 0, \psi'_{+++}, 0, 0, 0, 0]^T. \tag{3.122}$$

This implies that the ψ'_{+++} in Eq. (3.122) is the solution of the Dirac equation $D_{\bar{z}'^{\bar{j}}}\psi'_{+++} = 0$, $(j = 1, 2, 3)$. Also, We introduce the Laplacian Δ' in z' system

$$\Delta' \equiv -2 \sum_{j=1}^{3} \{D_{z'^j}, \bar{D}_{\bar{z}'^{\bar{j}}}\}. \tag{3.123}$$

Similarly to the ψ_{+++} in the z system, we get the eigen equation of Δ'

$$\Delta'\psi'_{+++} = 2 \cdot (\mathrm{tr}F'_{z^i\bar{z}^{\bar{j}}})\psi'_{+++}. \tag{3.124}$$

Therefore, when we denote $F'_{i\bar{j}} = F'_{z'^i\bar{z}'^{\bar{j}}}$, the $\mathcal{M}_{z'}'^2$ can be written as

$$\mathcal{M}_{z'}'^2 = \Delta' + 4 \begin{bmatrix} -F'_{i\bar{j}} & O \\ O & F'_{i\bar{j}} \end{bmatrix}. \tag{3.125}$$

By use of the eigenvalues of $F'_{i\bar{j}}$, λ'_i, we can derive

$$D'^{-1}\mathcal{M}_{z'}'^2 D' = 2(\lambda'_1 + \lambda'_2 + \lambda'_3) + 4 \cdot \mathrm{diag}[-\lambda'_1, -\lambda'_2, -\lambda'_3, +\lambda'_1, +\lambda'_2, +\lambda'_3]. \tag{3.126}$$

We finally obtain the D-term SUSY condition [81, 82]

$$\lambda'_I = \lambda'_J + \lambda'_K, \tag{3.127}$$

where λ'_I denotes the largest eigenvalue of $F'_{i\bar{j}}$. Also, the other two eigenvalues are written by λ'_J, λ'_K. In other words, we can write the D-term SUSY condition using the eigenvalues of $F_{z^i\bar{z}^{\bar{j}}}$, that is,

$$|\lambda_I| = |\lambda_J| + |\lambda_K|, \tag{3.128}$$

where λ_I denotes the largest absolute eigenvalue. Also, λ_J, λ_K denote the others absolute eigenvalues. Lastly, we comment on the D-term SUSY condition Eq. (3.128) in this paper. According to Refs. [96, 97], we can regard this D-term SUSY condition as the condition to vanish flux-induced Fayet-Iliopoulos (FI) terms $\xi = 0$ for $U(1)$ factors under vanishing vacuum expectation values (VEVs) of 4D complex scalar fields. In what follows, we will consider this D-term SUSY condition.

On the other hands, if we suppose non-vanishing VEVs of the 4D scalar fields with nonzero charges, the nonzero flux-induced FI-terms $\xi(\neq 0)$ can be canceled by the VEVs. We can therefore realize stable SUSY vacua if we take the $(1, 1)$-form background magnetic fluxes and nonzero ξ under the non-vanishing VEVs of 4D charged scalar fields.

3.3 Wave functions and $Sp(6,\mathbb{Z})$ modular symmetry

In what follows, we discuss the $Sp(6,\mathbb{Z})$ modular symmetry. The $Sp(6,\mathbb{Z})$ modular symmetry is one of geometrical symmetry derived from basis transformation on $T^6 \simeq \mathbb{C}^3/\Lambda$. In other words, we can consider the $Sp(6,\mathbb{Z})$ modular symmetry under the T^6 compactification [66, 94, 98–105]. The modular transformation is important to analyze $\mathbb{Z}_N$-twist and zero-mode numbers. By use of the modular transformation, we can verify whether three-generation models can be realized or not in magnetized orbifold models [58, 78, 80–82]. The modular symmetry is also important, because this symmetry can control 4D low energy effective field theory.[6] We will briefly show wave functions behaviors under the $Sp(6,\mathbb{Z})$ modular transformation. The details are described in the Appendix G.

According to Ref. [78], the $Sp(6,\mathbb{Z})$ modular transformations can be realized by the generators S and T. [7] The generators S and T can be written as

$$S = \begin{bmatrix} O_3 & \mathbf{1}_3 \\ -\mathbf{1}_3 & O_3 \end{bmatrix}, \quad T = \begin{bmatrix} \mathbf{1}_3 & B \\ 0_3 & \mathbf{1}_3 \end{bmatrix}, \tag{3.129}$$

where the B matrix denotes 3×3 integer symmetric matrix. Then, z and Ω behave as

$$S : (z,\Omega) \Rightarrow (-\Omega^{-1}z, -\Omega^{-1}), \quad T : (z,\Omega) \Rightarrow (z, \Omega + B). \tag{3.130}$$

In what follows, we will briefly show the wave functions behaviors under the modular S and T transformations, respectively.

3.3.1 Modular S transformation

First, we calculate the wave functions behaviors under the S transformation. The integer fluxes N are transformed as N^T, however, we suppose the S-symmetric fluxes, that is, $N^T = N$. Also, we suppose the S-invariant $R = N^{-1}M$. Under $R^2 = \mathbf{1}_3$ and $[\Omega, \bar{\Omega}] = 0$, since we can consider the wave functions with positive chirality $(+++)$ in the z' system, transformations of z' and Ω' can be related to

[6] Recently, modular flavor symmetries have been focused on [12]. (See for early works Refs. [14, 17, 18, 107–113] and for reviews Refs. [32, 33].)

[7] The $Sp(2g,\mathbb{Z})$ modular group is explained in Appendix G.

the original S transformation [8]

$$\begin{aligned} \Omega' &\Rightarrow -\Omega'^{-1} \quad \text{under S: } \Omega \to -\Omega^{-1}, \\ z' &\Rightarrow -\Omega'^{-1}z' \quad \text{under S: } z \to -\Omega^{-1}z. \end{aligned} \tag{3.131}$$

Then, we can describe the wave functions behaviors under the S transformation

$$\psi'^{J}_{+++}(-\Omega'^{-1}z', -\Omega'^{-1}) = \frac{\sqrt{\det(-i\Omega')}}{\sqrt{|\det N|}} \sum_{K\in\Lambda_N} e^{2\pi i J^T N^{-1} K} \cdot \psi'^{K}_{+++}(z', \Omega'). \tag{3.132}$$

If we define the Siegel modular form $f(\Omega)$ under the $Sp(2g,\mathbb{Z})$, $(g = 2, 3)$, modular transformation γ in the z system, that is,

$$f(\gamma\Omega) = |C\Omega + D|^k \rho(\gamma) f(\Omega), \quad \text{where} \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2g,\mathbb{Z}), \tag{3.133}$$

and we can regard k, $\rho(\gamma)$ as weight and the representation matrix, respectively [98–105], we find that the wave functions can be regarded as the Siegel modular form in the z' system.

3.3.2 Modular T transformation

Next, let us briefly show the T transformation of the wave functions. We require the following condition for the background magnetic fluxes F

$$(NB)^T = NB \quad \text{where} \quad (NB)_{ii} \in 2\mathbb{Z} \tag{3.134}$$

to be consistent with the wave functions and the T transformation. [9] The wave functions behaviors are written as

$$\psi^J_{M,N}(z, \Omega + B) = e^{\pi i J^T N^{-1} B J} \psi^J_{M,N}(z, \Omega) \tag{3.135}$$

under the T transformation. This can be also described as

$$\psi'^J_{+++}(z', \Omega' + B) = e^{\pi i J^T N^{-1} B J} \psi'^J_{+++}(z', \Omega') \tag{3.136}$$

in z' system.

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[8] See the details in Appendix G.1.
[9] See the details in Appendix G.2.

We mention $(ST\cdots)$-invariant moduli for $\mathbb{Z}_N$-twist. They satisfy $(ST\cdots)^N = \mathbf{1}_6$ and $\Omega = -(\Omega + B)^{-1}$. This is important to count the zero-mode numbers in magnetized $T^6/\mathbb{Z}_N$ orbifolds. Thus we derive

$$\psi^J_{N,R}(\Omega' z', \Omega') = \rho_{JK}(ST\cdots)\psi^K_{N,R}(z', \Omega') \tag{3.137}$$

under $(ST\cdots)$-transformation in the z' system. Here, $\rho_{JK}(ST\cdots)$ is representation derived from the modular transformations of the wave functions in magnetized T^6. We will use the modular transformation and the representation matrix when we consider magnetized $T^6/\mathbb{Z}_N$ orbifolds models. In particular, for $z' = z$, we can analyze the number of zero modes with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_N$. Similarly, for $z' \neq z$, we can analyze the number of zero modes with various chiralities in magnetized $T^6/\mathbb{Z}_N$.

In chapter 4, we will consider nonfactorizable $T^6/\mathbb{Z}_N$ with background magnetic fluxes and analyze the number of the zero modes in magnetized $T^6/\mathbb{Z}_N$.

Chapter 4

Magnetized $T^6/\mathbb{Z}_N$ orbifolds

We have discussed magnetized T^6 model. Then, we can extend the model to magnetized $T^6/\mathbb{Z}_N$ orbifold models which we may construct richer flavor models including three-generation structure. In this chapter, we introduce 6D lattices of $T^6/\mathbb{Z}_N$ orbifolds. In particular, we focus on magnetized $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ whose lattices are $SU(7)$, E_6 root lattices, respectively. This implies that they cannot be factorized as product of lower-dimensional compact spaces. We also introduce background magnetic fluxes in the 6D lattices. Shapes of the fluxes N are restricted by the F-term SUSY condition and generators of the $Sp(6,\mathbb{Z})$ modular symmetry. In what follows, we will see the details. By use of the modular transformations of the wave functions, we can analyze the number of zero modes in magnetized $T^6/\mathbb{Z}_N$ models. This chapter is along in Refs. [78, 81, 82].

4.1 Lattices of $T^6/\mathbb{Z}_N$

Here, we briefly review the $T^6/\mathbb{Z}_N$ orbifolds. To obtain 4D $\mathcal{N}=1$ SUSY, we need to classify the $\mathbb{Z}_N$ twists. They were studied in Refs. [46, 47]. Table 4.1 shows such $\mathbb{Z}_N$ twists [78]. Then, the condition of SUSY is written as

$$k_1 + k_2 + k_3 \equiv 0 \pmod N, \tag{4.1}$$

where we denote $e^{2\pi i k_i/N}$, $(i = 1, 2, 3)$, as eigenvalues of the orbifold twist. We divide the 6D flat space by a 6D lattice Λ_6 to construct the torus T^6. We divide T^6 by the $\mathbb{Z}_N$ twist so as to obtain the $T^6/\mathbb{Z}_N$ orbifold. In order to construct the 6D lattice Λ_6, we make use of Lie lattices. Since we obtain $T^6/\mathbb{Z}_N$ when we impose $\mathbb{Z}_N$ identifications on T^6, the 6D lattice has to have the $\mathbb{Z}_N$ symmetry.

Then, the Coxeter element C of Lie lattice can correspond to the $\mathbb{Z}_N$ twist [88–93]. In particular, $SU(N)$ root lattice has the Coxeter element C. The Coxeter element has the $\mathbb{Z}_N$ symmetry, that is, $C^N = 1$. In what follows, we use Lie root lattices. We also introduce complex structure moduli and magnetic fluxes to be consistent with the Lie root lattice. We can see that $T^2/\mathbb{Z}_N$ can construct the flavor

Orbifold	twists $(k_1, k_2, k_3)/N$	lattice
$T^6/\mathbb{Z}_3$	(1,1,-2)/3	$SU(3)^3$
$T^6/\mathbb{Z}_4$	(1,1,-2)/4	$(SO(4)^{[2]})^2 \times SU(2)^2$
$T^6/\mathbb{Z}_{6-I}$	(1,1,-2)/6	$(SU(3)^{[2]})^2 \times SU(3)$
$T^6/\mathbb{Z}_{6-II}$	(1,2,-3)/6	$SU(3)^{[2]} \times SU(3) \times SU(2)^2, \quad SO(8) \times SU(3)$
$T^6/\mathbb{Z}_7$	(1,2,-3)/7	$SU(7)$
$T^6/\mathbb{Z}_{8-I}$	(1,2,-3)/8	$(SO(8)^{[2]}) \times SO(4)^{[2]}$
$T^6/\mathbb{Z}_{8-II}$	(1,3,-4)/8	$(SO(8)^{[2]}) \times SU(2)^2$
$T^6/\mathbb{Z}_{12-I}$	(1,4,-5)/12	E_6
$T^6/\mathbb{Z}_{12-II}$	(1,5-6)/12	$(SO(8)^{[3]}) \times SU(2)^2$

Table 4.1: $T^6/\mathbb{Z}_N$ orbifolds and torus lattices.

structure in Refs. [70, 71]. Then, we focus on nonfactorizable $T^6/\mathbb{Z}_N$ orbifolds shown in Table 4.1. We particularly focus on the numbers of zero modes on $T^6/\mathbb{Z}_N$ with magnetic fluxes. Here, we consider nonfactorizable $T^6/\mathbb{Z}_N$ orbifolds, that is, the $T^6/\mathbb{Z}_7$ orbifold with the $SU(7)$ root lattice and the $T^6/\mathbb{Z}_{12-I}$ orbifold with E_6 root lattice. We can derive the zero-mode wave functions in magnetized $T^6/\mathbb{Z}_N$ orbifolds from linear combinations of the wave functions in magnetized T^6 . We then use generators of $Sp(6,\mathbb{Z})$ to realize the orbifold twists. That is, by use of $Sp(6,\mathbb{Z})$ modular transformation, we study the number of zero-mode wave functions in magnetized $T^6/\mathbb{Z}_N$. One can perform similar analysis in magnetized $T^2/\mathbb{Z}_N$ [58], $T^4/\mathbb{Z}_N$ orbifolds [80–82], respectively.

4.2 Nonfactorizable $T^6/\mathbb{Z}_N$ orbifolds

We define the boundary condition of the spinor $\Psi = \Psi(z, \bar{z})$ [80]

$$\Psi(\Omega z, \bar{\Omega}\bar{z}) \equiv V(z, \bar{z})\mathcal{S}^{twist}\Psi(z, \bar{z}), \tag{4.2}$$

where the identifications $z \sim \Omega z$ and $\Omega^N = \mathbf{1}_3$ on T^6 are imposed. $\mathcal{S}^{twist}$ denotes the 8×8 complex matrix and we will see its shape. The $V(z, \bar{z})$ denotes a transformation function. We use Eq. (3.81), that is,

$$\mathcal{S}^{spin} = \begin{bmatrix} \mathcal{S}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{S}_{22} & \mathcal{S}_{23} & 0 & \mathcal{S}_{25} & 0 & 0 & 0 \\ 0 & \mathcal{S}_{32} & \mathcal{S}_{33} & 0 & \mathcal{S}_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{S}_{44} & 0 & \mathcal{S}_{46} & \mathcal{S}_{47} & 0 \\ 0 & \mathcal{S}_{52} & \mathcal{S}_{53} & 0 & \mathcal{S}_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{S}_{64} & 0 & \mathcal{S}_{66} & \mathcal{S}_{67} & 0 \\ 0 & 0 & 0 & \mathcal{S}_{74} & 0 & \mathcal{S}_{76} & \mathcal{S}_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{S}_{88} \end{bmatrix}$$

$$= \mathcal{S}_{11} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{22} & \mathcal{S}_{11}^{-1}\mathcal{S}_{23} & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{25} & 0 & 0 & 0 \\ 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{32} & \mathcal{S}_{11}^{-1}\mathcal{S}_{33} & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{44} & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{46} & \mathcal{S}_{11}^{-1}\mathcal{S}_{47} & 0 \\ 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{52} & \mathcal{S}_{11}^{-1}\mathcal{S}_{53} & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{64} & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{66} & \mathcal{S}_{11}^{-1}\mathcal{S}_{67} & 0 \\ 0 & 0 & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{74} & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{76} & \mathcal{S}_{11}^{-1}\mathcal{S}_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{S}_{11}^{-1}\mathcal{S}_{88} \end{bmatrix}. \tag{4.3}$$

Then, we particularly define the boundary condition of the ψ_{+++} as follows

$$\psi_{+++}(\Omega z, \bar{\Omega}\bar{z}) \equiv V(z,\bar{z}) \cdot \psi_{+++}(z,\bar{z}) \equiv \mathcal{S}_{11} \cdot \psi_{+++}(z,\bar{z}). \tag{4.4}$$

We specify the $V(z,\bar{z})$ as follows under the $\mathbb{Z}_N$-twist

$$V(z,\bar{z}) = \mathcal{S}_{11} = e^{2\pi i\beta}. \tag{4.5}$$

Here, β is a real number which can be described by

$$\beta = -\frac{\theta_{1\bar{1}} + \theta_{2\bar{2}} + \theta_{3\bar{3}}}{2\pi}, \tag{4.6}$$

where $\theta_{i\bar{j}}$ are real numbers. Since the β must satisfies $\beta N \equiv 0 \pmod 1$, we find

$$\begin{aligned} \beta N &= -\frac{N}{2\pi}(\theta_{1\bar{1}} + \theta_{2\bar{2}} + \theta_{3\bar{3}}) = \ell \in \mathbb{Z}, \\ \theta_{1\bar{1}} + \theta_{2\bar{2}} + \theta_{3\bar{3}} &= -\frac{2\pi}{N}\ell. \end{aligned} \tag{4.7}$$

When we define and fix the above boundary condition of the ψ_{+++}, there exists degree of freedom of the $U(1)$ global symmetry derived from the action. Then, we can introduce the boundary conditions of the wave functions with 6D overall negative chirality taking the $U(1)$ global transformations into account. In this paper, since we particularly focus on analyzing the number of zero modes with 6D overall positive chirality, we do not consider the degree of freedom derived from the $U(1)$ global symmetry. In the following, we perform counting the zero modes with 6D overall positive chirality in magnetized $T^6/\mathbb{Z}_7$ and $T^6/\mathbb{Z}_{12}$ orbifold models.

4.2.1 Magnetized $T^6/\mathbb{Z}_7$ orbifold

First, we consider the number of zero modes with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_7$ orbifold [78]. We focus on how to construct the $T^6/\mathbb{Z}_7$ orbifold. To realize the $\mathbb{Z}_7$-twist and the modular transformation, we use generators of the $Sp(6,\mathbb{Z})$ modular symmetry Then, we can find the algebraic relation

$$(ST_3T_4T_5)^7 = \mathbf{1_6}. \tag{4.8}$$

We find that $ST_3T_4T_5$ transformation can lead to the $\mathbb{Z}_7$-twist. Thus, we can use the algebraic relation written in the S and T when we construct $T^6/\mathbb{Z}_7$ orbifold. When we consider the $ST_3T_4T_5$ transformation, we find

$$\begin{aligned} &\Omega \to -(\Omega + B_3 + B_4 + B_5)^{-1}, \\ &\vec{z} \to -(\Omega + B_3 + B_4 + B_5)^{-1}\vec{z}, \end{aligned} \tag{4.9}$$

for the complex structure moduli Ω and complex coordinates $\vec{z}$. Then, we find $ST_3T_4T_5$-invariant moduli Ω_7 under $\det(\mathrm{Im}\Omega_7) > 0$

$$\Omega_7 = \begin{bmatrix} \omega_1 & \omega_4 & \omega_6 \\ \omega_4 & \omega_2 & \omega_5 \\ \omega_6 & \omega_5 & \omega_3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{7}}i & -\frac{1}{2}+\frac{\sqrt{7}}{14}i & \frac{i}{\sqrt{7}} \\ -\frac{1}{2}+\frac{\sqrt{7}}{14}i & -\frac{i}{\sqrt{7}} & -\frac{1}{2}+\frac{3\sqrt{7}}{14}i \\ \frac{i}{\sqrt{7}} & -\frac{1}{2}+\frac{3\sqrt{7}}{14}i & -\frac{1}{2}-\frac{\sqrt{7}}{14}i \end{bmatrix}. \tag{4.10}$$

We can verify that we can construct the $SU(7)$ root lattice in Figure 4.1 [78]. Second, We can restrict the shape of flux N. From the F-term SUSY condition

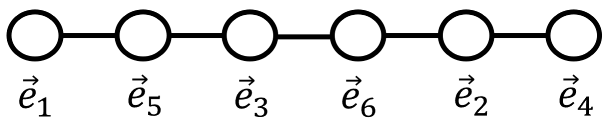

Figure 4.1: The lattice of $T^6/\mathbb{Z}_7$.

$(N\Omega_7)^T = N\Omega_7$, the N can be parameterized by three independent integer parameters n_{11}, n_{22}, and n_{33}

$$N = \begin{bmatrix} n_{11} & n_{33}-n_{22} & n_{22}-n_{11} \\ n_{33}-n_{22} & n_{22} & n_{33}-n_{11} \\ n_{22}-n_{11} & n_{33}-n_{11} & n_{33} \end{bmatrix}. \tag{4.11}$$

Furthermore, we consider the T-transformation. Then, the NB has to be symmetric. In particular, the diagonal components of the NB should be all even, that is, $(NB)_{ii} \in 2\mathbb{Z}$, $(i = 1,2,3)$. [1]. The NB, that is,

$$N(B_3+B_4+B_5) = \begin{bmatrix} n_{33}-n_{22} & n_{22} & n_{33}-n_{11} \\ n_{22} & 2n_{33}-n_{11}-n_{22} & n_{33}+n_{22}-n_{11} \\ n_{33}-n_{11} & n_{33}+n_{22}-n_{11} & 2n_{33}-n_{11} \end{bmatrix}, \tag{4.12}$$

leads to the condition

$$n_{11} \equiv n_{22} \equiv n_{33} \equiv 0 \qquad (\mathrm{mod}\ 2). \tag{4.13}$$

Then, the $\det N$ should be always a multiple of eight. We will count zero-mode numbers in $T^6/\mathbb{Z}_7$ with the flux N.

The number of zero modes

Next, we consider how to count the number of zero modes in magnetized $T^6/\mathbb{Z}_7$. Since $\Omega_7 = -(\Omega_7 + B_3 + B_4 + B_5)^{-1}$, we find the $ST_3T_4T_5$ transformation of zero-mode wave functions with positive chirality $(+++)$

$$\begin{aligned} &\psi_N^{\vec{J}}(\Omega_7 z, \Omega_7) \\ &\quad = \frac{\sqrt{\det[-i(\Omega_7+B_3+B_4+B_5)]}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}} e^{\pi i \vec{K}^T N^{-1}(B_3+B_4+B_5)\vec{K}} \psi_N^{\vec{K}}(z,\Omega_7) \\ &\quad \equiv \rho_{\vec{J}\vec{K}}(ST_3T_4T_5)\psi_N^{\vec{K}}(z,\Omega_7) \end{aligned} \tag{4.14}$$

Here, $\rho(ST_3T_4T_5) = \rho_{\vec{J}\vec{K}}(ST_3T_4T_5) = \rho$ denotes representation of $\mathbb{Z}_7 - twist$ of the wave functions. We note that the ρ has some $\mathbb{Z}_7$ eigenvalues. Also, we suppose a branch cut $\sqrt{t} > 0$ for $t > 0$. Then, we find

$$\sqrt{\det[-i(\Omega_7+B_3+B_4+B_5)]} = e^{-\pi i/4}. \tag{4.15}$$

[1]See the details in Appendix G.2

To classify the number of zero modes with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_7$, we focus on the eigenvalues of the $\rho(ST_3T_4T_5)$. Then, we take the trace of $\rho(ST_3T_4T_5)$, [2]

$$\mathrm{tr}\rho(ST_3T_4T_5) = \frac{e^{-\pi i/4}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{K}^T N^{-1}\left(\mathbf{1}_3+\frac{1}{2}(B_3+B_4+B_5)\right)\vec{K}}. \tag{4.16}$$

When we take $\det N = 8\ell, (\ell \in \mathbb{N})$, we find the following three values of $\mathrm{tr}\rho$

$$\mathrm{tr}\rho = -1, +1, -\sqrt{7}i, \tag{4.17}$$

where we perform numerical computations in the region $|n_{ii}| \leq 400$. On the other hand, we can represent the values of the $\mathrm{tr}\rho$

$$\mathrm{tr}\rho = \sum_{k=0}^{6} n_k \gamma^k, \quad \gamma = e^{2\pi i/7}, \tag{4.18}$$

where n_k denotes degenerated zero-mode numbers in $\mathbb{Z}_7$ sector with the eigenvalue γ^k. We also have $\sum_{k=0}^{6} n_k = \det N = 8\ell$.

First, let us discuss zero-mode number for tr $\rho = +1$.

One can obtain $[n_0, n_1, \cdots, n_5, n_6] = [m+1, m, m, m, m, m, m]$ for tr $\rho = +1$ because of $\sum_{k=0}^{6} \gamma^k = 0$. Then, we find

$$\det N = 8\ell = 1 + 7m = 7(m-1) + 8. \tag{4.19}$$

This implies that $m - 1$ should be a multiple of eight, that is,

$$m = 1, 9, \cdots . \tag{4.20}$$

Therefore we do not obtain three-generation model for $\mathrm{tr}\rho = +1$. Similarly, we do not find three-generation models for $\mathrm{tr}\rho = -1$.

Second, let us discuss zero-mode numbers for the $\mathrm{tr}\rho = -\sqrt{7}i$. Similarly to $\mathrm{tr}\rho = +1$, one can find $[n_0, n_1, \cdots, n_5, n_6] = [m+1, m, m, m+2, m, m+2, m+2]$ for $m \in \mathbb{Z}^+$ because of $\sum_{k=0}^{6} \gamma^k = 0$. For $m \in \mathbb{Z}^+$, Since $\det N$ is increased by $7m$, $\det N = 8\ell$ leads to the following equation

$$8\ell = 7 + 7m = 7(m+1). \tag{4.21}$$

[2]We can also calculate the trace $\mathrm{tr}\rho(ST_3T_4T_5)$ by using the g-dimensional Landsberg-Schaar relation in Appendix A.

Then $m+1$ should be a multiple of eight since 7 and 8 are coprime. Thus we obtain the possible m

$$m = 7, 15, \cdots . \tag{4.22}$$

This implies that we do not obtain three-generation model when $\mathrm{tr}\rho = -\sqrt{7}i$.

On the other hand, we can consider the D-term SUSY condition by tuning ratios of areas of T_i^2 on the $T_1^2 \times T_2^2 \times T_3^2$ [72, 95]. However, when we consider the D-term SUSY condition Eq. (3.119) in nonfactorizable T^6 with background magnetic fluxes, the D-term SUSY condition can be severe constraints. We will verify whether we can derive three-generation models with 4D $\mathcal{N}=1$ SUSY from magnetized $T^6/\mathbb{Z}_N$ or not. This means that we analyze models satisfying both the F-term and D-term SUSY conditions.

4D $\mathcal{N}=1$ SUSY in magnetized $T^6/\mathbb{Z}_7$

In the region $|n_{ii}| \leq 400$ and $|\det N| \leq 1.0 \times 10^{10}$, when we denote O as 3×3 zero matrix, there is no model satisfying the F-term and D-term SUSY conditions except for the trivial flux $N = O$. Therefore, we have no three-generation model with 4D $\mathcal{N}=1$ SUSY in magnetized $T^6/\mathbb{Z}_7$ model.

In conclusion, no three-generation model is derived from magnetized $T^6/\mathbb{Z}_7$ when we consider the wave functions with positive chirality $(+++)$. Similarly to the wave functions with positive chirality $(+++)$, we can consider the number of zero-mode wave functions with various chiralities in magnetized $T^6/\mathbb{Z}_7$. See the details in Appendix H. Lastly, so far, we have considered the wave functions with vanishing Scherk-Schwarz (SS) phases. In the future, we would like to analyze zero-mode wave functions with the non-vanishing SS phases in magnetized $T^6/\mathbb{Z}_7$, and verify whether three-generation model can be realized or not.

4.2.2 Magnetized $T^6/\mathbb{Z}_{12}$ orbifold

Next, we consider the number of zero modes with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_{12}$ orbifold [78]. We can construct the $\mathbb{Z}_{12}$ twist by the algebraic relation

$$(ST_1T_2T_3^{-1}T_5T_6)^{12} = \mathbf{1}_6, \tag{4.23}$$

where we denote $ST_1T_2T_3^{-1}T_5T_6$ as G. We can find the G-invariant complex structure moduli $\Omega = \Omega_{12}$

$$\Omega_{12} = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{6}i & \frac{\sqrt{3}}{3}i & -\frac{1}{2} + \frac{\sqrt{3}}{6}i \\ \frac{\sqrt{3}}{3}i & -\frac{1}{2} - \frac{\sqrt{3}}{6}i & -\frac{1}{2} + \frac{\sqrt{3}}{6}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{6}i & -\frac{1}{2} + \frac{\sqrt{3}}{6}i & \frac{1}{2} - \frac{\sqrt{3}}{6}i \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} e^{7\pi i/6} & e^{\pi i/2} & e^{5\pi i/6} \\ e^{\pi i/2} & e^{7\pi i/6} & e^{5\pi i/6} \\ e^{5\pi i/6} & e^{5\pi i/6} & e^{-\pi i/6} \end{bmatrix}. \tag{4.24}$$

This leads to the E_6 root lattice in Figure 4.2 [78]. We can restrict the shape of

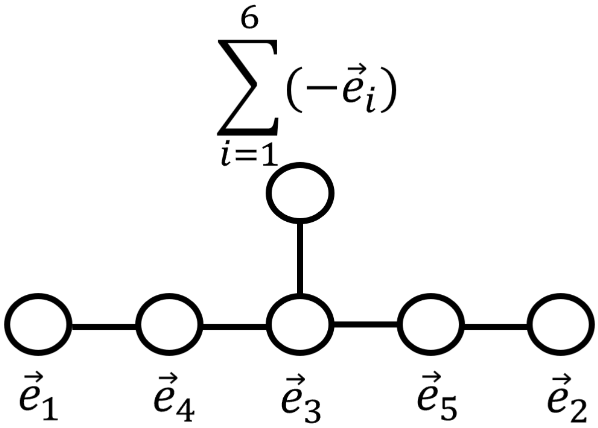

Figure 4.2: The lattice of $T^6/\mathbb{Z}_{12}$.

the flux N. From the F-term SUSY condition $(N\Omega_{12})^T = N\Omega_{12}$ and the algebraic relation, we find that the flux N is symmetric. Additionally, we consider the T-transformation, where $T = T_1T_2T_3^{-1}T_5T_6$. Then, $N(B_1 + B_2 - B_3 + B_5 + B_6)$ must be symmetric. Also, the diagonal elements of the $N(B_1 + B_2 - B_3 + B_5 + B_6)$ should be all even. Then, we find the flux N parametrized by n_{11}, n_{12}, and n_{13}

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{11} & n_{13} \\ n_{13} & n_{13} & n_{11} + n_{12} - 2n_{13} \end{bmatrix}, \tag{4.25}$$

where $n_{11} \equiv n_{12} \equiv n_{13} \pmod 2$. Here, we can regard G^2 as a $\mathbb{Z}_6$-twist. Similarly, we can see

$$G^{12} = (G^2)^6 = (G^3)^4 = (G^4)^3 = (G^6)^2 = \mathbf{1}_6. \tag{4.26}$$

Representation of modular transformation

We find $\mathbb{Z}_{12}$-twist generated by the G of wave functions with positive chirality $(+++)$

$$\psi_N^{\vec{J}}(\Omega_{12}z, \Omega_{12}) = \frac{\sqrt{\det[-i(\Omega_{12}+B)]}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{J}^T N^{-1}\vec{K}} e^{\pi i \vec{K}^T N^{-1} B\vec{K}} \psi_N^{\vec{K}}(z, \Omega_{12}), \tag{4.27}$$

where $B = B_1 + B_2 - B_3 + B_5 + B_6$. Then, we can describe the representation $\rho(G)$,

$$\rho_{\vec{K}_1\vec{K}_2}(G) = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{\det N}} e^{2\pi i \vec{K}_1^T N^{-1}\vec{K}_2} e^{i\pi(\vec{K}_1^T N^{-1} B\vec{K}_1)}. \tag{4.28}$$

Similarly, we obtain tr $\rho(G)$

$$\mathrm{tr}\rho(G) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i \vec{K}^T N^{-1}(\mathbf{1}_3 + B/2)\vec{K}}. \tag{4.29}$$

Then, we also find

$$tr\rho(G^n) = \frac{e^{-\frac{\pi i}{4}n}}{(\det N)^{n/2}} \sum_{\vec{K}_1,\vec{K}_2,\cdots\vec{K}_n\in\Lambda_N} e^{2\pi i(\vec{K}_1^T N^{-1}\vec{K}_2 + \vec{K}_2^T N^{-1}\vec{K}_3 + \cdots + \vec{K}_n^T N^{-1}\vec{K}_1)} \cdot e^{\pi i(\vec{K}_1^T N^{-1}B\vec{K}_1 + \cdots + \vec{K}_n^T N^{-1}B\vec{K}_n)}. \tag{4.30}$$

Wave functions with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_{12}$

We analyze the number of zero modes with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_{12}$. In what follows, the number of zero modes in $\mathbb{Z}_{12}$ sector with the charge $k \in \{0, 1, \cdots, 11\}$ can be written as n_k. That is, we obtain

$$\det N = \sum_{k=0}^{11} n_k. \tag{4.31}$$

Next, we find

$$\mathrm{tr}\rho(G) = \sum_{k=0}^{11} n_k e^{\frac{k\pi}{6}i}. \tag{4.32}$$

Similarly, we find the following relations between the trace and the n_k

$$\begin{aligned} \mathrm{tr}\rho(G^2) =& \sum_{k\equiv 0(mod6)} n_k + e^{\frac{\pi}{3}i} \sum_{k\equiv 1(mod6)} n_k + e^{\frac{2\pi}{3}i} \sum_{k\equiv 2(mod6)} n_k \\ &- \sum_{k\equiv 3(mod6)} n_k + e^{\frac{4\pi}{3}i} \sum_{k\equiv 4(mod6)} n_k + e^{\frac{5\pi}{3}i} \sum_{k\equiv 5(mod6)} n_k, \end{aligned} \tag{4.33}$$

$$\mathrm{tr}\rho(G^3) = \sum_{k\equiv 0(mod4)} n_k + i \sum_{k\equiv 1(mod4)} n_k - \sum_{k\equiv 2(mod4)} n_k - i \sum_{k\equiv 3(mod4)} n_k, \tag{4.34}$$

$$\mathrm{tr}\rho(G^4) = \sum_{k\equiv 0(mod3)} n_k + e^{\frac{2\pi}{3}i} \sum_{k\equiv 1(mod3)} n_k + e^{\frac{4\pi}{3}i} \sum_{k\equiv 2(mod3)} n_k, \tag{4.35}$$

$$\mathrm{tr}\rho(G^6) = \sum_{k\equiv 0(mod2)} n_k - \sum_{k\equiv 1(mod2)} n_k. \tag{4.36}$$

By use of the relations, we can classify the number of wave functions in magnetized $T^6/\mathbb{Z}_{12}$. When we perform numerical computations in the region $|n_{1i}| \leq 400$, we can find the zero modes satisfying the F-term SUSY condition. Their degenerated numbers are summarized in Tables 4.2 and 4.3 [78].

Table 4.2: The number of zero modes in magnetized $T^6/\mathbb{Z}_{12}$.

det$N$	4	8	12	16	20	24	24	28	32	32	36	36	40	$\cdots$
tr$\rho(G)$	-1	1	$-\sqrt{3}i$	-1	1	$-\sqrt{3}i$	-1	-1	1	1	$-\sqrt{3}i$	-1	-1	
tr$\rho(G^2)$	1	-1	$-\sqrt{3}i$	1	-1	$-\sqrt{3}i$	1	1	-1	-1	$-\sqrt{3}i$	1	1	
tr$\rho(G^3)$	2	4	6	8	10	12	2	14	16	4	18	2	20	
tr$\rho(G^4)$	1	-1	$\sqrt{3}i$	1	-1	$\sqrt{3}i$	-3	1	-1	-1	$\sqrt{3}i$	-3	1	
tr$\rho(G^6)$	4	8	12	16	20	24	4	28	32	8	36	4	40	
n_0	1	2	$\boxed{3}$	4	5	6	2	7	8	4	9	$\boxed{3}$	10	
n_1	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	
n_2	0	1	0	1	2	1	2	2	$\boxed{3}$	$\boxed{3}$	2	$\boxed{3}$	$\boxed{3}$	
n_3	0	0	0	0	0	0	1	0	0	2	0	2	0	
n_4	1	2	$\boxed{3}$	4	5	6	$\boxed{3}$	7	8	4	9	4	10	
n_5	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	
n_6	1	0	1	2	1	2	2	$\boxed{3}$	2	2	$\boxed{3}$	$\boxed{3}$	4	
n_7	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	
n_8	1	2	$\boxed{3}$	4	5	6	$\boxed{3}$	7	8	4	9	4	10	
n_9	0	0	0	0	0	0	1	0	0	2	0	2	0	
n_{10}	0	1	2	1	2	$\boxed{3}$	2	2	$\boxed{3}$	$\boxed{3}$	4	$\boxed{3}$	$\boxed{3}$	
n_{11}	0	0	0	0	0	0	2	0	0	2	0	$\boxed{3}$	0	

4D $\mathcal{N}=1$ SUSY in magnetized $T^6/\mathbb{Z}_{12}$

In magnetized $T^6/\mathbb{Z}_{12}$, we can take the consistent flux N

$$N = \begin{bmatrix} -5 & 13 & 3 \\ 13 & -5 & 3 \\ 3 & 3 & 2 \end{bmatrix}, \tag{4.37}$$

where the flux N satisfies both the F-term SUSY and the D-term SUSY conditions. We can see $\det N = 36$, $\mathrm{tr}\rho(G) = \mathrm{tr}\rho(G^2) = -\sqrt{3}i$, $\mathrm{tr}\rho(G^3) = 18$, $\mathrm{tr}\rho(G^4) = \sqrt{3}i$, $\mathrm{tr}\rho(G^6) = 36$. Then, we find the numbers of zero modes, that is, $[n_0, n_1, \cdots, n_{11}] = [9, 0, 2, 0, 9, 0, \boxed{3}, 0, 9, 0, 4, 0]$.

Also, we can take the other consistent flux N

$$N = \begin{bmatrix} -3 & 3 & 1 \\ 3 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{4.38}$$

Table 4.3: Continuation of Table 4.2.

$\det N$	44	48	52	52	56	60	64	64	68	72	72	72	$\cdots$
$\mathrm{tr}\rho(G)$	1	$-\sqrt{3}i$	-1	-1	1	$-\sqrt{3}i$	-1	-1	1	$-\sqrt{3}i$	1	$-\sqrt{3}i$	
$\mathrm{tr}\rho(G^2)$	-1	$-\sqrt{3}i$	1	1	-1	$-\sqrt{3}i$	1	1	-1	$-\sqrt{3}i$	-1	$-\sqrt{3}i$	
$\mathrm{tr}\rho(G^3)$	22	24	26	2	28	30	32	8	34	36	4	6	
$\mathrm{tr}\rho(G^4)$	-1	$\sqrt{3}i$	1	1	-1	$\sqrt{3}i$	1	1	-1	$\sqrt{3}i$	3	$-3\sqrt{3}i$	
$\mathrm{tr}\rho(G^6)$	44	48	52	4	56	60	64	16	68	72	8	12	
n_0	11	12	13	5	14	15	16	8	17	18	8	8	
n_1	0	0	0	4	0	0	0	4	0	0	5	4	
n_2	4	$\boxed{3}$	4	4	5	4	5	5	6	5	6	6	
n_3	0	0	0	4	0	0	0	4	0	0	6	5	
n_4	11	12	13	5	14	15	16	8	17	18	7	7	
n_5	0	0	0	4	0	0	0	4	0	0	5	6	
n_6	$\boxed{3}$	4	5	5	4	5	6	6	5	6	6	6	
n_7	0	0	0	4	0	0	0	4	0	0	5	4	
n_8	11	12	13	5	14	15	16	8	17	18	7	9	
n_9	0	0	0	4	0	0	0	4	0	0	6	5	
n_{10}	4	5	4	4	5	6	5	5	6	7	6	6	
n_{11}	0	0	0	4	0	0	0	4	0	0	5	6	

where the flux N satisfies both the F-term SUSY and the D-term SUSY conditions. We can see $\det N = 12$, $\mathrm{tr}\rho(G) = \mathrm{tr}\rho(G^2) = -\sqrt{3}i$, $\mathrm{tr}\rho(G^3) = 6$, $\mathrm{tr}\rho(G^4) = \sqrt{3}i$, $\mathrm{tr}\rho(G^6) = 12$. Then, we find the numbers of zero modes, that is, $[n_0, n_1, \cdots, n_{11}] = [\boxed{3}, 0, 0, 0, \boxed{3}, 0, 1, 0, \boxed{3}, 0, 2, 0]$. Thus, in magnetized $T^6/\mathbb{Z}_{12}$, when we consider the wave functions with positive chirality $(+++)$, we can obtain three-generation models satisfying both the F-term SUSY and the D-term SUSY conditions, and realize 4D $\mathcal{N} = 1$ SUSY.

Diagonalization in magnetized $T^6/\mathbb{Z}_{12}$

We consider the wave functions with various chiralities and their zero-mode numbers in magnetized $T^6/\mathbb{Z}_{12}$ [81, 82]. Similar analysis can be carried out in magnetized $T^6/\mathbb{Z}_7$. See the details in Appendix H. When we diagonalize $F_{i\bar{j}} =$

$\pi[N^T(\mathrm{Im}\Omega_{12})^{-1}]_{ij}$, we can use the diagonalization matrix $P \in SO(3)$

$$\begin{aligned}
P &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{3-\sqrt{3}}} & \frac{1+\sqrt{3}}{2\sqrt{3+\sqrt{3}}} \\ \frac{1}{\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{3-\sqrt{3}}} & \frac{1+\sqrt{3}}{2\sqrt{3+\sqrt{3}}} \\ 0 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \end{bmatrix} \qquad (4.39) \\
&= \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & \sin\theta_3 & \cos\theta_3 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1\cos\theta_2 & \sin\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_2\cos\theta_3 \\ \sin\theta_1 & \cos\theta_1\cos\theta_3 & -\cos\theta_1\sin\theta_3 \\ -\cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_2\sin\theta_3 & \cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3 \end{bmatrix},
\end{aligned}$$

where the P can be independent of the values of the magnetic fluxes. Here, we can diagonalize Ω_{12} as

$$P^{-1}\Omega_{12}P = \mathrm{diag}\left(e^{-2\pi i/3}, e^{-\pi i/6}, e^{5\pi i/6}\right). \qquad (4.40)$$

The P in Eq. (4.39) leads to $\theta_1 = \frac{3}{4}\pi$, $\theta_2 = 0$, that is, $q_1 = 1$ and $q_2 = 0$. This implies that we can consider the wave functions $\psi_{+--} = \psi_{-+-}$ and $\psi_{--+} = 0$ in magnetized $T^6/\mathbb{Z}_{12}$. By use of the P, we can find the number of zero modes with various chiralities. That is, we introduce the z' system. When we describe the spinor in z' system as Ψ', only ψ'_{+++} is excited and written in the Ψ'.

In z' system, we can describe the $\mathbb{Z}_N$-twist boundary condition of spinor $\Psi'(z')$

$$\Psi'(\Omega' z') = \mathcal{S}' V' \Psi'(z'). \qquad (4.41)$$

where we denote $\mathcal{S}'$, V' as the spinor representation and transformation function in z' system, respectively. In z' system, under the transformation $z \to \Omega z$, z' can be transformed as $\Omega' z'$, that is, we can impose the identification $z' \sim \Omega' z'$ such that $\Omega'^N = \mathbf{1}_3$ under $z \sim \Omega z$ such that $\Omega^N = \mathbf{1}_3$. On the other hands, ψ'^J_{+++}, $(J \in \Lambda_N)$ is written

$$\begin{aligned}
&\psi'^J_{+++}(\Omega'_{\mathrm{twist}} z', \Omega'_{\mathrm{twist}}) \qquad (4.42) \\
&= \frac{\sqrt{\det\left[-i(\Omega'_{\mathrm{twist}} + B)\right]}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'^K_{+++}(z', \Omega'_{\mathrm{twist}})
\end{aligned}$$

under $ST_1T_2T_3^{-1}T_5T_6$-transformations. Here, we describe $B = B_1 + B_2 - B_3 + B_5 + B_6$ and $\Omega'_{\mathrm{twist}} = \mathrm{Re}\Omega_{12} + iR\mathrm{Im}\Omega_{12}$.

We note that ψ'_{+++} can be invariant under the $\mathcal{S}'$ if we define the following V'

$$V' \equiv e^{2\pi i\beta'}, \quad \text{where} \quad \beta' N \equiv 0 \mod 1 \quad \text{and } \beta' \neq \beta \text{ in general.} \tag{4.43}$$

Here, we can see that this is consistent with the Dirac equation under the $\mathbb{Z}_N$-twist in z' system. Then, by using of the ψ'_{+++} and the $\mathbb{Z}_N$-twist boundary condition in z' system, we can analyze the number of zero modes by taking $U_1 = \mathrm{diag}(+,-,-), U_2 == \mathrm{diag}(-,+,-), U_3 = \mathrm{diag}(-,-,+)$.

The zero-mode number in magnetized $T^6/\mathbb{Z}_{12}$

We analyze the zero-mode numbers considering both the F-term and the D-term SUSY conditions in magnetized $T^6/\mathbb{Z}_{12}$. In regions $|n_{1i}| < 200$, $(i = 1, 2, 3)$, and $1 < \det N < 500$, we can find such three-generation models by using the modular transformations, and the U_i, $(i = 1, 2, 3)$. [3] For $U_1 = \mathrm{diag}(+,-,-)$, we can obtain the modular transformations

$$\begin{aligned} &\psi'^{J}_{+++}(\Omega'_{\mathrm{twist}} z', \Omega'_{\mathrm{twist}}) \\ &= \frac{e^{\frac{5}{12}\pi i}}{\sqrt{|\det N|}} \sum_{K\in\Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'^{K}_{+++}(z', \Omega'_{\mathrm{twist}}). \end{aligned} \tag{4.44}$$

Then, the consistent fluxes N and the M can be written as

$$N = \begin{bmatrix} -13 & 5 & -3 \\ 5 & -13 & -3 \\ -3 & -3 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} -5 & 13 & 3 \\ 13 & -5 & 3 \\ 3 & 3 & 2 \end{bmatrix}, \tag{4.45}$$

where $\det N = 36$. Then, we find zero-mode numbers in $\mathbb{Z}_N$ sectors n_i, $(i = 0, 1, \cdots, 11)$, that is,

$$[n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] = [9, 0, 4, 0, 9, 0, 2, 0, 9, 0, \boxed{3}, 0]. \tag{4.46}$$

The others are

$$N = \begin{bmatrix} -3 & 3 & -1 \\ 3 & -3 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} -3 & 3 & 1 \\ 3 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \tag{4.47}$$

---

[3] Similarly to zero-mode counting on magnetized $T^6/\mathbb{Z}_{12}$ using the U_i, $(i = 1, 2, 3)$, we can perform zero-mode counting on magnetized $T^6/\mathbb{Z}_7$ model. See the details in Appendix H.

where $\det N = 12$. We find zero-mode numbers

$$[n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] = [\boxed{3}, 0, 2, 0, \boxed{3}, 0, 0, 0, \boxed{3}, 0, 1, 0]. \tag{4.48}$$

For $U_2 = \mathrm{diag}(-, +, -)$, we can obtain the modular transformations

$$\begin{aligned} &\psi'^{J}_{+++}(\Omega'_{\mathrm{twist}} z', \Omega'_{\mathrm{twist}}) \\ &= \frac{e^{-\frac{\pi}{12} i}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'^{K}_{+++}(z', \Omega'_{\mathrm{twist}}). \end{aligned} \tag{4.49}$$

Then, the consistent fluxes N and the M can be written as

$$N = \begin{bmatrix} 1 & -5 & -1 \\ -5 & 1 & -1 \\ -1 & -1 & -2 \end{bmatrix}, \quad M = \begin{bmatrix} \sqrt{3}-3 & \sqrt{3}+3 & \sqrt{3} \\ \sqrt{3}+3 & \sqrt{3}-3 & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & 0 \end{bmatrix}, \tag{4.50}$$

where $\det N = 36$. Then, the zero-mode numbers are written as

$$[n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] = [5, 1, \boxed{3}, \boxed{3}, 4, 0, 5, \boxed{3}, \boxed{3}, 2, 4, \boxed{3}]. \tag{4.51}$$

Lastly, for $U_3 = \mathrm{diag}(-, -, +)$, we obtain the modular transformations

$$\begin{aligned} &\psi'^{J}_{+++}(\Omega'_{\mathrm{twist}} z', \Omega'_{\mathrm{twist}}) \\ &= \frac{e^{-\frac{\pi}{12} i}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B K} \cdot \psi'^{K}_{+++}(z', \Omega'_{\mathrm{twist}}). \end{aligned} \tag{4.52}$$

Then, the consistent fluxes N and the M can be written as

$$N = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} \sqrt{3}-3 & \sqrt{3}+3 & \sqrt{3} \\ \sqrt{3}+3 & \sqrt{3}-3 & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & 0 \end{bmatrix}, \tag{4.53}$$

where $\det N = 36$. and each zero-mode number is

$$[n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9, n_{10}, n_{11}] = [5, \boxed{3}, \boxed{3}, 2, 4, \boxed{3}, 5, 1, \boxed{3}, \boxed{3}, 4, 0]. \tag{4.54}$$

In conclusion, some three-generation models can be realized in magnetized $T^6/\mathbb{Z}_{12}$ orbifolds. Then, we have used the wave functions with positive chirality $(+++)$ or the wave functions with various chiralities.

Next, we would like to consider how to derive 4D Yukawa couplings from nonfactorizable T^6 with background magnetic fluxes. When we consider realistic models taking mass of matter particles into account, 4D Yukawa couplings can be important.

In chapter 5, we will consider the overlap integral of the wave functions in left-handed, right-handed fermions, and Higgs sectors. This leads to 4D Yukawa couplings.

Chapter 5

Phenomenological application

We have studied magnetized T^6, and its orbifolds $T^6/\mathbb{Z}_N$. We have found that three-generation models can be realized when we suppose magnetized T^6 or $T^6/\mathbb{Z}_{12}$. On the other hands, since the wave functions of magnetized $T^6/\mathbb{Z}_N$ can be expressed as linear combinations of the wave functions in magnetized T^6, we need to calculate explicitly overlap integral of three-point couplings leading to Yukawa couplings when we derive Yukawa couplings in 4D chiral theory with three-generation structure from magnetized $T^6/\mathbb{Z}_N$. We note that we need to take chirality into account in order to derive 4D Yukawa couplings properly. In this chapter, We calculate overlap integral of the zero-mode wave functions with various chiralities in left-handed, right-handed, and Higgs sectors [81,82]. Furthermore, we consider 4D effective field theory with $Sp(6,\mathbb{Z})$ modular symmetry and its superpotential. We will analyze the overlap integral behaviors under the $Sp(6,\mathbb{Z})$ modular symmetry. Lastly, by considering not only magnetized T^6, but magnetized T^4 derived from magnetized T^6, we can introduce non-Abelian discrete symmetries derived from magnetized T^4 or T^6 models.

5.1 Yukawa couplings in magnetized T^6 model

5.1.1 Calculation for overlap integral leading to Yukawa couplings

We would like to derive 4D Yukawa couplings when we consider realistic models. Then, we consider Yukawa couplings. Our discussion of the calculation for the overlap integral leading to Yukawa couplings is along in Refs. [55, 81, 82]. We need to take chirality into account because we would like to derive 4D Yukawa couplings in 4D $\mathcal{N}=1$ SUSY. Then, we should calculate the overlap integral of the wave functions with various chiralities, which leads to Yukawa couplings. Yukawa couplings can be described by three-point couplings of the spinor wave functions in left-handed, right-handed, and Higgs sectors. Then, let us consider chirality in each sector. For the Higgs sector, we denote the chirality of Higgsino fields. In what follows, background magnetic fluxes for left-handed and right-handed fermion sectors can be written by N_L and N_R, respectively. They may have different magnetic fluxes and chiralities. Here, we note that we need to consider the magnetic flux N_H in the Higgs sector satisfying $N_H = N_L + N_R$ because of the gauge invariance.

We also describe M_L, M_R, M_H, where $M_i = N_i R_i$, $(i = L, R, H)$. According to Ref. [114], when we describe the flux N under $[N, \Omega] = 0$, the $[N, \Omega] = 0$ leads to linear combinations of the unit matrix $\mathbf{1}_3$, the moduli Ω, and the inverse Ω^{-1}, that is,

$$N = k_1 \mathbf{1}_3 + k_2 \Omega + k_3 \Omega^{-1}, \tag{5.1}$$

where k_i, $(i = 1, 2, 3)$, denotes complex coefficients. Then, N can be the integer fluxes by choosing the k_i properly. Then, we can represent the N_L, N_R

$$\begin{aligned} N_L &= k_{1L} \mathbf{1}_3 + k_{2L} \Omega + k_{3L} \Omega^{-1}, \\ N_R &= k_{1R} \mathbf{1}_3 + k_{2R} \Omega + k_{3R} \Omega^{-1}. \end{aligned} \tag{5.2}$$

This leads to $[N_L, N_R] = 0$ under $[N, \Omega] = 0$. Then, different chiralities in each sector can be constructed. Similarly to $[N_L, N_R] = 0$, $[N, \Omega] = [R, \Omega] = 0$ leads to $[M_L, M_R] = 0$ and $[N, M] = 0$, respectively.

In what follows, we consider the wave functions in left-handed, right-handed, and Higgs sectors, and the overlap integral. The overlap integral can represent three-point couplings, which leads to Yukawa couplings.

Here, in magnetized T^6 with any complex structure moduli Ω, we calculate the overlap integral among other combinations of chiralities to obtain the 4D Yukawa couplings in 4D $\mathcal{N}=1$ SUSY. We denote wave functions of the left-handed matter field, the right-handed matter one, and the Higgs field by ψ_L^I, $(I \in \Lambda_{N_L})$, ψ_R^J, $(J \in \Lambda_{N_R})$, and ψ_H^K, $(K \in \Lambda_{N_H})$. First, we introduce the Yukawa couplings $Y^{I,J,K}$ as follows,

$$Y^{I,J,K} \equiv \sigma_{abc} g' \int_{T^6} d^3z d^3\bar{z} \; \psi_L^I \cdot \psi_R^J \cdot (\psi_H^K)^*, \tag{5.3}$$

where we denote g' and σ_{abc} as the gauge coupling constant in higher dimensional theory and a sign derived from the product of differences of fluxes between D-branes, respectively [54]. Under Eq. (5.3), we particularly consider the overlap integral of the wave functions with various chiralities. That is, we calculate the overlap integral taking chirality into account

$$\begin{aligned}
C \equiv & \int_{T^6} d^3z d^3\bar{z}\ \psi^I_{N_L,M_L}(z_L,\Omega_L)\cdot\psi^J_{N_R,M_R}(z_R,\Omega_R)\cdot(\psi^K_{N_H,M_H}(z_H,\Omega_H))^* \\
= & |-2i\mathrm{Im}\Omega| \qquad\qquad (5.4)\\
& \cdot\int_0^1 d^3x\int_0^1 d^3y\ \psi^I_{N_L,M_L}(z_L,\Omega_L)\cdot\psi^J_{N_R,M_R}(z_R,\Omega_R)\cdot(\psi^K_{N_L+N_R,M_H}(z_H,\Omega_H))^*.
\end{aligned}$$

Here, we use the following relation derived from the Jacobian

$$d^3zd^3\bar{z} = |-2i\mathrm{Im}\Omega|d^3xd^3y, \tag{5.5}$$

and the identifications, i.e., $x\sim x+e$ and $y\sim y+e$. Here, we define f_{N_ℓ} and $\Theta^J_{N_\ell}(z'_\ell,\Omega'_\ell)$ derived from Eq. (3.38) as follows,

$$f_{N_\ell}\equiv e^{i\pi[N_\ell z'_\ell]^T(\mathrm{Im}\Omega'_\ell)^{-1}\mathrm{Im}z'_\ell},\quad \Theta^J_{N_\ell}(z'_\ell,\Omega'_\ell)\equiv\theta\begin{bmatrix}J^TN_\ell^{-1}\\0\end{bmatrix}(N_\ell z'_\ell,N_\ell\Omega'_\ell), \tag{5.6}$$

where $J\in\Lambda_{N_\ell}$, $z'_\ell=\mathrm{Re}z+iR_\ell\mathrm{Im}z$, $\Omega'_\ell=\mathrm{Re}\Omega+iR_\ell\mathrm{Im}\Omega$, and $R_\ell=N_\ell^{-1}M_\ell$, $(\ell=L,R,H)$. In the Higgs sector, on the other hands, we can find

$$\left(\Theta^K_{N_H}(z'_H,\Omega'_H)\right)^* = \Theta^{-K}_{-N_H}(\bar{z}'_H,\bar{\Omega}'_H). \tag{5.7}$$

Here, we can see that the wave functions are well-defined because of $-M_H\mathrm{Im}\bar{\Omega}=M_H\mathrm{Im}\Omega>0$. Thus, we find

$$C\propto\int_{T^6}d^3zd^3\bar{z}\ \left[f_{N_L}\cdot\Theta^I_{N_L}(z'_L,\Omega'_L)\right]\cdot\left[f_{N_R}\cdot\Theta^J_{N_R}(z'_R,\Omega'_R)\right]\cdot\left[f^*_{N_H}\cdot\Theta^{-K}_{-N_H}(\bar{z}'_H,\bar{\Omega}'_H)\right]. \tag{5.8}$$

Since we would like to calculate Eq. (5.4), we first calculate the product of the wave functions, that is,

$$\psi^I_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^J_{N_R,M_R}(z'_R,\Omega'_R), \tag{5.9}$$

where $I=N_L\vec{i}\in\Lambda_{N_L}$ and $J=N_R\vec{j}\in\Lambda_{N_R}$. By noting that $z=\mathrm{Re}z+i\mathrm{Im}z=x+(\mathrm{Re}\Omega)y+i(\mathrm{Im}\Omega)y$, we can evaluate the product up to the normalization constant as follows:

$$
\begin{aligned}
&\psi^{I}_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^{J}_{N_R,M_R}(z'_R,\Omega'_R)\\
&\propto e^{i\pi[N_L z'_L]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z}\cdot e^{i\pi[N_R z'_R]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z}\cdot\Theta^{I}_{N_L}(z'_L,\Omega'_L)\cdot\Theta^{J}_{N_R}(z'_R,\Omega'_R)\\
&= e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\\
&\cdot\sum_{\vec{\ell}_1,\vec{\ell}_2\in\mathbb{Z}^g} e^{i\pi(i)[\vec{L}^T\hat{Q}_{N,M}\vec{L}]}\cdot e^{2\pi i[\vec{L}^T Q_N\vec{X}]}\cdot e^{2\pi i(i)[\vec{L}^T\hat{Q}_{N,M}\vec{Y}]},
\end{aligned}
\tag{5.10}
$$

where we have defined the $2g$-D real vectors:

$$
\vec{L}=\begin{bmatrix}\vec{i}+\vec{\ell}_1\\ \vec{j}+\vec{\ell}_2\end{bmatrix},\vec{X}=\begin{bmatrix}\vec{x}\\ \vec{x}\end{bmatrix},\vec{Y}=\begin{bmatrix}\vec{y}\\ \vec{y}\end{bmatrix},
\tag{5.11}
$$

and the $2g\times 2g$ matrices:

$$
\begin{aligned}
Q_N&=\begin{bmatrix}N_L & O\\ O & N_R\end{bmatrix},\\
\hat{Q}_{N,M}&=\begin{bmatrix}M_L\mathrm{Im}\Omega-iN_L\mathrm{Re}\Omega & O\\ O & M_R\mathrm{Im}\Omega-iN_R\mathrm{Re}\Omega\end{bmatrix}\equiv\begin{bmatrix}M'_L & O\\ O & M'_R\end{bmatrix}.
\end{aligned}
\tag{5.12}
$$

Here, we focus on the summation for indices $\vec{i}N_L,\vec{j}N_R,\vec{k}(N_L+N_R)$. $M'=M_L\mathrm{Im}\Omega-iN_L\mathrm{Re}\Omega$ is irrelavant to the index summation. According to Ref. [55], we can use the following transformation matrix T' under $[N_L,N_R]=[M_L,M_R]=0$

$$
\begin{aligned}
&T'\equiv\begin{bmatrix}1_g & 1_g\\ \alpha N_L^{-1} & -\alpha N_R^{-1}\end{bmatrix},(T')^T=\begin{bmatrix}1_g & N_L^{-1}\alpha^T\\ 1_g & -N_R^{-1}\alpha^T\end{bmatrix},\\
&(T')^{-1}=(N_L^{-1}+N_R^{-1})^{-1}\begin{bmatrix}N_R^{-1} & \alpha^{-1}\\ N_L^{-1} & -\alpha^{-1}\end{bmatrix},\\
&((T')^{-1})^T=(N_L^{-1}+N_R^{-1})^{-1}\begin{bmatrix}N_R^{-1} & N_L^{-1}\\ (\alpha^{-1})^T & -(\alpha^{-1})^T\end{bmatrix},
\end{aligned}
\tag{5.13}
$$

with $g\times g$ matrix α, and $g\times g$ unit matrix 1_g. In what follows, we consider how to calculate the series in the product $\psi^{I}_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^{J}_{N_R,M_R}(z'_R,\Omega'_R)$. First, we can take $\alpha=|N_L||N_R|1_g$ when $|N_L|$ and $|N_R|$ are coprime. This is because $\alpha(N_L^{-1}+N_R^{-1})$ takes integer values when both N_L and N_R are integer matrices. We need this condition for the consistency with the series. Second, we can take α as the least common multiple (L.C.M.) of $|N_L|$ and $|N_R|$ when they are not coprime. In general, we can find the α

$$
\alpha=\begin{cases}|N_L||N_R|1_g & \text{for } |N_L|,\ |N_R| \text{ coprime},\\ \mathrm{L.C.M.}(|N_L|,|N_R|)1_g & \text{for the others.}\end{cases}
\tag{5.14}
$$

We can then calculate the overlap integral. See the details of the calculation for the overlap integral leading to Yukawa couplings in Appendix I. For $\alpha = |N_L||N_R|1_g$ and $g = 3$, we can obtain the overlap integral in magnetized T^6

$$\begin{aligned}
C =& |-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i\mathcal{N}_j\mathcal{N}_k \cdot \\
&\int_0^1 d^3y [e^{-\pi[y^T[M_L+M_R+M_H](\mathrm{Im}\Omega)y]} \cdot \\
&\sum_{\vec{\ell}_3,\vec{\ell}_4\in\mathbb{Z}^3}\sum_{\vec{p},\vec{\tilde{p}}} e^{\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T\cdot\hat{\mathbf{Q}}'^{|N_L||N_R|}_{N,M'}\cdot[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]} \cdot e^{2\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T\vec{\mathbf{Y}}'^{|N_L||N_R|}_{N,M'}}] \\
=& |-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i\mathcal{N}_j\mathcal{N}_k \cdot \\
&\sum_{\vec{p},\vec{\tilde{p}}}\int_0^1 d^3y \left[e^{-\pi[y^T[M_L+M_R+M_H](\mathrm{Im}\Omega)y]} \cdot \theta \begin{bmatrix} \vec{\mathbf{K}} \\ 0 \end{bmatrix} (i\vec{\mathbf{Y}}'^{|N_L||N_R|}_{N,M'}, i\hat{\mathbf{Q}}'^{|N_L||N_R|}_{N,M'}) \right],
\end{aligned} \tag{5.15}$$

where

$$\vec{\mathbf{L_1}} = \begin{bmatrix} \vec{\ell}_3 \\ \vec{\ell}_4 \end{bmatrix}, \quad \vec{\mathbf{K}} = \begin{bmatrix} \vec{k} \\ \frac{N_R(N_L+N_R)^{-1}N_L}{|N_L||N_R|}(\vec{i} - \vec{j} + \vec{\tilde{m}}') \end{bmatrix}, \tag{5.16}$$

$$\begin{aligned}
\vec{\mathbf{Y}}'^{|N_L||N_R|}_{N,M'} &\equiv \begin{bmatrix} [M'_L + M'_R + (M'_H)^*]y \\ [(|N_L||N_R|)(M'_L N_L^{-1} - M'_R N_R^{-1})]y \end{bmatrix} \\
&= \begin{bmatrix} [(M_L + M_R + M_H)\mathrm{Im}\Omega]y \\ |N_L||N_R|[(R_L - R_R)\mathrm{Im}\Omega]y \end{bmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\hat{\mathbf{Q}}'^{|N_L||N_R|}_{N,M'} \\
&\equiv \begin{bmatrix} M'_L + M'_R + (M'_H)^* & |N_L||N_R|(M'_L N_L^{-1} - M'_R N_R^{-1}) \\ |N_L||N_R|(N_L^{-1}M'_L - N_R^{-1}M'_R) & (|N_L||N_R|)^2(N_L^{-1}M'_L N_L^{-1} + N_R^{-1}M'_R N_R^{-1}) \end{bmatrix}.
\end{aligned} \tag{5.17}$$

Here, we denote $\mathrm{Re}\hat{\mathbf{Q}}'^{|N_L||N_R|}_{N,M'} > 0$, $\alpha = |N_L||N_R|1_g$, $M'_L = M_L\mathrm{Im}\Omega - iN_L\mathrm{Re}\Omega$, $M'_R = M_R\mathrm{Im}\Omega - iN_R\mathrm{Re}\Omega$, and $(M'_H)^* = M_H\mathrm{Im}\Omega + i(N_L + N_R)\mathrm{Re}\Omega$. When we apply $M_i = N_iR_i$, $(i = L, R, H)$, to numerical computations, we can obtain the specific values of the overlap integral.

Furthermore, $\vec{\tilde{m}}$ is the solution for $(N_L + N_R)\vec{k} = N_L\vec{i} + N_R\vec{j} + N_L\vec{\tilde{m}}$. Also, we take $\vec{p} \in \Lambda_{|N_R|N_R^{-1}}, \vec{\tilde{p}} \in \Lambda_{|N_L|N_L^{-1}}$. Then, we can replace $\vec{m}$ satisfying $\vec{m}N_L(N_L + N_R)^{-1} \in \mathbb{Z}$ with $\vec{\tilde{m}}' = \vec{\tilde{m}} + \vec{p}|N_L|(N_L + N_R)N_L^{-1} + \vec{\tilde{p}}|N_R|(N_L + N_R)N_R^{-1}$.

For $\alpha = \text{L.C.M.}(|N_L|, |N_R|)1_g \equiv L_R 1_g$ and $g = 3$, we can also calculate the overlap integral

$$
\begin{aligned}
C = & |-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \\
& \cdot \int_0^1 d^3y [e^{-\pi[y^T[M_L+M_R+M_H](\mathrm{Im}\Omega)y]} \cdot \\
& \sum_{\vec{\ell}_3, \vec{\ell}_4 \in \mathbb{Z}^3} \sum_{\vec{p}, \vec{\tilde{p}}} e^{\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T \cdot \hat{\mathbf{Q}}'^{L_R}_{N,M'} \cdot [\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]} \cdot e^{2\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T \vec{\mathbf{Y}}'^{L_R}_{N,M'}}] \\
= & |-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \\
& \cdot \sum_{\vec{p}, \vec{\tilde{p}}} \int_0^1 d^3y \left[e^{-\pi[y^T[M_L+M_R+M_H](\mathrm{Im}\Omega)y]} \cdot \theta \begin{bmatrix} \vec{\mathbf{K}} \\ 0 \end{bmatrix} (i\vec{\mathbf{Y}}'^{L_R}_{N,M'}, i\hat{\mathbf{Q}}'^{L_R}_{N,M'}) \right],
\end{aligned} \tag{5.18}
$$

where

$$
\vec{\mathbf{L_1}} = \begin{bmatrix} \vec{\ell}_3 \\ \vec{\ell}_4 \end{bmatrix}, \quad \vec{\mathbf{K}} = \begin{bmatrix} \vec{k} \\ \frac{N_R(N_L+N_R)^{-1}N_L}{L_R}(\vec{i} - \vec{j} + \vec{\tilde{m}}') \end{bmatrix}, \tag{5.19}
$$

$$
\vec{\mathbf{Y}}'^{L_R}_{N,M'} \equiv \begin{bmatrix} [M'_L + M'_R + (M'_H)^*]y \\ [L_R(M'_L N_L^{-1} - M'_R N_R^{-1})]y \end{bmatrix} = \begin{bmatrix} [(M_L + M_R + M_H)\mathrm{Im}\Omega]y \\ L_R[(R_L - R_R)\mathrm{Im}\Omega]y \end{bmatrix}, \tag{5.20}
$$

and

$$
\hat{\mathbf{Q}}'^{L_R}_{N,M'} \equiv \begin{bmatrix} M'_L + M'_R + (M'_H)^* & L_R(M'_L N_L^{-1} - M'_R N_R^{-1}) \\ L_R(N_L^{-1}M'_L - N_R^{-1}M'_R) & (L_R)^2(N_L^{-1}M'_L N_L^{-1} + N_R^{-1}M'_R N_R^{-1}) \end{bmatrix}. \tag{5.21}
$$

Here, we denote $\mathrm{Re}\hat{\mathbf{Q}}'^{L_R}_{N,M'} > 0$, $M'_L = M_L\mathrm{Im}\Omega - iN_L\mathrm{Re}\Omega$, $M'_R = M_R\mathrm{Im}\Omega - iN_R\mathrm{Re}\Omega$, and $(M'_H)^* = M_H\mathrm{Im}\Omega + i(N_L + N_R)\mathrm{Re}\Omega$.

Furthermore, $\vec{\tilde{m}}$ is the solution for $(N_L + N_R)\vec{k} = N_L\vec{i} + N_R\vec{j} + N_L\vec{\tilde{m}}$. Also, we take $\vec{p} \in \Lambda_{L_R N_R^{-1}}, \vec{\tilde{p}} \in \Lambda_{L_R N_L^{-1}}$. Then, we can replace $\vec{m}$ satisfying $\vec{m}N_L(N_L + N_R)^{-1} \in \mathbb{Z}$ with $\vec{\tilde{m}}' = \vec{\tilde{m}} + \vec{p}(N_L + N_R)N_L^{-1} + \vec{\tilde{p}}(N_L + N_R)N_R^{-1}$.

When we impose the following conditions taking chirality and symmetries into account

$$\begin{aligned}
&N^T = N, M^T = M, \Omega^T = \Omega,\\
&R = N^{-1}M = PUP^{-1} = PUP^T = R^T = R^{-1},\\
&[N, \Omega] = [N_L, N_R] = [R, \Omega] = [N, R] = [\Omega, \bar{\Omega}] = 0,\\
&N_H = N_L + N_R, (M\mathrm{Im}\Omega)^T = M\mathrm{Im}\Omega > 0,\\
&N_L\vec{i} + N_R\vec{j} + N_L\vec{\tilde{m}} = (N_L + N_R)\vec{k},\\
&\vec{p} \in \Lambda_{L_R N_R^{-1}}, \vec{\tilde{p}} \in \Lambda_{L_R N_L^{-1}},\\
&L_R = \mathrm{L.C.M.}(|N_L|, |N_R|), \mathcal{N}_i = |\det N|^{\frac{1}{4}} \cdot [\mathrm{Vol}(T^6)]^{-\frac{1}{2}},
\end{aligned} \tag{5.22}$$

the specific C is written explicitly

$$\begin{aligned}
C =& \frac{|-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L + M_R + M_H)\mathrm{Im}\Omega|}} \cdot \sum_{\vec{p},\vec{\tilde{p}}} \theta \begin{bmatrix} \frac{N_R(N_L+N_R)^{-1}N_L}{L_R}(\vec{i} - \vec{j} + \vec{\tilde{m}}') \\ \vec{0} \end{bmatrix} (\vec{0}, \tilde{\Omega}_Y) \\
&\cdot \delta_{(N_L+N_R)\vec{k}, N_L\vec{i}+N_R\vec{j}+N_L\vec{\tilde{m}}},
\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}
\tilde{\Omega}_Y \equiv& (L_R)^2[(N_L^{-1} + N_R^{-1})\mathrm{Re}\Omega \\
&+ i[(R_L N_L^{-1} + R_R N_R^{-1}) - (R_L - R_R)(M_L + M_R + M_H)^{-1}(R_L - R_R)]\mathrm{Im}\Omega],
\end{aligned} \tag{5.24}$$

and $\tilde{\Omega}_Y \in \mathcal{H}^3$ [82]. We prove Eq. (5.23) in the Appendix I. We can obtain the modular T transformation of the C,

$$T : C \Rightarrow e^{i\pi(\vec{i}-\vec{j}+\vec{\tilde{m}})^T B N_R(N_L+N_R)^{-1}N_L(\vec{i}-\vec{j}+\vec{\tilde{m}})} \cdot C, \tag{5.25}$$

where B satisfies $(L_R)^2[(N_L^{-1} + N_R^{-1})B]_{ii} \in 2\mathbb{Z}$ under $T : \Omega \Rightarrow \Omega + B$. In what follows, we will consider the modular transformations of the C. In the future, we would like to perform the numerical computations and verify whether realistic flavor models can be realized or not from nonfactorizable T^6 with background magnetic fluxes.

5.1.2 Proof of $\tilde{R}^2 = \mathbf{1}_3$

The $C = C^{\vec{i}\vec{j}\vec{k}}$ can lead to Yukawa couplings. Then, if we consider the $Sp(6, \mathbb{Z})$ modular transformations of the C, we may discuss modular flavor symmetries by computations of these representations, so we would like to calculate these transformations under the well-defined Riemann-Theta functions. We have found the explicit form of the $C^{\vec{i}\vec{j}\vec{k}}$

$$C^{\vec{i}\vec{j}\vec{k}} = \frac{|-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k}{\sqrt{|(M_L + M_R + M_H)\mathrm{Im}\Omega|}} \cdot \sum_{\vec{p},\vec{\tilde{p}}} \theta \begin{bmatrix} \frac{N_R N_H^{-1} N_L}{L_R}(\vec{i} - \vec{j} + \vec{\tilde{m}}') \\ \vec{0} \end{bmatrix} (\vec{0}, \tilde{\Omega}_Y) \cdot \delta_{N_H\vec{k}, N_L\vec{i} + N_R\vec{j} + N_L\vec{\tilde{m}}}, \tag{5.26}$$

where

$$\begin{aligned} \tilde{\Omega}_Y \equiv\ & (L_R)^2[(N_L^{-1} + N_R^{-1})\mathrm{Re}\Omega \\ & + i[(R_L N_L^{-1} + R_R N_R^{-1}) - (R_L - R_R)(M_L + M_R + M_H)^{-1}(R_L - R_R)]\mathrm{Im}\Omega]. \end{aligned} \tag{5.27}$$

Then, we can calculate the $C^{\vec{i}\vec{j}\vec{k}}$. Specifically, we rewrite $\tilde{\Omega}_Y$ by a new matrix $\tilde{\Omega}'_Y$

$$\tilde{\Omega}_Y \equiv (L_R)^2 N_L^{-1} N_H N_R^{-1}(\mathrm{Re}\Omega + i\tilde{R}\mathrm{Im}\Omega) \equiv (L_R)^2 N_L^{-1} N_H N_R^{-1} \tilde{\Omega}'_Y, \tag{5.28}$$

where both $\tilde{\Omega}_Y^T = \tilde{\Omega}_Y$ and $\mathrm{Im}\tilde{\Omega}_Y > 0$ have to be satisfied as the Riemann condition, and

$$\begin{aligned} \tilde{R} \equiv\ & (N_L^{-1} N_H N_R^{-1})^{-1} \\ & \times \left[R_L N_L^{-1} + R_R N_R^{-1} - (R_L - R_R)[N_L(R_L + R_H) + N_R(R_R + R_H)]^{-1}(R_L - R_R) \right], \\ \tilde{\Omega}'_Y =\ & \mathrm{Re}\Omega + i\tilde{R}\mathrm{Im}\Omega. \end{aligned}$$

Since we have supposed $[N_L, N_R] = [N, \Omega] = [N, R] = [R, \Omega] = [\Omega, \bar{\Omega}] = 0$, we find $[\tilde{R}, \Omega] = 0$ and can consider the following R_L, R_R, and R_H

$$R_L = P U_L P^{-1}, \ R_R = P U_R P^{-1}, \ R_H = P U_H P^{-1}. \tag{5.29}$$

Here, $P \in SO(3)$ denotes joint-diagonalization matrix. Also, U_a, $(a = L, R, H)$, represents a sign of chirality in each sector such that $U_a^2 = \mathbf{1}_3$. In what follows, we describe $u_a^i \in \{-1, +1\}$ as the components of U_a on $\mathbb{C}^i$, $(i = 1, 2, 3)$ in order to prove $\tilde{R}^2 = \mathbf{1}_3$. We note that $(u_a^i)^2 = 1$. Furthermore, we describe λ_a^i as the eigenvalues of N_a on $\mathbb{C}^i$, where $\lambda_H^i = \lambda_L^i + \lambda_R^i$.

When we perform the diagonalization of $\tilde{R}$, we obtain the following component

$$[P^{-1}\tilde{R}P]_{ii} = \frac{1}{\lambda_H^i} \cdot \left[\lambda_L^i u_R^i + \lambda_R^i u_L^i - 2\lambda_L^i \lambda_R^i \frac{1 - u_L^i u_R^i}{\lambda_L^i(u_L^i + u_H^i) + \lambda_R^i(u_R^i + u_H^i)} \right], \tag{5.30}$$

where $\lambda^i_L(u^i_L + u^i_H) + \lambda^i_R(u^i_R + u^i_H) \neq 0$. In the following, we consider two cases, namely $u^i_L = u^i_R$ or $u^i_L \neq u^i_R$.

(I): $u^i_L = u^i_R$
When we consider $u^i_L = u^i_R$, we can evaluate

$$\begin{aligned}\lambda^i_L(u^i_L + u^i_H) + \lambda^i_R(u^i_R + u^i_H) &= \lambda^i_L(u^i_L + u^i_H) + \lambda^i_R(u^i_L + u^i_H) \\ &= (\lambda^i_L + \lambda^i_R)(u^i_L + u^i_H) \\ &= \lambda^i_H(u^i_L + u^i_H).\end{aligned} \tag{5.31}$$

Under $\lambda^i_H \neq 0$, since $u^i_L + u^i_H \neq 0$ to satisfy $\lambda^i_L(u^i_L + u^i_H) + \lambda^i_R(u^i_R + u^i_H) \neq 0$, we find $u^i_L = u^i_R = u^i_H$, that is, all the chiralities on $\mathbb{C}^i$ have to coincide including the Higgs sector under $u^i_L = u^i_R$. Furthermore, since $1 - u^i_L u^i_R = 1 - (u^i_L)^2 = 0$, we finally obtain

$$[P^{-1}\tilde{R}P]_{ii} = \frac{1}{\lambda^i_H} \cdot \left(\lambda^i_L u^i_R + \lambda^i_R u^i_L\right) = \frac{1}{\lambda^i_H} \cdot \left(\lambda^i_L u^i_L + \lambda^i_R u^i_L\right) = u^i_L. \tag{5.32}$$

Thus, we find $[P^{-1}\tilde{R}^2 P]_{ii} = (u^i_L)^2 = 1$.

(II): $u^i_L \neq u^i_R$
When we consider $u^i_L \neq u^i_R$, namely $u^i_L = -u^i_R$, we find

$$u^i_L + u^i_R = 0 \qquad \text{and} \qquad u^i_L u^i_R = -1. \tag{5.33}$$

Then, we can calculate $[P^{-1}\tilde{R}P]_{ii}$ as follows

$$
\begin{aligned}
[P^{-1}\tilde{R}P]_{ii} &= \frac{1}{\lambda_H^i}\cdot\left[\lambda_L^i u_R^i + \lambda_R^i u_L^i - \frac{4\lambda_L^i\lambda_R^i}{\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)}\right] \\
&= \frac{1}{\lambda_H^i}\cdot\frac{(\lambda_L^i u_R^i + \lambda_R^i u_L^i)[\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)] - 4\lambda_L^i\lambda_R^i}{\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)} \\
&= \frac{-[(\lambda_L^i)^2+(\lambda_R^i)^2] + (\lambda_L^i+\lambda_R^i)\lambda_R^i u_L^i u_H^i + (\lambda_L^i+\lambda_R^i)\lambda_L^i u_R^i u_H^i - 2\lambda_L^i\lambda_R^i}{\lambda_H^i[\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)]} \\
&= \frac{-(\lambda_L^i+\lambda_R^i)^2 + (\lambda_L^i+\lambda_R^i)(\lambda_R^i u_L^i + \lambda_L^i u_R^i)u_H^i}{\lambda_H^i[\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)]} \\
&= \frac{(\lambda_R^i u_L^i + \lambda_L^i u_R^i)u_H^i - (\lambda_L^i+\lambda_R^i)}{\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)} \\
&= \frac{[(\lambda_L^i+\lambda_R^i)(u_L^i+u_R^i) - (\lambda_L^i u_L^i + \lambda_R^i u_R^i)]u_H^i - (\lambda_L^i+\lambda_R^i)}{\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)} \\
&= -\frac{(\lambda_L^i u_L^i + \lambda_R^i u_R^i)u_H^i + (\lambda_L^i+\lambda_R^i)}{\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)} \\
&= -\frac{\lambda_L^i(1+u_L^i u_H^i) + \lambda_R^i(1+u_R^i u_H^i)}{\lambda_L^i(u_L^i+u_H^i)+\lambda_R^i(u_R^i+u_H^i)}.
\end{aligned} \tag{5.34}
$$

Thus, even if $u_L^i \neq u_R^i$, we obtain

$$
\begin{aligned}
&[P^{-1}\tilde{R}^2P]_{ii} \\
&= \frac{(\lambda_L^i)^2(1+u_L^i u_H^i)^2 + (\lambda_R^i)^2(1+u_R^i u_H^i)^2 + 2\lambda_L^i\lambda_R^i(1+u_L^i u_H^i)(1+u_R^i u_H^i)}{(\lambda_L^i)^2(u_L^i+u_H^i)^2 + (\lambda_R^i)^2(u_R^i+u_H^i)^2 + 2\lambda_L^i\lambda_R^i(u_L^i+u_H^i)(u_R^i+u_H^i)} \\
&= \frac{2(\lambda_L^i)^2(1+u_L^i u_H^i) + 2(\lambda_R^i)^2(1+u_R^i u_H^i)}{2(\lambda_L^i)^2(1+u_L^i u_H^i) + 2(\lambda_R^i)^2(1+u_R^i u_H^i)} \\
&= 1.
\end{aligned} \tag{5.35}
$$

In conclusion, regardless of whether $u_L^i = u_R^i$ or not, we obtain $[P^{-1}\tilde{R}^2P]_{ii} = 1$ for any i. This means that we obtain $\tilde{R}^2 = 1$. From previous calculations, the eigenvalues of $\tilde{R}$ become $+1$ or -1. This is because $\tilde{R}$ is real since all R_a are elements of $SO(3)$. Also, we can diagonalize $\tilde{R}$ simultaneously by rotation matrix $P \in SO(3)$, so we find $\tilde{R}^T = \tilde{R}$.

Examples of chiralities for magnetized T^6 models

We take $U_L = \mathrm{diag}(+,-,-)$, $U_R = \mathrm{diag}(-,+,-)$ under $\mathrm{sign}[P^{-1}(\mathrm{Im}\Omega)P] \equiv U_{\mathrm{Im}\Omega} = (+,+,+)$. Then we find that u_H^3 is taken as -1 since this matches

$u_L^3 = u_R^3 = -1$. Furthermore, by considering fixed 6D positive overall chirality, we should take the eigenvalues of R_H as $(+,+,-)$ since the sign of the chirality matrix R_H is flipped, or $-R_H$ when taking complex conjugate of the wave functions in Higgs sector into account. Then, when we take $u_H^1 = +1$ and $u_H^2 = +1$, we obtain

$$[P^{-1}\tilde{R}P]_{11} = [P^{-1}\tilde{R}P]_{22} = [P^{-1}\tilde{R}P]_{33} = -1. \tag{5.36}$$

Thus, we can see that the eigenvalue in each sector and $\tilde{R}$ are given by

$$\begin{aligned} &N_L : (+,-,-)\,, \quad N_R : (-,+,-)\,, \quad N_H : (+,+,-)\,, \\ &\text{Higgs}^* : (-,-,+) \Rightarrow (0,0,+1), \quad \tilde{R} : (-1,-1,-1), \quad \mathrm{Im}\Omega : (+,+,+). \end{aligned} \tag{5.37}$$

where Higgs* denotes complex conjugate of the wave functions in Higgs sector. Here note that $\tilde{R} = -\mathbf{1}_3$ under this condition, which means $\tilde{R}\mathrm{Im}\Omega = -\mathrm{Im}\Omega = \mathrm{Im}\bar{\Omega}$. Also, from Eqs. (5.28) and (5.37), we find that

$$\mathrm{sign}[P^{-1}(\mathrm{Im}\tilde{\Omega}_Y)P]_{ii} = (L_R)^2 \mathrm{sign}[P^{-1}[N_L^{-1}N_H N_R^{-1}\tilde{R}\mathrm{Im}\Omega]P]_{ii} = +1 \tag{5.38}$$

for all i, which means that we have verified the Riemann condition, namely $\mathrm{Im}\tilde{\Omega}_Y > 0$. In conclusion, we obtain

$$\tilde{\Omega}_Y = (L_R)^2 N_L^{-1} N_H N_R^{-1} \bar{\Omega} \tag{5.39}$$

under $U_L = \mathrm{diag}(+,-,-)$, $U_R = \mathrm{diag}(-,+,-)$, and $U_{\mathrm{Im}\Omega} = (+,+,+)$.

Similarly, we can verify the other cases under $U_{\mathrm{Im}\Omega} = (+,+,+)$. When we consider (I): $U_L = \mathrm{diag}(+,-,-)$, $U_R = \mathrm{diag}(-,-,+)$ or (II): $U_L = \mathrm{diag}(-,+,-)$, $U_R = \mathrm{diag}(-,-,+)$, we obtain $\tilde{R} = -\mathbf{1}_3$.

$$\begin{aligned} &N_L : (+,-,-)\,, \quad N_R : (-,-,+)\,, \quad N_H : (+,-,+)\,, \\ &\text{Higgs}^* : (-,+,-) \Rightarrow (0,+1,0), \quad \tilde{R} : (-1,-1,-1), \quad \mathrm{Im}\Omega : (+,+,+), \end{aligned} \tag{5.40}$$

$$\begin{aligned} &N_L : (-,+,-)\,, \quad N_R : (-,-,+)\,, \quad N_H : (-,+,+)\,, \\ &\text{Higgs}^* : (+,-,-) \Rightarrow (+1,0,0), \quad \tilde{R} : (-1,-1,-1), \quad \mathrm{Im}\Omega : (+,+,+). \end{aligned} \tag{5.41}$$

Magnetized $T^6/\mathbb{Z}_{12}$

In magnetized $T^6/\mathbb{Z}_{12}$, the complex structure moduli are fixed as

$$P^{-1}\Omega_{12}P = \mathrm{diag}(e^{-\frac{2}{3}\pi i}, e^{-\frac{\pi}{6}i}, e^{\frac{5}{6}\pi i}). \tag{5.42}$$

Thus, we can take $U_{\mathrm{Im}\Omega_{12}} = (-,-,+)$. Even if we consider the above $U_{\mathrm{Im}\Omega_{12}}$ and assign $U_L = (+,-,-)$ and $U_R = (-,+,-)$, we obtain

$$\begin{aligned}
&N_L : (-,+,-)\,, \quad N_R : (+,-,-)\,, \quad N_H : (-,-,-),\\
&\text{Higgs}^* : \left(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}\right) \Rightarrow (0,0,+1), \quad \tilde{R} : (-1,-1,-1), \quad \mathrm{Im}\Omega : (-,-,+),
\end{aligned} \tag{5.43}$$

or $\tilde{R} = -\mathbf{1}_3$.

5.1.3 Modular transformation of $C = C^{\vec{i}\vec{j}\vec{k}}$

On the other hand, we can also see the modular S and T transformations of $C^{\vec{i}\vec{j}\vec{k}}$ by making use of $\tilde{\Omega}'_Y$ and $\tilde{R}^2 = 1$

$$\begin{aligned}
&S : \Omega \to -\Omega^{-1} \Rightarrow \tilde{\Omega}'_Y \to -\tilde{\Omega}'^{-1}_Y,\\
&T : \Omega \to \Omega + B \Rightarrow \tilde{\Omega}'_Y \to \tilde{\Omega}'_Y + B.
\end{aligned} \tag{5.44}$$

Because of $\mathrm{Im}(-\tilde{\Omega}'^{-1}_Y) = (\bar{\tilde{\Omega}}'_Y \tilde{\Omega}'_Y)^{-1}\mathrm{Im}\tilde{\Omega}'_Y = (\bar{\Omega}\Omega)^{-1}\mathrm{Im}\tilde{\Omega}'_Y$ under $[\tilde{R},\Omega] = [\bar{\Omega},\Omega] = 0$ and $\tilde{R}^2 = \mathbf{1}_3$, we can calculate the modular S transformation of the Riemann-Theta function part on the $C^{\vec{i}\vec{j}\vec{k}}$ under $N_H\vec{k} = N_L\vec{i} + N_R\vec{j} + N_L\vec{m}$

$$\begin{aligned}
&\sum_{\vec{p},\vec{\tilde{p}}} \theta \begin{bmatrix} \frac{N_R N_H^{-1} N_L}{L_R}(\vec{i} - \vec{j} + \vec{m}') \\ \vec{0} \end{bmatrix} (\vec{0}, (L_R)^2 N_L^{-1} N_H N_R^{-1} \tilde{\Omega}'_Y)\\
&= \sum_{\vec{p},\vec{\tilde{p}}} \theta \begin{bmatrix} \tilde{N}_Y^{-1} \vec{L}_R \\ \vec{0} \end{bmatrix} (\vec{0}, \tilde{N}_Y \tilde{\Omega}'_Y),
\end{aligned} \tag{5.45}$$

where

$$\tilde{N}_Y \equiv (L_R)^2 N_L^{-1} N_H N_R^{-1} = (L_R)^2 (N_L^{-1} + N_R^{-1}), \tag{5.46}$$

$$\vec{L}_R \equiv L_R(\vec{i} - \vec{j} + \vec{m}'). \tag{5.47}$$

Because of the definitions of $L_R = \mathrm{L.C.M.}(|N_L|, |N_R|)$, $\tilde{N}_Y$ and $\vec{L}_R$ are integer matrix, vector, respectively. Strictly speaking, the periodicity of the Riemann-

Theta function is generally described as

$$\begin{aligned}\theta\begin{bmatrix}\vec{a}+\vec{n}\\ \vec{b}\end{bmatrix}(\vec{z},\Omega) &= \sum_{\vec{m}\in\mathbb{Z}^3} e^{i\pi(\vec{m}+\vec{a}+\vec{n})^T\Omega(\vec{m}+\vec{a}+\vec{n})}\cdot e^{2\pi i(\vec{m}+\vec{a}+\vec{n})^T(\vec{z}+\vec{b})}\\ &= \sum_{\vec{m}'\in\mathbb{Z}^3} e^{i\pi(\vec{m}'+\vec{a})^T\Omega(\vec{m}'+\vec{a})}\cdot e^{2\pi i(\vec{m}'+\vec{a})^T(\vec{z}+\vec{b})}\\ &= \theta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z},\Omega),\end{aligned} \tag{5.48}$$

where $\vec{n}$ represents arbitrary constant vector with all integer components, so we find $\vec{m}'=\vec{m}+\vec{n}\in\mathbb{Z}^3$. In other words, we can take $\vec{a}$ as an element of $\mathbb{R}^3/\mathbb{Z}^3$. Thus, by using unit vector $\vec{e}$, we obtain

$$\begin{aligned}\sum_{\vec{p},\vec{\tilde{p}}}\theta\begin{bmatrix}\tilde{N}_Y^{-1}(\vec{L}_R+\tilde{N}_Y\vec{e})\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{N}_Y\tilde{\Omega}'_Y) &= \sum_{\vec{p},\vec{\tilde{p}}}\theta\begin{bmatrix}\tilde{N}_Y^{-1}\vec{L}_R+\vec{e}\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{N}_Y\tilde{\Omega}'_Y)\\ &= \sum_{\vec{p},\vec{\tilde{p}}}\theta\begin{bmatrix}\tilde{N}_Y^{-1}\vec{L}_R\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{N}_Y\tilde{\Omega}'_Y),\end{aligned} \tag{5.49}$$

where $\vec{L}_R\in\Lambda_{\tilde{N}_Y}$. That is, we can obtain the modular S transformation under $\Omega\to-\Omega^{-1}$

$$\begin{aligned}&\sum_{\vec{p},\vec{\tilde{p}}}\theta\begin{bmatrix}\tilde{N}_Y^{-1}\vec{L}_R\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{N}_Y(-\tilde{\Omega}'_Y)^{-1}) \qquad (5.50)\\ &=\sqrt{\det(\tilde{N}_Y^{-1}\tilde{\Omega}'_Y/i)}\cdot\sum_{\vec{p},\vec{\tilde{p}}}\sum_{\vec{D}_R\in\Lambda_{\tilde{N}_Y}} e^{2\pi i\vec{L}_R^T\tilde{N}_Y^{-1}\vec{D}_R}\cdot\theta\begin{bmatrix}\tilde{N}_Y^{-1}\vec{D}_R\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{N}_Y\tilde{\Omega}'_Y).\end{aligned}$$

Similarly, we consider the behavior of following coefficient derived from the Gaussian integral and its Jacobian

$$\frac{|-2i\mathrm{Im}\Omega|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L+M_R+M_H)\mathrm{Im}\Omega|}} \tag{5.51}$$

under the modular S transformation. Since $\mathrm{Im}\Omega_S=\mathrm{Im}(-\Omega^{-1})=(\bar{\Omega}\Omega)^{-1}\mathrm{Im}\Omega$ and $\mathcal{N}_i=||N||^{\frac{1}{4}}\cdot[\mathrm{Vol}(T^6)]^{-\frac{1}{2}}$ is invariant under the modular S transformation, the coefficient becomes

$$\begin{aligned}\frac{|-2i\mathrm{Im}\Omega_S|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L+M_R+M_H)\mathrm{Im}\Omega_S|}} &= \frac{|-2i(\bar{\Omega}\Omega)^{-1}\mathrm{Im}\Omega|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L+M_R+M_H)(\bar{\Omega}\Omega)^{-1}\mathrm{Im}\Omega|}}\\ &= |\bar{\Omega}\Omega|^{-\frac{1}{2}}\frac{|-2i\mathrm{Im}\Omega|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L+M_R+M_H)\mathrm{Im}\Omega|}}.\end{aligned} \tag{5.52}$$

In conclusion, we obtain the modular S transformation of the $C^{\vec{i}\vec{j}\vec{k}} \equiv C^{\vec{L}_R}$ in magnetized T^6

$$S : C^{\vec{L}_R} \Rightarrow e^{\frac{\pi}{4}i} \cdot \frac{|\bar{\tilde{\Omega}}'_Y \tilde{\Omega}'_Y|^{-\frac{1}{2}} \cdot |\tilde{\Omega}'_Y|^{\frac{1}{2}}}{\sqrt{|\det \tilde{N}_Y|}} \cdot \sum_{\vec{D}_R \in \Lambda_{\tilde{N}_Y}} e^{2\pi i \vec{L}_R^T \tilde{N}_Y^{-1} \vec{D}_R} \cdot C^{\vec{D}_R} \tag{5.53}$$
$$= e^{\frac{\pi}{4}i} \cdot \frac{|\bar{\Omega}\Omega|^{-\frac{1}{2}} \cdot |\tilde{\Omega}'_Y|^{\frac{1}{2}}}{\sqrt{|\det \tilde{N}_Y|}} \cdot \sum_{\vec{D}_R \in \Lambda_{\tilde{N}_Y}} e^{2\pi i \vec{L}_R^T \tilde{N}_Y^{-1} \vec{D}_R} \cdot C^{\vec{D}_R}.$$

For example, when we take 6D overall positive chirality such as $U_L = (+,-,-)$, $U_R = (-,+,-)$, and $U_{\mathrm{Im}\Omega} = (+,+,+)$, we have found $\tilde{R}\mathrm{Im}\Omega = \mathrm{Im}\bar{\Omega}$, namely $\tilde{\Omega}'_Y = \bar{\Omega}$. $\tilde{R} = -\mathbf{1}_3$ always holds when we consider 6D overall positive chirality in left-handed and right-handed sectors. Thus, we obtain the following transformation

$$S : C^{\vec{L}_R} \Rightarrow e^{\frac{\pi}{4}i} \cdot \frac{|\bar{\Omega}\Omega|^{-\frac{1}{2}} \cdot |\bar{\Omega}|^{\frac{1}{2}}}{\sqrt{|\det \tilde{N}_Y|}} \cdot \sum_{\vec{D}_R \in \Lambda_{\tilde{N}_Y}} e^{2\pi i \vec{L}_R^T \tilde{N}_Y^{-1} \vec{D}_R} \cdot C^{\vec{D}_R} \tag{5.54}$$
$$= \frac{e^{\frac{\pi}{4}i} \cdot |\Omega|^{-\frac{1}{2}}}{\sqrt{|\det \tilde{N}_Y|}} \cdot \sum_{\vec{D}_R \in \Lambda_{\tilde{N}_Y}} e^{2\pi i \vec{L}_R^T \tilde{N}_Y^{-1} \vec{D}_R} \cdot C^{\vec{D}_R},$$

where $|\bar{\Omega}\Omega|^{-\frac{1}{2}}|\bar{\Omega}|^{\frac{1}{2}} = |\bar{\Omega}|^{-\frac{1}{2}}|\Omega|^{-\frac{1}{2}}|\bar{\Omega}|^{\frac{1}{2}} = |\Omega|^{-\frac{1}{2}}$. This implies that the $C^{\vec{L}_R}$ can be holomorphic functions of the Ω. This is consistent with 4D $\mathcal{N} = 1$ supersymmetric and modular invariant effective field theory (EFT).

According to Ref. [12], in 4D $\mathcal{N} = 1$ global supersymmetric modular invariant theory, the superpotential $W(\Phi)$ in the action $\mathcal{S}$

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\ K(\Phi, \bar{\Phi}) + \int d^4x d^2\theta\ W(\Phi) + \mathrm{h.c.} \tag{5.55}$$

is given by

$$W(\Phi) = \sum_n Y_{I_1 \cdots I_n}(\tau) \cdot \phi^{(I_1)} \cdots \phi^{(I_n)}, \tag{5.56}$$

where $\Phi = (\tau, \phi^{(I_n)})$ denotes the chiral superfield. That is, τ, $\phi^{(I_n)}$ denote complex structure moduli, chiral supermultiplets separated into each sector, respectively. This is $SL(2,\mathbb{Z})$-invariant EFT. Specifically, when we take following γ

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}), \tag{5.57}$$

we can assume the following modular transformation in 4D EFT

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \phi^{(I)} \to (c\tau + d)^{-k_I}\rho^{(I)}(\gamma)\phi^{(I)}. \tag{5.58}$$

We note that the chiral supermultiplets $\phi^{(I)}$ are not necessarily modular form, so we can take any values for k_I. Furthermore, since the $\mathcal{S}$ has modular invariance, this requires the following transformations for $W(\Phi)$ and $K(\Phi, \bar{\Phi})$

$$\begin{aligned} &W(\Phi) \to W(\Phi), \\ &K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + f(\Phi) + f(\bar{\Phi}), \end{aligned} \tag{5.59}$$

where $f(\Phi)$ denotes Kähler function appearing in Kähler transformation. On the other hand, we can extend these approaches to $Sp(2g, \mathbb{Z})$ by using Siegel modular form, according to Ref. [106].

According to the discussion so far, when replacing τ with Ω, we can construct 4D $\mathcal{N} = 1$ SUSY and $Sp(6, \mathbb{Z})$ invariant EFT derived from magnetized T^6 model. Take an example for EFT with 4D $\mathcal{N} = 1$ global SUSY and $Sp(6, \mathbb{Z})$ modular symmetry. With $\Phi = (\Omega, \phi^{(I)})$, when we assume the following transformations for Ω and $\phi^{(I)}$

$$\begin{aligned} &\Omega \to \gamma\Omega = (A\Omega + B)(C\Omega + D)^{-1}, \\ &\Omega' \to \gamma\Omega' = (A\Omega' + B)(C\Omega' + D)^{-1}, \\ &\phi^{(I)} \to |C\Omega' + D|^{k_I}\rho^{(I)}(\gamma)\phi^{(I)}, \end{aligned} \tag{5.60}$$

where

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(6, \mathbb{Z}), \quad \Omega' = \mathrm{Re}\Omega + iR\mathrm{Im}\Omega, \tag{5.61}$$

coefficient $Y_{I_1 \cdots I_n}$ in the superpotential $W(\Phi)$

$$W(\Phi) = \sum_n Y_{I_1 \cdots I_n}(\Omega) \cdot \phi^{(I_1)} \cdots \phi^{(I_n)} \tag{5.62}$$

can become holomorphic function of Ω. Furthermore, we can consider a following behavior under the modular transformation

$$Y_{I_1 \cdots I_n} \to |C\Omega + D|^{k_Y(n)}\rho^{(Y)}(\gamma)Y_{I_1 \cdots I_n}. \tag{5.63}$$

Lastly, the modular T transformation of the $C^{\vec{L}_R}$ is written as

$$T : C^{\vec{L}_R} \Rightarrow e^{\pi i \vec{L}_R^T \tilde{N}_Y^{-1} B \vec{L}_R} \cdot C^{\vec{L}_R} \tag{5.64}$$

under $\tilde{\Omega}'_Y \to \tilde{\Omega}'_Y + B$, where $(\tilde{N}_Y B)_{ii} \in 2\mathbb{Z}$. Based on Refs. [61, 63, 114], we will use these modular transformations to derive non-Abelian discrete symmetries from the background magnetic fluxes in the next section 5.2.

5.2 Non-Abelian discrete symmetries

5.2.1 Modular representation $\rho^{(I)}(\gamma)$

In 4D $\mathcal{N}=1$ SUSY model under $Sp(6,\mathbb{Z})$ modular symmetry, the chiral supermultiplet in I sector, $\phi^{(I)}$, can correspond to fermion (and its superpartner) zero modes when we consider dimensional reduction from magnetized D-brane model. Let us review dimensional reduction and the KK decomposition briefly.

In order to obtain 4D chiral theory including properties of the Standard model from extra dimension, we have considered ten-dimensional (10D) gauge theory with $\mathcal{N}=1$ SUSY. The 10D action has been given by

$$S=\int_{\mathbb{R}^{3,1}} d^4x \left[\int_{\mathcal{M}_6} d^3z d^3\bar{z} \left[-\frac{1}{4g^2}\mathrm{Tr}(F^{MN}F_{MN})+\frac{i}{2g^2}\mathrm{Tr}(\bar{\lambda}\Gamma^M D_M\lambda)\right]\right], \tag{5.65}$$

where $F_{MN}=\partial_M A_N-\partial_N A_M-i[A_M,A_N]$ and $D_M\lambda=\partial_M\lambda-i[A_M,\lambda]$ with $M,N\in\{0,1,\cdots,9\}$. Here, we have supposed 10D flat background space-time $\mathcal{M}_{10}$ as product of 4D Minkowski space-time $\mathbb{R}^{3,1}$ and the 6D compactification $\mathcal{M}_6$, namely

$$\mathcal{M}_{10}=\mathbb{R}^{3,1}\times\mathcal{M}_6. \tag{5.66}$$

Furthermore, we have supposed a flat compact geometry such as torus or its orbifolds. When we consider the KK decomposition with magnetic fluxes background on T^6, this would be a key for us to understand whether we can realize the flavor structure in lepton and quark sectors. This is because analyzing chiral superfield and the superpotential gives us Yukawa couplings. When we take $\Omega'=\mathrm{Re}\Omega+iR\mathrm{Im}\Omega$, we obtain the wave function after the S transformation under $N\Omega', N^{-1}\Omega'\in\mathcal{H}^3$ and $N^T=N$ [114]

$$\begin{aligned}\psi^J(-\Omega'^{-1}z',-\Omega'^{-1}) &= \sqrt{\det(-iN^{-1}\Omega')}\sum_{K\in\Lambda_N} e^{2\pi i J^T N^{-1}K}\cdot\psi^K(z',\Omega')\\ &\equiv \sqrt{\det(-\Omega')}\sum_{K\in\Lambda_N}[\rho(S)]_{JK}\cdot\psi^K(z',\Omega'),\end{aligned} \tag{5.67}$$

and the T transformation

$$\begin{aligned}\psi^J(z',\Omega'+B) &= e^{\pi i J^T N^{-1}BK}\cdot\delta_{J,K}\cdot\psi^K(z',\Omega')\\ &\equiv [\rho(T)]_{JK}\cdot\psi^K(z',\Omega'), \quad \text{where} \quad (NB)_{ii}\in 2\mathbb{Z},\end{aligned} \tag{5.68}$$

where $J, K \in \Lambda_N$ and

$$[\rho(T)]_{JK} = \rho^{(N)}_{JK}(T) \equiv e^{\pi i J^T N^{-1} BK} \cdot \delta_{J,K}. \tag{5.69}$$

We consider N with $\det N > 0$. Since we can obtain zero-mode wave functions with negative chirality when replacing N with $-N$, we obtain

$$\rho^{(-N)}_{IJ}(T) = e^{-\pi i J^T N^{-1} BK} \cdot \delta_{J,K} = [\rho^{(N)}_{IJ}(T)]^*. \tag{5.70}$$

In the following, we consider finite discrete groups generated from $\{\rho(S), \rho(T)\}$, and we will describe them as $G =< \rho(S), \rho(T) >$ [114]. First, we focus on the values of representation of the S transformation, $\rho_{IJ}(S)$, in particular, how to calculate $\sqrt{\det(-iN^{-1}\Omega')}$ properly. Since we calculate products of square roots of complex numbers $\det(-iN^{-1}\Omega')$, we have to consider the branch cut. Note that, in the following, when choosing the branch cut of $\sqrt{z} = a + bi$, we take it such that $\sqrt{z} > 0$ when $z > 0$, that is,

$$\sqrt{z} = \sqrt{r} e^{i\theta/2}, \quad \text{where} \quad r > 0, \quad \arg(z) = \theta \in (-\pi, +\pi]. \tag{5.71}$$

Let us consider specific values of $\rho_{IJ}(S)$ When we consider 6D overall positive chirality, that is, $\det N > 0$ under $N^{-1}\Omega' \in \mathcal{H}^3$ and $N^T = N$ on magnetized T^6, we can obtain

$$\begin{aligned}
\sqrt{\det(-iN^{-1}\Omega')} &= \sqrt{\det(iN^{-1}(-\Omega'))} \\
&= \sqrt{\det(i\mathbf{1}_3) \cdot \det(N^{-1}) \cdot \det(-\Omega')} \\
&= \sqrt{\frac{\det(i\mathbf{1}_3) \cdot \det(-\Omega')}{\det N}} \\
&= \sqrt{\frac{(-i) \cdot \det(-\Omega')}{\det N}} \\
&= \frac{1}{\sqrt{\det N}} \cdot \sqrt{(-i) \cdot \det(-\Omega')} \\
&= \frac{\sqrt{-i}}{\sqrt{\det N}} \cdot \sqrt{\det(-\Omega')} \\
&= \frac{e^{-\frac{\pi i}{4}}}{\sqrt{\det N}} \cdot \sqrt{\det(-\Omega')}
\end{aligned} \tag{5.72}$$

if we choose the above branch cut. Here, we have supposed

$$\sqrt{(-i) \cdot \det(-\Omega')} = +\sqrt{-i} \cdot \sqrt{\det(-\Omega')} = +e^{-\frac{\pi i}{4}} \cdot \sqrt{\det(-\Omega')}, \tag{5.73}$$

where $\arg(\det(-\Omega')) \in \left(-\frac{\pi}{2}, +\pi\right]$. Otherwise, we simply multiply a minus sign -1 for the whole formula Eq. (5.72), namely

$$\sqrt{\det(-iN^{-1}\Omega')} = -\frac{e^{-\frac{\pi i}{4}}}{\sqrt{\det N}} \cdot \sqrt{\det(-\Omega')}, \text{ where } \arg(\det(-\Omega')) \in \left(-\pi, -\frac{\pi}{2}\right]. \tag{5.74}$$

In this chapter, we take $\arg(\det(-\Omega')) \in \left(-\frac{\pi}{2}, +\pi\right]$. Similarly, when we take 6D overall negative chirality, that is, $\det N < 0$ under $N^{-1}\Omega' \in \mathcal{H}^3$ and $N^T = N$ on magnetized T^6, we can obtain

$$\begin{aligned}
\sqrt{\det(-iN^{-1}\Omega')} &= \sqrt{\det(iN^{-1}(-\Omega'))} \\
&= \sqrt{\det(i\mathbf{1}_3) \cdot \det(N^{-1}) \cdot \det(-\Omega')} \\
&= \sqrt{\frac{\det(i\mathbf{1}_3) \cdot \det(-\Omega')}{\det N}} \\
&= \sqrt{\frac{(-i) \cdot \det(-\Omega')}{\det N}} \\
&= \sqrt{\frac{i \cdot \det(-\Omega')}{-\det N}} \\
&= \frac{1}{\sqrt{-\det N}} \cdot \sqrt{i \cdot \det(-\Omega')} \\
&= \frac{\sqrt{i}}{\sqrt{-\det N}} \cdot \sqrt{\det(-\Omega')} \\
&= \frac{e^{+\frac{\pi i}{4}}}{\sqrt{-\det N}} \cdot \sqrt{\det(-\Omega')},
\end{aligned} \tag{5.75}$$

where we have used $-\det N > 0$ and chosen the branch cut. Additionally, we have supposed $\arg(\det(-\Omega')) \in \left(-\pi, +\frac{\pi}{2}\right]$. Otherwise, we have to multiply a minus sign -1 for the whole formula above. In this section, we take the above argument of $\det(-\Omega')$. We can therefore take the values of $\rho_{IJ}(S)$ as follows

$$\rho_{IJ}(S) = \begin{cases} \rho_{IJ}^{(+,N)}(S) \equiv \dfrac{e^{-\frac{\pi}{4}i}}{\sqrt{\det N}} \cdot e^{2\pi i J^T N^{-1} K} & \text{for} \quad \det N > 0, \\ \rho_{IJ}^{(-,N)}(S) \equiv \dfrac{e^{+\frac{\pi}{4}i}}{\sqrt{-\det N}} \cdot e^{2\pi i J^T N^{-1} K} & \text{for} \quad \det N < 0. \end{cases} \tag{5.76}$$

Here, in magnetized T^6 model, we can obtain zero-mode wave functions with negative chirality by flipping fluxes N with $\det N > 0$ to $-N$, that is,

$$N \Rightarrow N' \equiv -N, \tag{5.77}$$

we then find $\det N' = \det(-N) = -\det N < 0$ and $|\det N'| = |\det N|$. [1] We can therefore compare $\rho(S)$ for $\det N > 0$ with $\rho(S)$ for $\det N < 0$.

The $\rho(S)$ for $\det N > 0$ is written as

$$\rho_{IJ}^{(+,N)}(S) = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{\det N}} \cdot e^{2\pi i J^T N^{-1} K} = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{|\det N|}} \cdot e^{2\pi i J^T N^{-1} K} \quad . \tag{5.78}$$

On the other hand, $\rho(S)$ for $\det N' = \det(-N) < 0$, where $N' = -N$ and $\det N > 0$, is written as

$$\begin{aligned} \rho_{IJ}^{(-,N')}(S) &= \frac{e^{+\frac{\pi}{4}i}}{\sqrt{-\det N'}} \cdot e^{2\pi i J^T N'^{-1} K} \\ &= \frac{e^{+\frac{\pi}{4}i}}{\sqrt{|\det N|}} \cdot e^{2\pi i J^T (-N)^{-1} K} \\ &= \frac{e^{+\frac{\pi}{4}i}}{\sqrt{|\det N|}} \cdot e^{-2\pi i J^T N^{-1} K}. \end{aligned} \tag{5.79}$$

When comparing Eq. (5.78) with Eqs. (5.70), (5.79), we obtain a following relation

$$\begin{aligned} \rho_{IJ}^{(-,N')}(S) &= \rho_{IJ}^{(-,(-N))}(S) = \left[\rho_{IJ}^{(+,N)}(S)\right]^*, \\ \rho_{IJ}^{(-,N')}(T) &= \rho_{IJ}^{(-,(-N))}(T) = \left[\rho_{IJ}^{(+,N)}(T)\right]^*. \end{aligned} \tag{5.80}$$

This means that these complex conjugates $[\rho(S)]^*$ and $[\rho(T)]^*$ for positive chirality $\det N > 0$ are equivalent to $\rho(S)$ and $\rho(T)$ for negative chirality $\det N' = \det(-N) < 0$. Thus, we can construct isomorphic groups, that is,

$$< \rho(S), \rho(T) > \simeq < [\rho(S)]^*, [\rho(T)]^* > \tag{5.81}$$

whether we take 6D chirality as positive or negative. Similarly, we can obtain the $\rho_{IJ}(S)$ in magnetized T^4 as follows

$$\rho_{IJ}(S) = \begin{cases} +\dfrac{i}{\sqrt{\det N}} \cdot e^{2\pi i J^T N^{-1} K} & \text{for} \quad \det N > 0, \\ +\dfrac{1}{\sqrt{-\det N}} \cdot e^{2\pi i J^T N^{-1} K} & \text{for} \quad \det N < 0, \end{cases} \tag{5.82}$$

[1] This corresponds to shifting positive chirality to negative chirality by replacing a flux $M > 0$ with $M' \equiv -M < 0$ in magnetized T^2, where $|M'| = |M|$ [54]. Note that we cannot realize such flipping of the chirality in T^4 models because $\det(-N) = \det N$ always holds in magnetized T^4.

where we have supposed $\arg(\det(-\Omega')) \in (-\pi, 0]$ under $\det N > 0$. Meanwhile, the minus sign -1 would be multiplied for the whole formula when $\det N > 0$ and $0 < \arg(\det(-\Omega')) \leq +\pi$. Note that when $\det N < 0$ and $-\pi < \arg(\det(-\Omega')) \leq +\pi$ on T^4, we find the following constant value

$$\begin{aligned}
\sqrt{\det(-iN^{-1}\Omega')} &= \sqrt{\det(iN^{-1}(-\Omega'))} \\
&= \sqrt{\det(i\mathbf{1}_2) \cdot \det(N^{-1}) \cdot \det(-\Omega')} \\
&= \sqrt{\frac{\det(i\mathbf{1}_2) \cdot \det(-\Omega')}{\det N}} \\
&= \sqrt{\frac{(-1) \cdot \det(-\Omega')}{\det N}} \\
&= \sqrt{\frac{(+1) \cdot \det(-\Omega')}{-\det N}} \\
&= \frac{+1}{\sqrt{-\det N}} \cdot \sqrt{\det(-\Omega')}.
\end{aligned} \tag{5.83}$$

Additionally, note that $\rho(S)$ depends on the integer fluxes N, on the other hands, $\rho(T)$ depends on not only the N but the B matrix in the T_i generators. Then, let us consider finite groups $G =< \rho(S), \rho(T) >$ generated by (5.76) and (5.82).

5.2.2 Magnetic flux on $T^6/\mathbb{Z}_{12}$

In this paper, we have constructed $T^6/\mathbb{Z}_{12}$ with E_6 root Lie-lattice derived from the algebraic structure $(ST_1T_2T_3^{-1}T_5T_6)^{12} = \mathbf{1}_6$. Then, the integer fluxes N, the complex structure moduli Ω_{12}, these joint diagonalization matrix P, and the B matrix of the T transformations are given by [78, 81, 82].

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{11} & n_{13} \\ n_{13} & n_{13} & n_{11}+n_{12}-2n_{13} \end{bmatrix}, \quad \Omega_{12} = \begin{bmatrix} -\frac{1}{2}-\frac{\sqrt{3}}{6}i & \frac{\sqrt{3}}{3}i & -\frac{1}{2}+\frac{\sqrt{3}}{6}i \\ \frac{\sqrt{3}}{3}i & -\frac{1}{2}-\frac{\sqrt{3}}{6}i & -\frac{1}{2}+\frac{\sqrt{3}}{6}i \\ -\frac{1}{2}+\frac{\sqrt{3}}{6}i & -\frac{1}{2}+\frac{\sqrt{3}}{6}i & \frac{1}{2}-\frac{\sqrt{3}}{6}i \end{bmatrix},$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{3-\sqrt{3}}} & \frac{1+\sqrt{3}}{2\sqrt{3+\sqrt{3}}} \\ \frac{1}{\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{3-\sqrt{3}}} & \frac{1+\sqrt{3}}{2\sqrt{3+\sqrt{3}}} \\ 0 & \frac{1}{\sqrt{3-\sqrt{3}}} & \frac{1}{\sqrt{3+\sqrt{3}}} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

where $\det P = 1$ and we find the P is independent of the background magnetic fluxes. Note that the shape of the N is determined by the F-term SUSY condition

$(N\Omega)^T = N\Omega$. The diagonalized Ω_{12} are given by

$$P^{-1}\Omega_{12}P = \mathrm{diag}\left(e^{-2\pi i/3}, e^{-\pi i/6}, e^{5\pi i/6}\right), \tag{5.84}$$

that is, a matrix representing the sign of eigenvalues of $\mathrm{Im}\Omega_{12}$, $U_{\mathrm{Im}\Omega}$, can become $U_{\mathrm{Im}\Omega} = \mathrm{diag}(-,-,+)$. Additionally, since $n_{11} \equiv n_{12} \equiv n_{13} \pmod 2$, at least we find

$$\det N = (n_{11} - n_{12})[(n_{11} + n_{12})(n_{11} + n_{12} - 2n_{13}) - 2n_{13}^2] \equiv 0 \mod 4. \tag{5.85}$$

Thus, we would like to analyze the representations $\rho(S)$, $\rho(T) = \rho(T_1T_2T_3^{-1}T_5T_6)$ in each positive $\det N$. In particular, we will focus on the wave functions with either positive or negative chirality. We will show two examples for $|\det N| = 4$ because order of finite groups we can obtain is too large to specify the group.

Case 1: Positive chirality

When we consider $z' = z$ and $\Omega' = \Omega$, we obtain wave functions with chirality $(+++)$ in z system, namely $\psi^J_{+++}(z,\Omega)$. We can regard $\psi^J_{+++}(z,\Omega)$ as the Siegel modular form with modular weight $\frac{1}{2}$ [114]. When we define U_N as the sign of eigenvalues of the fluxes N, and take $U_N = \mathrm{diag}(-,-,+)$, we choose one of the following N_1, N_2, and N_3 matrices,

$$N_1 = \begin{bmatrix} -5 & -3 & -3 \\ -3 & -5 & -3 \\ -3 & -3 & -2 \end{bmatrix}, N_2 = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 2 \end{bmatrix}, N_3 = \begin{bmatrix} 3 & 5 & -11 \\ 5 & 3 & -11 \\ -11 & -11 & 30 \end{bmatrix}, \tag{5.86}$$

where $\det N = 4$. We can see that finite groups generated by $\rho(S)$ and $\rho(T)$ are the same and they are $\Delta(48) \rtimes (\mathbb{Z}_8 \times \mathbb{Z}_2)$ whose order is 768. This can be consistent with $\tilde{\Delta}(96) \simeq \Delta(48) \rtimes \mathbb{Z}_8$ realized in Ref. [114] because we consider another T_i-generator. We can also have at least two non-Abelian discrete subgroups $\Sigma(192)$, $\Delta(48)$.

Case 2: Negative chirality

We choose one of the following N matrices,

$$N = -N_1, -N_2, -N_3, \tag{5.87}$$

where $\det N = -4$. Finite group generated by $\rho(S)$ and $\rho(T)$ is $\Delta(48) \rtimes (\mathbb{Z}_8 \times \mathbb{Z}_2)$ whose order is 768. Also, we have at least two non-Abelian discrete subgroups $\Sigma(192)$, $\Delta(48)$. From the above results and Eq. (5.81), even if we fix 6D overall as positive or negative, we can realize the same finite groups.

5.2.3 Magnetized T^6

We analyze finite groups derived from the modular representation of magnetized T^6. In what follows, we impose conditions $(NB)^T = NB$ and $[NB]_{ii} \in 2\mathbb{Z}$ for the modular T-transformations. We will call these conditions consistency condition.

$\det N = +3$

When we take $\det N = +3$ and the sign of eigenvalues of N, $U_N = \mathrm{diag}(-,+,-)$, we choose the following N-matrix

$$N = \begin{bmatrix} -3 & -3 & 2 \\ -3 & -3 & 3 \\ 2 & 3 & -3 \end{bmatrix}. \tag{5.88}$$

We use the following T_i to generate finite groups

$$\begin{aligned} T_i &= \begin{bmatrix} \mathbf{1}_3 & B_i \\ O_3 & \mathbf{1}_3 \end{bmatrix}, \\ \text{where} \quad B_1 &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \\ B_3 &= \begin{bmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned} \tag{5.89}$$

These B matrices satisfy the consistency conditions. When we choose a $B = B_i$, $(i = 1, 2, 3, 4)$ or B_1 and B_4, we can construct a finite group $G_i =< \rho(S), \rho(T_i) >= T' \times \mathbb{Z}_8$, whose order is 192. [2]

$\det N = +4$

When we take $\det N = +4$, $U_N = \mathrm{diag}(-,+,-)$, we choose the following N-matrix

$$N = \begin{bmatrix} -3 & -2 & -1 \\ -2 & -2 & -2 \\ -1 & -2 & -1 \end{bmatrix}. \tag{5.90}$$

[2] Even if we choose the same N-matrix, B_i-matrix, and the other sign of $\sqrt{\det(-iN^{-1}\Omega')}$ in this model, we can see the same finite group $T' \times \mathbb{Z}_8$.

We use the T_i-generators with B_i-matrices

$$T_i = \begin{bmatrix} \mathbf{1}_3 & B_i \\ O_3 & \mathbf{1}_3 \end{bmatrix}, \quad \text{where}$$

$$B_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix}, B_3 = \begin{bmatrix} -1 & -2 & -1 \\ -2 & 0 & -2 \\ -1 & -2 & 1 \end{bmatrix},$$

$$B_4 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 1 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & -2 & -2 \\ -2 & -1 & 0 \\ -2 & 0 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}, B_8 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & -1 \end{bmatrix}, B_9 = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 1 & 0 \\ -2 & 0 & 2 \end{bmatrix},$$

$$B_{10} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

These B matrices satisfy the consistency conditions, and we can see $B_j + B_{11-j} = 0$, $(j = 1, \cdots, 5)$, namely $T_j^{-1} = T_{11-j}$. Then, let us consider the group G. When we choose a $B = B_1$ or $B = B_{10} = -B_1$, we can construct a finite group $G_1 =< \rho(S), \rho(T_1) >= (\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, whose order is 32. This is the same as $G_{10} =< \rho(S), \rho(T_{10}) >$. When we choose a $B = B_2$ or $B = B_9 = -B_2$, we can construct a finite group $G_2 =< \rho(S), \rho(T_2) >= T' \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$, whose order is 192. This is the same as $G_9 =< \rho(S), \rho(T_9) >$. When we choose a $B = B_3$ or $B = B_8 = -B_3$, we can construct a finite group $G_3 =< \rho(S), \rho(T_3) >= \mathbb{Z}_8$, whose order is 8. This is the same as $G_8 =< \rho(S), \rho(T_8) >$. When we choose a $B = B_i$, $(i = 4, 5, 6, 7)$, we cannot specify finite groups $G_i =< \rho(S), \rho(T_i) >$, whose order is 384. Lastly, when we choose a $B = B_1$ and B_2, we cannot specify finite groups $G_{1,2} =< \rho(S), \rho(T_1), \rho(T_2) >$, whose order is 768.

$\det N = -3$

Next we consider models with negative determinant $\det N < 0$. When we take $\det N = -3$, $U_N = \mathrm{diag}(-, +, +)$, we choose the following N-matrix

$$N = \begin{bmatrix} -3 & -3 & 0 \\ -3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.91}$$

We use the following T_i with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B_3 = -B_2, B_4 = -B_1. \tag{5.92}$$

When we choose a $B = B_i$, $(i = 1, \cdots, 4)$ or $B = B_1$ and B_3, we can construct a finite group $G_i =< \rho(S), \rho(T_i) >= T' \times \mathbb{Z}_8$, whose order is 192. Next we choose the following N-matrix

$$N = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}. \tag{5.93}$$

We use the T_i-generators with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 2 & -1 \\ 2 & 0 & 2 \\ -1 & 2 & -2 \end{bmatrix}, B_3 = \begin{bmatrix} -2 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & -2 \end{bmatrix}$$
$$B_4 = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & -2 \end{bmatrix}, B_5 = \begin{bmatrix} -2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & -2 \end{bmatrix}, B_6 = \begin{bmatrix} -2 & -2 & 0 \\ -2 & 2 & -2 \\ 0 & -2 & -2 \end{bmatrix},$$
$$B_7 = \begin{bmatrix} -2 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -2 \end{bmatrix}.$$

When we choose a $B = B_i$, $(i = 1, 2, 3, 5, 6)$, we can construct a finite group $G_i =< \rho(S), \rho(T_i) >= T' \times \mathbb{Z}_8$, whose order is 192. Lastly, when we choose a $B = B_i$, $(i = 4, 7)$, we can construct a finite group $G_i =< \rho(S), \rho(T_i) >= \mathbb{Z}_8$, whose order is 8.

$\det N = -4$

When we take $\det N = -4$, $U_N = \mathrm{diag}(-, +, +)$, we choose the following N-matrix

$$N = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -1 & 1 \end{bmatrix}. \tag{5.94}$$

We use the T_i-generators with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & -2 & -2 \\ -2 & -1 & -1 \\ -2 & -1 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$
$$B_4 = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix}, B_5 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, B_6 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & -2 \end{bmatrix},$$
$$B_7 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, B_8 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

When we choose a $B = B_i$, $(i = 1, 2, 7, 8)$, we can construct a finite group $G_i =< \rho(S), \rho(T_i) >= (\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, whose order is 32. When we choose a $B = B_i$, $(i = 3, 6)$, we cannot specify finite groups G_i, whose order is 384. When we choose a $B = B_i$, $(i = 4, 5)$, we can construct a finite group $G_i = T' \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_2)$, whose order is 192. Lastly, when we choose $B = B_1, B_3$, and B_4, we cannot specify

$$G_{1,3,4} =< \rho(S), \rho(T_1), \rho(T_3), \rho(T_4) >, \tag{5.95}$$

whose order is 768.

5.2.4 Magnetized T^4

Reduction from T^6 to T^4

According to Refs. [81,82], we can discuss magnetized T^4 models from magnetized T^6. In order to consider magnetized T^4, we introduce the magnetic fluxes on T^4. We particularly realize magnetized $T^4 \times T^2$. When we take $\tilde{N}'_{12} = 0$ and $q_2 = 0$, or $\theta_2 = 0$, we find the following formulae

$$\begin{aligned}
F^{\rm part}_{1\bar{1}} &= [F_{1\bar{1}} + F_{2\bar{2}} q_1^2 - 2F_{1\bar{2}} q_1] \cos^2\theta_1 = \frac{\hat{N}'}{\cos^2\theta_1}, \\
F^{\rm part}_{2\bar{2}} &= [2F_{1\bar{2}} q_1 + F_{1\bar{1}} q_1^2 + F_{2\bar{2}}] \cos^2\theta_1 = \frac{\tilde{N}'_{11}}{\cos^2\theta_1}, \\
F^{\rm part}_{3\bar{3}} &= F_{3\bar{3}} = \tilde{N}'_{22}, \\
F^{\rm part}_{2\bar{3}} &= [F_{1\bar{3}} q_1 + F_{2\bar{3}}] \cos\theta_1 = \frac{\tilde{N}'_{12}}{\cos\theta_1} = 0,
\end{aligned} \tag{5.96}$$

where $q_1 = -\tan\theta_1$ under $q_2 = 0$. Next, we calculate $F_{i\bar{j}} = PF^{\mathrm{part}}_{i\bar{j}}P^{-1}$ using Eq. (5.96). We can then derive

$$\begin{aligned}
F_{i\bar{j}} &= PF^{\mathrm{part}}_{i\bar{j}}P^{-1} \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\hat{N}'}{\cos^2\theta_1} & 0 & 0 \\ 0 & \frac{\tilde{N}'_{11}}{\cos^2\theta_1} & \frac{\tilde{N}'_{12}}{\cos\theta_1} \\ 0 & \frac{\tilde{N}'_{12}}{\cos\theta_1} & \tilde{N}'_{22} \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \hat{N}' + \tilde{N}'_{11}\tan^2\theta_1 & \hat{N}'\tan\theta_1 - \tilde{N}'_{11}\tan\theta_1 & 0 \\ \hat{N}'\tan\theta_1 - \tilde{N}'_{11}\tan\theta_1 & \hat{N}'\tan^2\theta_1 + \tilde{N}'_{11} & 0 \\ 0 & 0 & \tilde{N}'_{22} \end{bmatrix} \\
&= \begin{bmatrix} \hat{N}' + \tilde{N}'_{11}q_1^2 & -\hat{N}'q_1 + \tilde{N}'_{11}q_1 & 0 \\ -\hat{N}'q_1 + \tilde{N}'_{11}q_1 & \hat{N}'q_1^2 + \tilde{N}'_{11} & 0 \\ 0 & 0 & \tilde{N}'_{22} \end{bmatrix}.
\end{aligned}$$

In particular, the $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$ components of the $F_{i\bar{j}}$ can be regarded as T^4 part of magnetic fluxes. That is, we can find the magnetic fluxes on T^4 with $q_1 = -\tan\theta_1$, $q_2 = 0$, $\hat{N}'$, and $\tilde{N}'_{11}$ [55]

$$F^{T^4}_{i\bar{j}} = \hat{N}' \begin{bmatrix} 1 & -q_1 \\ -q_1 & q_1^2 \end{bmatrix} + \tilde{N}'_{11} \begin{bmatrix} q_1^2 & q_1 \\ q_1 & 1 \end{bmatrix}. \tag{5.97}$$

According to the fluxes in Eq. (5.97), spinors ψ_{+-} and ψ_{-+} on magnetized T^4 are excited simultaneously when we consider the nonzero off-diagonal components of the fluxes [55]. The number of zero modes has been studied in magnetized $T^4/\mathbb{Z}_N$ orbifold models. (See Refs. [80, 81]) In what follows, we will consider finite groups generated by magnetized T^4. We particularly focus on differences of chiralities between $(++)$ and $(+-)$. Similarly to the previous case on magnetized T^6, we will describe finite groups generated by $\rho(S)$ and $\rho(T)$.

Lastly, we impose the consistency condition under the T transformation, namely $(NB)^T = NB$ and $[NB]_{ii} \in 2\mathbb{Z}$. Imposing these conditions based on symmetries generally leads to physical constraints of the values of the fluxes. We can analyze the representation of the S transformation, $\rho_{IJ}(S)$, by the sign of $\det N$.

$\det N = +3$

When we take $\det N = +3$, $U_N = \mathrm{diag}(+,+)$, we choose the following N-matrix

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{5.98}$$

We use the T_i-generators with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$B_4 = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},$$
$$B_7 = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, B_8 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_9 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B_{10} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

When we choose a $B = B_i$, $(i = 1, 3, \cdots, 8, 10)$ or B_1, B_2, and B_3, we can construct a finite group $G = T'$, whose order is 24. When we choose a $B = B_i$, $(i = 2, 9)$, we can construct a finite group $G = \mathbb{Z}_4$.

$\det N = +4$

When we take $\det N = +4$, $U_N = \mathrm{diag}(+,+)$, we choose the following N-matrix

$$N = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}. \tag{5.99}$$

We use the T_i-generators with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix},$$
$$B_4 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, B_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B_6 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix},$$
$$B_7 = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}, B_8 = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}, B_9 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, B_{9+j} = -B_{10-j},\ (j = 1, \cdots 9).$$

We find a symmetric structure on B_j. When we choose a $B = B_i$, or B_{19-i}, $(i = 1, 3, 5, 7, 8)$ we can find a relation on finite groups $G_i =< \rho(S), \rho(T_i) >= G_{19-i}$.

For $i = 1$ we find

$$G_1 =< \rho(S), \rho(T_1) >= G_{18} = (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2. \tag{5.100}$$

For $i = 3, 8$ we find

$$G_3 =< \rho(S), \rho(T_3) >= G_{16} = G_8 = G_{11} = QD(16) = \mathbb{Z}_8 \rtimes \mathbb{Z}_2. \tag{5.101}$$

For $i = 5$ we find

$$G_5 =< \rho(S), \rho(T_5) >= G_{14} = A_4 \rtimes \mathbb{Z}_4. \tag{5.102}$$

For $i = 7$ we find

$$G_7 =< \rho(S), \rho(T_7) >= G_{12} = S_4 \times \mathbb{Z}_4. \tag{5.103}$$

For $i = 2, 4, 15, 17$, we cannot specify finite groups whose order is 640.

For $i = 6, 9, 10, 13$, we cannot specify finite groups whose order is 320.

When we take $B = B_1$ and B_3, we can construct finite group

$$G_{1,3} \equiv< \rho(S), \rho(T_1), \rho(T_3) >= \Sigma(32) \rtimes \mathbb{Z}_2 \simeq (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2), \tag{5.104}$$

whose order is 64. When we take $B = B_1, B_3, B_5$, and B_7, we cannot specify finite group whose order is 384.

$\det N = +5$

We choose the following N-matrix

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \tag{5.105}$$

We use the T_i-generators with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & -2 \\ -2 & -2 \end{bmatrix},$$
$$B_4 = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}, B_5 = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

When we take $B = B_i$, $(i = 1, \cdots 6)$, we cannot specify finite groups whose order is 480.

$\det N = +6$

We choose the following N-matrix

$$N = \begin{bmatrix} 10 & 2 \\ 2 & 1 \end{bmatrix} \tag{5.106}$$

We use the T_i-generators with B_i-matrices

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{5.107}$$

When we take $B = B_i$, $(i = 1, 2)$, we specify finite groups

$$T' \times D_8 \tag{5.108}$$

whose order is 384.

$\det N = -3$

Next we consider negative determinant of the N, namely $\det N < 0$. When we take $\det N = -3$ and $U_N = \mathrm{diag}(+,-)$, we choose the following N-matrix

$$N = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}. \tag{5.109}$$

We use the T_i-generators with B_i-matrices

$$\begin{aligned} B_1 &= \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_3 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \\ B_4 &= \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, B_6 = -B_5, \\ B_7 &= -B_4, B_8 = -B_3, B_9 = -B_2, B_{10} = -B_1. \end{aligned}$$

When we choose a $B = B_i$, $(i = 1, 2, 4, \cdots, 7, 910)$ or B_1, B_2, and B_3, we can construct a finite group $G = T' \times \mathbb{Z}_4$, whose order is 96. Lastly, when we choose a $B = B_i$, $(i = 3, 8)$, we can construct a finite group $G = \mathbb{Z}_4$.

$\det N = -4$

Next we consider models with $\det N = -4$ and $U_N = \mathrm{diag}(+,-)$. We choose the following N-matrix

$$N = \begin{bmatrix} 15 & -13 \\ -13 & 11 \end{bmatrix}. \tag{5.110}$$

We use the following T_i-generators

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{5.111}$$

When we choose a $B = B_i$, $(i = 1, 2$ or B_1 and B_2, we can construct a finite group $G = (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$, whose order is 64.

We choose the following N-matrix

$$N = \begin{bmatrix} 14 & -2 \\ -2 & 0 \end{bmatrix}. \tag{5.112}$$

We use the following T_i-generators

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{5.113}$$

When we choose a $B = B_i$, $(i = 1, 4)$, we can construct a finite group $G = D_4$, whose order is 8.

When we choose a $B = B_i$, $(i = 2, 3)$ or B_1, B_2, and B_3, we can construct a finite group $G = S_4$, whose order is 24.

We choose the following N-matrix

$$N = \begin{bmatrix} -2 & -4 \\ -4 & -6 \end{bmatrix}. \tag{5.114}$$

We use the following T_i-generators

$$\begin{aligned} &B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, B_3 = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}, \\ &B_5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B_6 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, B_7 = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, B_8 = \begin{bmatrix} 0 & -2 \\ -2 & -2 \end{bmatrix}, \\ &B_9 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, B_{9+j} = -B_{10-j}, (j = 1, 2, \cdots, 9). \end{aligned} \tag{5.115}$$

We dercribe the case studies. When we choose a $B = B_i$, $(i = 1, 18)$, we can construct a finite group $G = D_4$, whose order is 8. When we choose a $B = B_i$, $(i = 2, 4, 15, 17)$, we cannot specify finite groups whose order is 640. When we choose a $B = B_i$, $(i = 3, 8, 11, 16)$, we can specify finite group $G = D_8$ whose order is 16. When we choose a $B = B_i$, $(i = 5, 14)$, we can specify finite group $G = S_4$ whose order is 24. When we choose a $B = B_i$, $(i = 6, 9, 10, 13)$, we cannot specify finite group whose order is 320. When we choose a $B = B_i$, $(i = 7, 12)$ or B_1 and B_{12}, we can specify finite group $G = S_4 \times \mathbb{Z}_4$ whose order is 96. When we choose $B = B_1$ and B_3, we can specify finite group

$$G = \Sigma(32) \rtimes \mathbb{Z}_2 \simeq Q_4 \rtimes D_4 \simeq D_4 \rtimes D_4 \tag{5.116}$$

whose order is 64. Lastly, when we choose $B = B_1$, B_3, B_5 and B_7, we cannot specify finite group whose order is 384.

We choose the following N-matrix

$$N = \begin{bmatrix} -4 & 10 \\ 10 & -24 \end{bmatrix}. \tag{5.117}$$

We use the following T_i-generators

$$\begin{aligned} &B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} -2 & -2 \\ -2 & 2 \end{bmatrix}, B_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ &B_5 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, B_{6+j} = -B_{7-j}, (j = 1, 2, \cdots, 9). \end{aligned} \tag{5.118}$$

When we choose a $B = B_i$, $(i = 1, 12)$, we can construct a finite group $G = \mathbb{Z}_2$, whose order is 2. When we choose a $B = B_i$, $(i = 2, 6, 7, 11)$ or B_2 and B_3, we can construct a finite group $G = S_4 \times \mathbb{Z}_4$. When we choose a $B = B_i$, $(i = 3, 10)$, we can construct a finite group $G = D_4$. When we choose a $B = B_i$, $(i = 4, 9)$, we can construct a finite group $G = D_3$. When we choose a $B = B_i$, $(i = 5, 8)$, we can construct a finite group

$$G = (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4 \tag{5.119}$$

whose order is 64. When we choose $B = B_3$ and B_4, we can construct a finite group $G = S_4$. Lastly, when we choose $B = B_1, B_2, B_3,\ B_4,\ B_5,\ B_{12}$ we can construct a finite group $G = (\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_4$, whose order is 384.

$\det N = -5$

We choose the following N-matrix

$$N = \begin{bmatrix} 11 & 7 \\ 7 & 4 \end{bmatrix}. \tag{5.120}$$

We use the following T_i-generators

$$\begin{aligned} &B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & -2 \\ -2 & 2 \end{bmatrix}, \\ &B_5 = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned} \tag{5.121}$$

When we choose a $B = B_i$, $(i = 1, \cdots 6)$, we cannot specify finite groups whose order is 240.

$\det N = -6$

We choose the following N-matrix

$$N = \begin{bmatrix} 15 & -9 \\ -9 & 5 \end{bmatrix}. \tag{5.122}$$

We use the following T_i-generators

$$B_1 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{5.123}$$

Lastly, when we choose a $B = B_i$, $(i = 1, 2)$, we can specify a finite group $T' \times QD(16)$ whose order is 384.

In conclusion, we can see that various finite groups are realized in regardless of the sign of $\det N$. First we have performed numerical computations in magnetized $T^6/\mathbb{Z}_{12}$ model with the fixed T_i-generators by the algebraic structure. We find that we can obtain $\Delta(48) \rtimes (\mathbb{Z}_8 \times \mathbb{Z}_2)$ with the order 768 in not only $\det N = +4$ but $\det N = -4$. We have calculated details of $\rho(S)$ and $\rho(T)$. Then, we expect that the same finite groups can be realized in regardless of the 6D overall chirality because of Eq. (5.81).

Next, we have performed numerical computations in magnetized T^6 model with the $Sp(6, \mathbb{Z})$ modular symmetry, in particular, the consistency conditions

$$(NB)^T = NB \quad \text{and} \quad [NB]_{ii} \in 2\mathbb{Z} \tag{5.124}$$

for the modular T symmetry since we have supposed $p_{xx} = p_{yy} = 0$ in magnetic fluxes and the chiral wave functions with the vanishing Wilson lines (or equivalently the Scherk-Schwarz phases). When we take $|\det N| = 3$ where three-generation models can be realized, we obtain $T' \times \mathbb{Z}_8$ whose order is 192. For $|\det N| = 4$, we can at least obtain the following two finite groups

$$G = (\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2,\ T' \rtimes (\mathbb{Z}_4 \times \mathbb{Z}_2). \tag{5.125}$$

In magnetized T^4, there are definite differences of generated finite groups by the values and sign of $\det N$. For $\det N = +3$, we can obtain T' or $\mathbb{Z}_4$. On the other hand, we obtain $T' \times \mathbb{Z}_4$ or $\mathbb{Z}_4$ for $\det N = -3$. For $\det N = +4$, we can at least obtain the following finite groups

$$\begin{aligned} &(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2,\ \mathbb{Z}_8 \rtimes \mathbb{Z}_2,\ A_4 \rtimes \mathbb{Z}_4, \\ &S_4 \times \mathbb{Z}_4,\ \Sigma(32) \rtimes \mathbb{Z}_2 \simeq (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2). \end{aligned} \tag{5.126}$$

On the other hands, for $\det N = -4$, we can obtain the following groups

$$\begin{aligned} &\mathbb{Z}_2,\ (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4,\ D_3,\ D_4,\ S_4,\ D_8, \\ &S_4 \times \mathbb{Z}_4,\ \Sigma(32) \rtimes \mathbb{Z}_2 \simeq Q_4 \rtimes D_4 \simeq D_4 \rtimes D_4, \\ &(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_4. \end{aligned} \tag{5.127}$$

We just find a common group $S_4 \times \mathbb{Z}_4$ in $|\det N| = 4$. The others are different each other. We can also see the D_N discrete symmetry in $\det N = -4$ only. Since there are lots of combinations of the fluxes N, we would realize other interesting finite groups.

Chapter 6

Conclusion

In this paper, we have constructed magnetized T^6 model with fluxes N and complex structure moduli Ω, and considered the zero-mode wave functions. Our main purpose is to verify whether we can realize four-dimensional (4D) chiral effective field theory (EFT) with three-generation structure and 4D Yukawa couplings considering chirality and symmetries derived from nonfactorizable compact spaces such as T^6 or its $T^6/\mathbb{Z}_N$ orbifolds with background magnetic fluxes. Then, we have introduced the Ansatz $\psi_{+i} = \alpha_i \psi$, $(i = 1, 2, 3)$, and the notation $q_1 = \frac{\alpha_2}{\alpha_1}$ and $q_2 = -\frac{\alpha_3}{\alpha_1}$. When we introduce M and the Riemann condition $N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega \in \mathcal{H}^3$, the wave functions become well-defined and satisfy both the Dirac equation and the boundary conditions in T^6. We therefore obtain not only the spinor wave functions with positive chirality $(+++)$, but the wave functions with various chiralities in magnetized T^6. When we consider the $SO(6)$ rotation, we can also derive the $Spin(6)$ spinor rotation, which can lead to the values of the coefficients $\alpha_i \in \mathbb{C}$ and the notation $q_1, q_2 \in \mathbb{R}$. By use of the $R = N^{-1}M = PUP^{-1} \in SO(3)$ derived from the diagonalization for the magnetic fluxes, we can define z' system where the background magnetic fluxes become positive definite. In both the z system and the z' system, we find the Laplacian, its eigen equation, and the D-term SUSY condition. When we consider the modular transformations in the z system, we can also obtain the modular transformations of the wave functions under $Sp(6, \mathbb{Z})$ modular symmetry in the z' system. The modular transformations are important to analyze the number of zero modes in magnetized $T^6/\mathbb{Z}_N$.

We have also studied the number of zero modes in nonfactorizable $T^6/\mathbb{Z}_N$ orbifold models with background magnetic flux. We have constructed magnetized $T^6/\mathbb{Z}_N$ twisted orbifolds by generators of $Sp(6, \mathbb{Z})$, symmetric flux N, and complex structure moduli Ω. By use of the $Sp(6,\mathbb{Z})$ modular transformation, we have considered zero-mode numbers with the $\mathbb{Z}_N$ charge in magnetized $T^6/\mathbb{Z}_N$ models. First, we have focused on the zero modes with positive chirality $(+++)$ and verified whether three-generation models can be realized or not. By use of representation derived from modular transformations of the zero-mode wave functions, we can classify zero mode number in $\mathbb{Z}_N$ sector. For $T^6/\mathbb{Z}_7$, we do not obtain three-generation model when the tr $\rho \in \{+1,-1,-\sqrt{7}i\}$. This can be proved by properties of number theory. In addition, we have no solution satisfying both the F-term SUSY and the D-term SUSY conditions in magnetized $T^6/\mathbb{Z}_7$. For $T^6/\mathbb{Z}_{12}$, on the other hand, some three-generation models can be realized. These three-generation models can satisfy the F-term SUSY condition. Furthermore, three-generation models satisfying both the F-term SUSY and the D-term SUSY conditions can be realized. This can be important to consider 4D realistic models. Similarly to wave functions with positive chirality $(+++)$, we can also analyze the number of zero modes with various chiralities by using the z' system. Then, we have some three-generation models satisfying both the F-term SUSY and the D-term SUSY conditions in magnetized $T^6/\mathbb{Z}_{12}$. This can also be important to consider 4D realistic models including 4D Yukawa couplings. It is also important to verify whether the results are consistent with the Atiyah-Singer index theorem on orbifolds [83–85], e.g. through the blowup with localized flux and curvature [115–117].

On the other hands, we have also calculated overlap integral of left-handed, right-handed fermions, and Higgs sectors leading to 4D Yukawa couplings. We take both chirality and symmetries into account. We note that we can derive the wave functions with different chiralities in each sector. Since we can calculate the overlap integral explictly, it would be interesting if we can apply our results to realization of masses and mixing angles in quark and lepton.

Furthermore, we have considered its phenomenological applications such as Yukawa couplings, non-Abelian discrete symmetries. The overlap integral leading to Yukawa couplings is written by the Riemann-theta functions under $\tilde{\Omega}_Y \in \mathcal{H}^3$ and Eq. (5.22). We can also describe the overlap integral behaviors under $Sp(6,\mathbb{Z})$ modular symmetry when we focus on the chiralities on magnetized T^{2g}, $(g = 2,3)$, models. When taking the chiralities in left-handed, right-handed, and Higgs (Higgsino) sectors into account, we find that the overlap integral leading to Yukawa couplings can be holomorphic on Ω in 4D $\mathcal{N}=1$ SUSY EFT. Under 4D $\mathcal{N}=1$ SUSY EFT, we can also consider non-Abelian discrete symmetries derived from $Sp(2g,\mathbb{Z})$ modular symmetry on magnetized T^{2g}, $(g = 2,3)$, when taking chirality into account. We find that we can derive isomorphic groups between positive and negative chirality in magnetized T^6. We have found some non-Abelian discrete symmetries, that is, $\Delta(48) \rtimes (\mathbb{Z}_8 \times \mathbb{Z}_2)$, $T' \times \mathbb{Z}_8$, $(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, $T' \times (\mathbb{Z}_4 \times \mathbb{Z}_2)$ for $|\det N| = 3, 4$ in magnetized T^6. On the other hands, magnetized T^4 can also lead to some non-Abelian discrete symmetries, that is, T', $T' \times \mathbb{Z}_4$, $(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_8 \rtimes \mathbb{Z}_2$, $A_4 \rtimes \mathbb{Z}_4$, $S_4 \times \mathbb{Z}_4$, $\Sigma(32) \rtimes \mathbb{Z}_2 \simeq (\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$, $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_4$, D_3, D_4, S_4, D_8, $S_4 \times \mathbb{Z}_4$, $\Sigma(32) \rtimes \mathbb{Z}_2 \simeq Q_4 \rtimes D_4 \simeq D_4 \rtimes D_4$, $(\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_4$.

Lastly, we have discussed magnetized D-brane models compactified on T^6, its orbifolds $T^6/\mathbb{Z}_N$. Meanwhile, there are lots candidates for the compact space such as $K3$ surface, CY_3 manifold. It would be interesting to analyze localized modes in $T^4/\mathbb{Z}_N$ and $T^6/\mathbb{Z}_N$. On the other hands, non-invertible selection rules in magnetized D-brane models have been proposed (See Ref. [118]). We are not sure non-invertible selection rules on nonfactorizable $T^6/\mathbb{Z}_N$ or T^6/G_n, (G_n: non-Abelian groups [119]), with background magnetic fluxes. Then, by studying them, we may consider another realistic models in the future.

Appendix A

Landsberg-Schaar relation

We introduce the g-dimensional Landsberg-Schaar relation. This Appendix is along in Ref. [66]. First, we consider N and B matrices, satisfying

$$\begin{aligned} &N^T = N, \\ &B^T = B, \\ &(NB)^T = NB, \\ &(NB)_{ii} \in 2\mathbb{Z}, \end{aligned} \tag{A.1}$$

where $i, j \in \{1, 2, \cdots, g\}$. We use the Poisson resummation formula to derive the g-dimensional Landsberg-Schaar relation. Also, we introduce an infinitesimal positive number ϵ ($0 < \epsilon << 1$).

First, we introduce a function $f(A)$. This is important to derive the Poisson resummation formula with $g \times g$ symmetric matrix A,

$$f(A) = \sum_{K \in \mathbb{Z}^g} e^{-\pi^t KAK}. \tag{A.2}$$

The Poisson resummation formula is written by

$$f(A) = \frac{1}{\sqrt{\det A}} f(A^{-1}). \tag{A.3}$$

Next, we introduce following A^{-1}

$$A^{-1} = iB^{-1}N + \epsilon I_g. \tag{A.4}$$

In what follows, we will consider the limit $\epsilon \to +0$ when we derive the Landsberg-Schaar relation. Then, we do not consider higher-order terms of ϵ because they are irrelevant to the Landsberg-Schaar relation. Then, we obtain A

$$A = -iBN^{-1} + \epsilon B^2 N^{-2}. \tag{A.5}$$

On the other hands, we describe $f(A)$, that is,

$$
\begin{aligned}
f(A) &= \sum_{K\in\mathbb{Z}^g} e^{-\pi{}^tK(-iBN^{-1}+\epsilon B^2N^{-2})K} \\
&= \sum_{K\in\mathbb{Z}^g} e^{-\pi\epsilon{}^t(BN^{-1}K)(BN^{-1}K)} e^{\pi i{}^tKBN^{-1}K} \\
&= \sum_{L\in\Lambda_N}\sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon({}^tJB+{}^tLBN^{-1})(BJ+BN^{-1}L)} e^{\pi i({}^tJN+{}^tL)BN^{-1}(NJ+L)} \\
&= \sum_{L\in\Lambda_N} e^{\pi i{}^tLBN^{-1}L} \sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon{}^t(J+N^{-1}L)B^2(J+N^{-1}L)},
\end{aligned}
\tag{A.6}
$$

where we take $K = NJ + L$ ($J \in \mathbb{Z}^g, L \in \Lambda_N$) in the third equality. We also note $(NB)_{ii} \in 2\mathbb{Z}$ in the fourth equality. By taking the limit $\epsilon \to +0$, we finally derive

$$
\lim_{\epsilon\to+0} f(A) = \lim_{\epsilon\to+0} \frac{1}{|\det B|}\frac{1}{\epsilon^{g/2}} \sum_{L\in\Lambda_N} e^{\pi i{}^tLBN^{-1}L}, \tag{A.7}
$$

where we consider the Gaussian integral. Furthermore, we can rewrite $f(A^{-1})/\sqrt{\det A}$ as

$$
\begin{aligned}
&\frac{1}{\sqrt{\det A}} f(A^{-1}) \\
&= \frac{1}{\sqrt{\det A}} \sum_{K\in\mathbb{Z}^g} e^{-\pi{}^tK(iB^{-1}N+\epsilon I_g)K} \\
&= \frac{1}{\sqrt{\det A}} \sum_{K\in\mathbb{Z}^g} e^{-\pi\epsilon{}^tKK} e^{-\pi i{}^tKB^{-1}NK} \\
&= \frac{1}{\sqrt{\det A}} \sum_{L\in\Lambda_B}\sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon{}^t(BJ+L)(BJ+L)} e^{-\pi i{}^t(BJ+L)B^{-1}N(BJ+L)} \\
&= \frac{1}{\sqrt{\det A}} \sum_{L\in\Lambda_B} e^{-\pi i{}^tLB^{-1}NL} \sum_{J\in\mathbb{Z}^g} e^{-\pi\epsilon{}^t(J+B^{-1}L)B^2(J+B^{-1}L)},
\end{aligned}
\tag{A.8}
$$

where we take $K = BJ + L$ ($J \in \mathbb{Z}^g$, $L \in \Lambda_B$) in the third equality, and we use $(NB)_{ii} \in 2\mathbb{Z}$ in the fourth equality. By taking the limit $\epsilon \to +0$, we finally derive

$$
\lim_{\epsilon\to+0} \frac{1}{\sqrt{\det A}} f(A^{-1}) = \lim_{\epsilon\to+0} \sqrt{|\det N|}\frac{e^{\pi i(g+2(n_-^N-n_-^B))/4}}{\sqrt{|\det B|}} \frac{1}{|\det B|}\frac{1}{\epsilon^{g/2}} \sum_{L\in\Lambda_B} e^{-\pi i{}^tLB^{-1}NL}, \tag{A.9}
$$

where the Gaussian integral is considered. we also take $(\pm i)^{x/2} = e^{\pm\pi ix/4}$. We describe the numbers of negative eigenvalues of B (N) matrix, n_-^B (n_-^N), respectively.

Lastly the g-dimensional Landsberg-Schaar relation is written as

$$\frac{e^{-\pi i n_-^N/2}}{\sqrt{|\mathrm{det} N|}} \sum_{K \in \Lambda_N} e^{\pi i {}^t K N^{-1} B K} = \frac{e^{\pi i g/4} e^{-\pi i n_-^B/2}}{\sqrt{|\mathrm{det} B|}} \sum_{K \in \Lambda_B} e^{-\pi i {}^t K N B^{-1} K}. \tag{A.10}$$

Appendix B

Non-trivial scale factors in magnetized T^6

We consider magnetized T^6 model with the non-trivial scale factors. This Appendix is along in Ref. [81].

The following discussion is consistent with the discussion in the main text. We introduce the asymmetric fluxes N and complex structure moduli Ω under the F-term SUSY condition $(N\Omega)^T = N\Omega$ and $(\Omega^T\bar{\Omega})^T = \Omega^T\bar{\Omega}$. Specifically, when we consider complex coordinates on $\mathbb{C}^3$, Z^i

$$Z^i = 2\pi r_i z^i = 2\pi r_i (x^i + \Omega^{ik} y^k), \quad \bar{Z}^{\bar{i}} = 2\pi r_i \bar{z}^{\bar{i}} = 2\pi r_i (x^i + \bar{\Omega}^{ik} y^k), \tag{B.1}$$

where we denote the complex coordinates on T^6 as z^i, We can then introduce the flat metric g on $\mathbb{C}^3$

$$\begin{aligned} g &= \delta_{i,\bar{j}} dZ^i d\bar{Z}^{\bar{j}} \\ &= (2\pi r_i)^2 dz^i d\bar{z}^{\bar{i}} \\ &= (2\pi r_i)^2 \begin{bmatrix} dx^T & dy^T \end{bmatrix} \begin{bmatrix} \mathbf{1}_3 & \mathrm{Re}\Omega \\ (\mathrm{Re}\Omega)^T & (\mathrm{Re}\Omega)^T\mathrm{Re}\Omega + (\mathrm{Im}\Omega)^T\mathrm{Im}\Omega \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}. \end{aligned} \tag{B.2}$$

In particular, we take the non-trivial factors $2\pi r_i$ into account. Also, we note the background magnetic fluxes, the Dirac operator, and the Laplacian.

First, we can describe the background magnetic flux F under the non-trivial factors

$$\begin{aligned} F &= F_{i\bar{j}}(idz^i \wedge d\bar{z}^{\bar{j}}) \\ &= \frac{F_{i\bar{j}}}{(2\pi r_i)(2\pi r_j)}[i(2\pi r_i)dz^i \wedge (2\pi r_j)d\bar{z}^{\bar{j}}] \\ &\equiv G_{i\bar{j}}(idZ^i \wedge d\bar{Z}^{\bar{j}}), \end{aligned} \tag{B.3}$$

where $F_{i\bar{j}} = \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{ij}$, and $G_{i\bar{j}} = \frac{F_{i\bar{j}}}{(2\pi r_i)(2\pi r_j)}$. This is consistent with the Dirac quantization condition

$$\int_{T^6} dz^i d\bar{z}^{\bar{j}} F_{i\bar{j}} = \int_{T^6} dZ^i d\bar{Z}^{\bar{j}} G_{i\bar{j}} = 2\pi N^T. \tag{B.4}$$

Here, We can represent $G_{i\bar{j}}$ in magnetized T^6

$$\begin{aligned} G_{i\bar{j}} &\equiv \frac{[\pi \hat{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij}}{(2\pi r_i)(2\pi r_j)} + \frac{[\pi \tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij}}{(2\pi r_i)(2\pi r_j)} \\ &\equiv \hat{N}' \begin{bmatrix} 1 & -q_1 & -q_2 \\ -q_1 & q_1^2 & q_1 q_2 \\ -q_2 & q_1 q_2 & q_2^2 \end{bmatrix} \\ &+ \begin{bmatrix} \tilde{N}'_{11} q_1^2 + 2\tilde{N}'_{12} q_1 q_2 + \tilde{N}'_{22} q_2^2 & \tilde{N}'_{11} q_1 + \tilde{N}'_{12} q_2 & \tilde{N}'_{12} q_1 + \tilde{N}'_{22} q_2 \\ \tilde{N}'_{11} q_1 + \tilde{N}'_{12} q_2 & \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{12} q_1 + \tilde{N}'_{22} q_2 & \tilde{N}'_{12} & \tilde{N}'_{22} \end{bmatrix}, \end{aligned} \tag{B.5}$$

where $N = \hat{N} + \tilde{N}$. Then, we replace $F_{i\bar{j}}$ with $G_{i\bar{j}}$. We obtain $U(1)$ gauge fields $A_{Z^i}, A_{\bar{Z}^{\bar{i}}}$

$$D_{Z^i} = \partial_{Z^i} - iA_{Z^i}, \quad \bar{D}_{\bar{Z}^{\bar{i}}} = \partial_{\bar{Z}^{\bar{i}}} - iA_{\bar{Z}^{\bar{i}}}, \tag{B.6}$$

and

$$[D_{Z^i}, \bar{D}_{\bar{Z}^{\bar{j}}}] = G_{i\bar{j}}, \quad D_{Z^i} = \frac{1}{2\pi r_i} D_i, \quad \bar{D}_{\bar{Z}^{\bar{i}}} = \frac{1}{2\pi r_i} \bar{D}_i. \tag{B.7}$$

We can derive the wave functions in magnetized T^6 model with $2\pi r_i$ when we see the following replacements

$$\begin{aligned} &z^i \Rightarrow Z^i, \quad z^i + e^i \Rightarrow Z^i + e^i, \quad z^i + \Omega^{ij} e^j \Rightarrow Z^i + \Omega^{ij} e^j, \\ &F_{i\bar{j}} \Rightarrow G_{i\bar{j}}. \end{aligned} \tag{B.8}$$

For the boundary condition of the wave functions $\Phi_r(Z,\bar{Z})$, by using Z^i, $G_{i\bar{j}}$, N, we find

$$\begin{aligned}
&\Phi_r(Z^i+e^i,\bar{Z}^i+e^i)=e^{i[G_{i\bar{j}}\mathrm{Im}Z^j]}\Phi_r(Z^i,\bar{Z}^i),\\
&\Phi_r(Z^i+\Omega^{ij}e^j,\bar{Z}^i+\bar{\Omega}^{ij}e_j)=e^{i\mathrm{Im}[\bar{\Omega}_{ji}G_{j\bar{k}}Z^k]}\Phi_r(Z^i,\bar{Z}^i).
\end{aligned} \tag{B.9}$$

Then, we find the wave functions with various chiralities and the scale factors, $\Phi_r^J(Z,\bar{Z})$,

$$\Phi_r^J(Z,\bar{Z})=\mathcal{N}_r\cdot f_r(Z,\bar{Z})\cdot\Theta_r(Z,\bar{Z}), \tag{B.10}$$

where $\mathcal{N}_r$ denotes the normalization constant and

$$\begin{aligned}
f_r(Z,\bar{Z})&\equiv\exp\left[i\left[Z^i\frac{[\pi\hat{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij}}{(2\pi r_i)(2\pi r_j)}\mathrm{Im}Z^j+\bar{Z}^i\frac{[\pi\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{ij}}{(2\pi r_i)(2\pi r_j)}\mathrm{Im}Z^j\right]\right],\\
\Theta_r(Z,\bar{Z})&\equiv\theta\begin{bmatrix}J^TN^{-1}\\0\end{bmatrix}(\hat{N}Z+\tilde{N}\bar{Z},\hat{N}\Omega+\tilde{N}\bar{\Omega})\\
&=\theta\begin{bmatrix}J^TN^{-1}\\0\end{bmatrix}(NZ',N\Omega'),
\end{aligned} \tag{B.11}$$

with $J\in\Lambda_{N^T}$, $(N\Omega)^T=N\Omega$, and $N\Omega'\in\mathcal{H}^3$. The Dirac equation is written in the Dirac operator

$$i\not{D}_{r_i}\equiv i(\Gamma^{Z^i}D_{Z^i}+\Gamma^{\bar{Z}^{\bar{i}}}\bar{D}_{\bar{Z}^{\bar{i}}}),\quad i\not{D}_{r_i}\Psi=0. \tag{B.12}$$

On the other hand, we also define the Laplacian Δ_{r_i}

$$\Delta_{r_i}\equiv-2\sum_{i=1}^{3}\{D_{Z^i},\bar{D}_{\bar{Z}^{\bar{i}}}\}. \tag{B.13}$$

Then we find

$$\not{D}\Rightarrow\not{D}_{r_i},\quad\Delta\Rightarrow\Delta_{r_i}. \tag{B.14}$$

Then, we may derive the D-term SUSY conditions in magnetized T^6 with the non-trivial scales.

Appendix C

Boundary conditions in magnetized T^6

We consider the boundary conditions of wave functions with various chiralities. This Appendix is along in Ref. [81].

we can see that the wave functions $\psi = \psi^{j}_{N,M}$ ($j = J^T N^{-1}$, $J \in \Lambda_{N^T}$) in Eq. (3.38) satisfy the boundary conditions Eq. (3.21) under the F-term SUSY condition $(N\Omega)^T = N\Omega$ and the Riemann condition $N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega \in \mathcal{H}^3$. [1] For the boundary conditions of the e_k, ($k = 1, 2, 3$), we find

$$\begin{aligned}
\psi(z + e_k) &= \mathcal{N} \cdot e^{i\pi[N\mathrm{Re}(z+e_k)+iM\mathrm{Im}z]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z} \\
&\quad \cdot \sum_{m\in\mathbb{Z}^3} e^{\pi i(m+j)^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j)} e^{2\pi i(m+j)^T(N\mathrm{Re}z+iM\mathrm{Im}z)} \cdot e^{2\pi i(m+j)^T N e_k} \\
&= e^{i\pi e_k^T(N^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z)}\psi(z) \\
&= e^{i\chi_{e_k}(z)}\psi(z),
\end{aligned} \tag{C.1}$$

where $e^{2\pi i(m+j)^T N e_k} = 1$ for any e_k, ($k = 1, 2, 3$). Next, for the boundary conditions of the Ωe_k, ($k = 1, 2, 3$), the components $f(z, \bar{z})$ and $\hat{\Theta}(z, \bar{z})$ can be written as

$$\begin{aligned}
f(z + \Omega e_k, \bar{z} + \bar{\Omega} e_k) &= e^{i\pi[N\mathrm{Re}(z+\Omega e_k)+iM\mathrm{Im}(z+\Omega e_k)]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z+\Omega e_k)} \\
&= e^{i\pi[[\hat{N}(z+\Omega e_k)]^T+[\tilde{N}(\bar{z}+\bar{\Omega}e_k)]^T](\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z+\Omega e_k)} \\
&= e^{i\pi[\hat{N}z+\tilde{N}\bar{z}]^T e_k} \cdot e^{i\pi[(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z+\Omega e_k)} \cdot f(z, \bar{z}),
\end{aligned} \tag{C.2}$$

[1] We can verify that even if we introduce the asymmetric N and Ω under the $(1,1)$-form background magnetic fluxes and $(N\Omega)^T = N\Omega$.

$$
\begin{aligned}
&\hat{\Theta}(z+\Omega e_k, \bar{z}+\bar{\Omega} e_k) \\
&= \sum_m e^{\pi i(m+j)^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j)} e^{2\pi i(m+j)^T[\hat{N}(z+\Omega e_k)+\tilde{N}(\bar{z}+\bar{\Omega}e_k)]} \\
&= \sum_m e^{\pi i(m+j)^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j)} e^{2\pi i(m+j)^T[\hat{N}z+\tilde{N}\bar{z}]} e^{2\pi i(m+j)^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k} \\
&= \sum_m e^{\pi i(m+j+e_k)^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j+e_k)} e^{2\pi i(m+j+e_k)^T[\hat{N}z+\tilde{N}\bar{z}]} e^{2\pi i(m+j)^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k} \\
&\cdot e^{-\pi i e_k^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)e_k} \cdot e^{-2\pi i(m+j)^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k} \cdot e^{-2\pi i e_k^T(\hat{N}z+\tilde{N}\bar{z})} \\
&= e^{-\pi i e_k^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)e_k} \cdot e^{-2\pi i e_k^T(\hat{N}z+\tilde{N}\bar{z})} \cdot \hat{\Theta}(z,\bar{z}),
\end{aligned}
\tag{C.3}
$$

where $(N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega)^T = N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega$. We then obtain

$$
\begin{aligned}
&\psi(z+\Omega e_k) \\
&= e^{i\pi[\hat{N}z+\tilde{N}\bar{z}]^T e_k} \cdot e^{i\pi[(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z+\Omega e_k)} \cdot e^{-\pi i e_k^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)e_k} \\
&\cdot e^{-2\pi i e_k^T(\hat{N}z+\tilde{N}\bar{z})} \cdot \psi(z) \\
&= e^{-i\pi(N\mathrm{Re}z+iM\mathrm{Im}z)e_k} \cdot e^{\pi i(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z e_k} \cdot \psi(z) \\
&= e^{i\pi e_k^T[N(\mathrm{Re}\Omega(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z-\mathrm{Re}z)]}\psi(z) \\
&= e^{i\chi_{\Omega e_k}(z)}\psi(z).
\end{aligned}
$$

Therefore, we can see that the wave functions satisfy the boundary conditions.

Appendix D

Solutions of Dirac equation in magnetized T^6

We consider the Dirac equation and its solutions. In the main text, we take the Ansatz, that is, $\psi_{i+} = \alpha_i \psi$, $(i = 1, 2, 3)$. By use of Eq. (3.37), we verify that the wave functions in Eq. (3.38) are consistent with the Dirac equation Eq. (3.34). This Appendix is along in Ref. [81]. We introduce the Dirac equation and the $(1,1)$-form background magnetic fluxes $F = F_{z\bar{z}}(idz \wedge d\bar{z})$. According to Eq. (3.34), the Dirac equation is given by

$$\begin{cases} (\alpha_1 D_2 + \alpha_2 D_1)\psi = 0, \\ (\alpha_1 D_3 - \alpha_3 D_1)\psi = 0, \\ (\alpha_1 \bar{D}_1 - \alpha_2 \bar{D}_2 + \alpha_3 \bar{D}_3)\psi = 0. \end{cases} \tag{D.1}$$

To verify the consistency, let us take an example for $(\alpha_1 \bar{D}_1 - \alpha_2 \bar{D}_2 + \alpha_3 \bar{D}_3)\psi = 0$. We denote the wave functions as $\psi = \mathcal{N} \cdot f(z, \bar{z}) \cdot \hat{\Theta}(z, \bar{z})$. We then calculate $f(z, \bar{z})$ and $\hat{\Theta}(z, \bar{z})$, respectively. For $k = 1, 2, 3$, we take covariant derivative $\bar{D}_k = \bar{\partial}_k - i\bar{A}_k$, where $\bar{\partial}_k = \partial_{\bar{z}_k}$ and $\bar{A}_k = A_{\bar{k}}$:

$$\begin{aligned} \bar{D}_k \psi &= (\bar{\partial}_k - i\bar{A}_k)\psi \\ &= \mathcal{N} \cdot \bar{\partial}_k (f \cdot \hat{\Theta}) - i\bar{A}_k \psi \\ &= \mathcal{N}[(\bar{\partial}_k f) \cdot \hat{\Theta} + f \cdot (\bar{\partial}_k \hat{\Theta})] - i\bar{A}_k \psi, \end{aligned} \tag{D.2}$$

$$\begin{aligned} \bar{\partial}_k f &= f \cdot i\pi \bar{\partial}_k [(\hat{N}z)^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z - (\tilde{N}\bar{z})^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im}\bar{z}] \\ &= -f \cdot \pi i \left[\frac{1}{2i} [(\hat{N}z)^T (\mathrm{Im}\Omega)^{-1} + (\tilde{N}\bar{z})^T (\mathrm{Im}\Omega)^{-1}]_k + (\tilde{N}^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im}\bar{z})_k \right], \end{aligned} \tag{D.3}$$

where $\mathrm{Im}z = (z-\bar{z})/2i$, $\mathrm{Im}\bar{z} = -\mathrm{Im}z$. Then, we find

$$\begin{aligned}
\bar{\partial}_k \hat{\Theta}(z,\bar{z}) &= \sum_m e^{\pi i(m+j)^T(N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega)(m+j)} \cdot \bar{\partial}_k(e^{2\pi i(m+j)^T(\hat{N}z+\tilde{N}\bar{z})}) \\
&= \sum_m e^{\pi i(m+j)^T(N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega)(m+j)} \cdot e^{2\pi i(m+j)^T(\hat{N}z+\tilde{N}\bar{z})} \cdot \bar{\partial}_k(2\pi i(m+j)^T\tilde{N}\bar{z}) \\
&= 2\pi i \sum_m (m+j)^T_\ell \tilde{N}_{\ell k} e^{\pi i(m+j)^T(N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega)(m+j)} \cdot e^{2\pi i(m+j)^T(\hat{N}z+\tilde{N}\bar{z})} \\
&\equiv 2\pi i \sum_m (m+j)^T_\ell \tilde{N}_{\ell k} \cdot g(z,\bar{z}),
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
&\alpha_1\bar{\partial}_1\hat{\Theta} - \alpha_2\bar{\partial}_2\hat{\Theta} + \alpha_3\bar{\partial}_3\hat{\Theta} \\
&= 2\pi i \sum_m (m+j)^T_\ell(\alpha_1\tilde{N}_{\ell 1} - \alpha_2\tilde{N}_{\ell 2} + \alpha_3\tilde{N}_{\ell 3}) \cdot g(z,\bar{z}) \\
&= 2\pi i \sum_m (m+j)^T_k (\mathrm{Im}\Omega)_{kr} \\
&\cdot (\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{1r} - \alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{2r} + \alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{3r}) \cdot g(z,\bar{z}) \\
&= 0,
\end{aligned} \tag{D.5}$$

where we can directly verify a following identity using Eq. (3.37)

$$\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{1r} - \alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{2r} + \alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{3r} = 0. \tag{D.6}$$

Thus we find

$$
\begin{aligned}
&(\alpha_1\bar{D}_1-\alpha_2\bar{D}_2+\alpha_3\bar{D}_3)\psi\\
&=\mathcal{N}[\alpha_1\bar{\partial}_1 f-\alpha_2\bar{\partial}_2 f+\alpha_3\bar{\partial}_3 f]\cdot\hat{\Theta}-i(\alpha_1\bar{A}_1-\alpha_2\bar{A}_2+\alpha_3\bar{A}_3)\psi\\
&=-\mathcal{N}\cdot f\cdot\pi i\{\alpha_1\left[\frac{1}{2i}[z^T\hat{N}^T(\mathrm{Im}\Omega)^{-1}+\bar{z}^T\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_1+[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\bar{z}]_1\right]\\
&-\alpha_2\left[\frac{1}{2i}[z^T\hat{N}^T(\mathrm{Im}\Omega)^{-1}+\bar{z}^T\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_2+[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\bar{z}]_2\right]\\
&+\alpha_3\left[\frac{1}{2i}[z^T\hat{N}^T(\mathrm{Im}\Omega)^{-1}+\bar{z}^T\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_3+[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}\bar{z}]_3\right]\}\cdot\hat{\Theta}\\
&-i(\frac{\pi}{2}i[\alpha_1[z^TN^T(\mathrm{Im}\Omega)^{-1}]_1-\alpha_2[z^TN^T(\mathrm{Im}\Omega)^{-1}]_2+\alpha_3[z^TN^T(\mathrm{Im}\Omega)^{-1}]_3])\psi\\
&=-\frac{\pi}{2}\cdot\psi\cdot[\bar{z}_j^T(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j1}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j2}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j3})\\
&-z_j^T(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j1}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j2}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j3})]\\
&=0,
\end{aligned}
\tag{D.7}
$$

where we can also verify

$$
\begin{aligned}
&\bar{z}_j^T(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j1}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j2}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j3})\\
&-z_j^T(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j1}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j2}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j3})\\
&=(\bar{z}_j^T-z_j^T)(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j1}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j2}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{j3})\\
&=(-2i\mathrm{Im}z_1)\cdot(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{11}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{12}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{13})\\
&+(-2i\mathrm{Im}z_2)\cdot(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{21}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{22}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{23})\\
&+(-2i\mathrm{Im}z_3)\cdot(\alpha_1[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{31}-\alpha_2[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{32}+\alpha_3[\tilde{N}^T(\mathrm{Im}\Omega)^{-1}]_{33})\\
&=(-2i\mathrm{Im}z_1)\cdot 0+(-2i\mathrm{Im}z_2)\cdot 0+(-2i\mathrm{Im}z_3)\cdot 0\\
&=0
\end{aligned}
\tag{D.8}
$$

when we use $q_1=\alpha_2/\alpha_1$, $q_2=-\alpha_3/\alpha_1$, F, and Eqs. (3.37) and (D.6).

We thus can prove $(\alpha_1\bar{D}_1-\alpha_2\bar{D}_2+\alpha_3\bar{D}_3)\psi=0$. Similarly, we can also verify $(\alpha_1 D_2+\alpha_2 D_1)\psi=0$ and $(\alpha_1 D_3-\alpha_3 D_1)\psi=0$. Thus, the wave functions ψ in the main text satisfy both the boundary conditions and the Dirac equation in magnetized T^6.

Appendix E

Wave functions with SS phases and Wilson lines

We study magnetized T^6 model with the non-trivial Scherk-Schwarz (SS) phases and the complex Wilson lines. (See Refs. [66,81].) Note that we suppose that they are independent of both z and $\bar{z}$ in this Appendix.

Similarly to the main text, when we consider the $U(1)$ background magnetic fluxes in Eq. (3.16), they are independent of the complex Wilson lines η, $\bar{\eta}$. Then, We obtain the $U(1)$ background gauge potential in Eq. (3.17). Here, the complex Wilson lines are independent of both the complex coordinates z and their complex conjugate $\bar{z}$ because of the $U(1)$ gauge-invariant constant. We define following complex variables to describe wave functions with the non-trivial SS phases and the complex Wilson lines

$$z_w \equiv z + \eta, \quad \bar{z}_w \equiv \bar{z} + \bar{\eta}, \tag{E.1}$$

and stress that η is independent of both z and $\bar{z}$, namely

$$\frac{\partial \eta}{\partial z} = \frac{\partial \eta}{\partial \bar{z}} = \frac{\partial \bar{\eta}}{\partial z} = \frac{\partial \bar{\eta}}{\partial \bar{z}} = 0, \tag{E.2}$$

or equivalently

$$\frac{\partial z_w}{\partial z} = 1, \ \frac{\partial z_w}{\partial \bar{z}} = 0, \ \frac{\partial \bar{z}_w}{\partial \bar{z}} = 1, \ \frac{\partial \bar{z}_w}{\partial z} = 0. \tag{E.3}$$

Then, when we consider the covariant derivatives Eq. (3.10), we obtain following commutation relations which is the same as Eq. (3.18), namely

$$\begin{aligned}
[D_i, \bar{D}_j] &= F_{i\bar{j}}, \\
[D_i, D_j] &= F_{ij} = 0, \\
[\bar{D}_i, \bar{D}_j] &= F_{\bar{i}\bar{j}} = 0.
\end{aligned} \tag{E.4}$$

That is, even if the $U(1)$ gauge fields have the non-vanishing complex Wilson lines $\eta \neq \vec{0}$, the relation Eq. (3.18) is invariant. In particular, we prove $[D_i, \bar{D}_j] = F_{i\bar{j}}$ using an arbitrary smooth complex function $\hat{f} = \hat{f}(z, \bar{z})$:

$$\begin{aligned}
[D_j, \bar{D}_k]\hat{f} &= [\partial_{z^j} - iA_{z^j_w}, \partial_{\bar{z}^{\bar{k}}} - iA_{\bar{z}^{\bar{k}}_w}]\hat{f} \\
&= [(\partial_{z^j} - iA_{z^j_w})(\partial_{\bar{z}^{\bar{k}}} - iA_{\bar{z}^{\bar{k}}_w}) - (\partial_{\bar{z}^{\bar{k}}} - iA_{\bar{z}^{\bar{k}}_w})(\partial_{z^j} - iA_{z^j_w})]\hat{f} \\
&= [(\partial_{z^j}\partial_{\bar{z}^{\bar{k}}} - \partial_{\bar{z}^{\bar{k}}}\partial_{z^j})\hat{f} - i\partial_{z^j}(A_{\bar{z}^{\bar{k}}_w}\hat{f}) - iA_{z^j_w}\partial_{\bar{z}^{\bar{k}}}\hat{f} - A_{z^j_w}A_{\bar{z}^{\bar{k}}_w}\hat{f} \\
&\quad + i\partial_{\bar{z}^{\bar{k}}}(A_{z^j_w}\hat{f}) + iA_{\bar{z}^{\bar{k}}_w}\partial_{z^j}\hat{f} + A_{\bar{z}^{\bar{k}}_w}A_{z^j_w}\hat{f}] \\
&= [-i(\partial_{z^j}A_{\bar{z}^{\bar{k}}_w}) + i(\partial_{\bar{z}^{\bar{k}}}A_{z^j_w}) + [A_{\bar{z}^{\bar{k}}_w}, A_{z^j_w}]]\hat{f} \\
&= i(\partial_{\bar{z}^{\bar{k}}}A_{z^j_w} - \partial_{z^j}A_{\bar{z}^{\bar{k}}_w})\hat{f} \\
&= \frac{\pi}{2}\left[\partial_{\bar{z}^{\bar{k}}}\left[(N\bar{z}_w)^T(\mathrm{Im}\Omega)^{-1}\right]_j + \partial_{z^j}\left[(Nz_w)^T(\mathrm{Im}\Omega)^{-1}\right]_k\right] \\
&= \pi[N^T(\mathrm{Im}\Omega)^{-1}]_{jk} = F_{j\bar{k}},
\end{aligned} \tag{E.5}$$

where we have used the $U(1)$ gauge symmetry and Eq. (E.3). Similarly to Eq. (E.5), we can also prove the other commutation relations in Eq. (E.4). We note that since the Gamma matrices $\Gamma^{z^j}, \Gamma^{\bar{z}^j}$ are invariant under the non-vanishing complex Wilson lines, the Dirac operator $i\not{D}$ is also invariant. Thus, the Dirac equation is the same as Eq. (3.23). Furthermore, we consider the $U(1)$ gauge background $A(z, \bar{z}) = A(z)$ with the non-vanishing complex Wilson lines η satisfying the boundary conditions Eq. (3.19). When we introduce the non-trivial Scherk-Schwarz (SS) phases, we can obtain following boundary conditions [66]

$$\begin{aligned}
\psi(z + e_k) &= e^{2\pi i(\alpha^S)^T e_k} \cdot e^{i\zeta_{e_k}(z)}\psi(z), \\
\psi(z + \Omega e_k) &= e^{2\pi i(\beta^S)^T e_k} \cdot e^{i\zeta_{\Omega e_k}(z)}\psi(z),
\end{aligned} \tag{E.6}$$

where $\alpha^S = [\alpha^S_1, \alpha^S_2, \alpha^S_3]^T$ and $\beta^S = [\beta^S_1, \beta^S_2, \beta^S_3]^T$, $(0 \leq \alpha^S_i, \beta^S_i \leq 1)$, denote the SS phases for e_k, Ωe_k directions, respectively. We can replace non-vanishing Wilson lines with the non-trivial SS phases because we can redefine the SS phases through redefinitions of the wave functions and an appropriate gauge transformations.

Then, both the Dirac equation (3.23) and the commutation relations (3.18) are invariant under the non-trivial SS phases and Wilson lines. This means that we can consider background magnetic fluxes which are consistent with the Dirac equation even if we have the non-trivial (α^S, β^S) and the complex Wilson lines η. Therefore, wave functions with the non-trivial (α^S, β^S) and the η satisfy the Dirac equation on T^6. See the details in Appendix D.

When we consider the wave functions with positive chirality and the non-trivial SS phases satisfying both the Dirac equation and the boundary conditions Eq. (E.6), we can find the following solution (See Ref. [66]):

$$\psi_{+++}^{(J+\alpha^S,\beta^S)} = \mathcal{N} \cdot e^{-2\pi i (J+\alpha^S)^T N^{-1} \beta^S} \cdot e^{i\pi [Nz]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z} \cdot \theta \begin{bmatrix} (J+\alpha^S)^T N^{-1} \\ -(\beta^S)^T \end{bmatrix} (Nz, N\Omega). \tag{E.7}$$

Similarly to Eq. (3.26), we can also find periodicity of the fluxes N in the wave functions

$$\psi_N^{(\vec{J}+N^T\vec{e}_n+\alpha^S,\beta^S)} = \psi_N^{(\vec{J}+\alpha^S,\beta^S)}. \tag{E.8}$$

Additionally, by use of z_w derived from Eq. (E.1), we can obtain the wave functions with the non-trivial SS phases (α^S, β^S), the non-vanishing Wilson lines η, both of which are independent of complex coordinates $(z, \bar{z})$, and various chiralities in magnetized T^6, ψ:

$$\begin{aligned} \psi &= \mathcal{N}'_{(\alpha^S,\beta^S)} \cdot f(z_w, \bar{z}_w) \cdot \theta \begin{bmatrix} (J+\alpha^S)^T N^{-1} \\ -(\beta^S)^T \end{bmatrix} (Nz'_w, N\Omega') \\ &\equiv \mathcal{N}'_{(\alpha^S,\beta^S)} \cdot f(z_w, \bar{z}_w) \cdot \hat{\Theta}_{(\alpha^S,\beta^S)}(z_w, \bar{z}_w) \end{aligned}, \tag{E.9}$$

where $z_w = z + \eta$, $\bar{z}_w = \bar{z} + \bar{\eta}$, and

$$\begin{aligned} &\mathcal{N}'_{(\alpha^S,\beta^S)} = \mathcal{N} \cdot e^{-2\pi i (J+\alpha^S)^T N^{-1} \beta^S} \equiv \mathcal{N} \cdot e^{-2\pi i j_\alpha \beta^S} \equiv \mathcal{N}', \\ &j_{\alpha^S}^N \equiv (J+\alpha^S)^T N^{-1},\ \mathcal{N} : \text{normalization constant}, \\ &f(z_w, \bar{z}_w) \equiv e^{i\pi [Nz'_w]^T (\mathrm{Im}\Omega')^{-1} \mathrm{Im} z'_w} = e^{i\pi [Nz'_w]^T (\mathrm{Im}\Omega)^{-1} \mathrm{Im} z_w}, \\ &Nz'_w = N\mathrm{Re} z_w + iM\mathrm{Im} z_w = N(\mathrm{Re} z_w + iR\mathrm{Im} z_w), \\ &R = N^{-1}M \in SO(3), \quad \Omega' = \mathrm{Re}\Omega + iR\mathrm{Im}\Omega. \end{aligned} \tag{E.10}$$

Note that the coefficient $\mathcal{N}'_{(\alpha^S,\beta^S)}$ is independent of z and $\bar{z}$. Thus, the $\mathcal{N}'_{(\alpha^S,\beta^S)}$ is invariant under shift transformations $z \to z+e_k$ and $z \to z+\Omega e_k$, $(k = 1, 2, 3)$. The wave functions in Eq. (E.9) satisfy the Dirac equation (3.34) and the boundary conditions (E.6). Similarly to Appendices C and D, we can directly verify that.

On the boundary conditions for e_k, $(k=1,2,3)$ directions, we obtain

$$\begin{aligned}\psi(z+e_k) &= \mathcal{N}' \cdot e^{i\pi[N\mathrm{Re}(z_w+e_k)+iM\mathrm{Im}z_w]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z_w} \\ &\quad \cdot \sum_{m\in\mathbb{Z}^3} e^{\pi i(m+j^N_{\alpha^S})^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j^N_{\alpha^S})} \\ &\qquad \cdot e^{2\pi i(m+j^N_{\alpha^S})^T(N\mathrm{Re}z_w+iM\mathrm{Im}z_w-(\beta^S)^T)} \cdot e^{2\pi i(m+j^N_{\alpha^S})^T Ne_k} \\ &= e^{i\pi e_k^T(N^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z_w)} \cdot e^{2\pi i(\alpha^S)^T e_k} \cdot \psi(z) \\ &= e^{2\pi i(\alpha^S)^T e_k} \cdot e^{i\chi_{e_k}(z_w)} \cdot \psi(z),\end{aligned} \tag{E.11}$$

where we have used $j^N_{\alpha^S} = (J+\alpha^S)^T N^{-1}$ and $e^{2\pi i(m+j^N_{\alpha^S})^T Ne_k} = e^{2\pi i(\alpha^S)^T e_k}$ for any e_k, $(k=1,2,3)$.

Next we consider the boundary conditions for Ωe_k directions. Under $z \to z + \Omega e_k$, $(z_w \to z_w + \Omega e_k)$, when we calculate each component $f(z_w,\bar{z}_w)$, $\hat{\Theta}_{(\alpha^S,\beta^S)}(z_w,\bar{z}_w)$, we find

$$\begin{aligned}f(z_w+\Omega e_k, \bar{z}_w+\bar{\Omega}e_k) &= e^{i\pi[N\mathrm{Re}(z_w+\Omega e_k)+iM\mathrm{Im}(z_w+\Omega e_k)]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z_w+\Omega e_k)} \\ &= e^{i\pi[[\hat{N}(z_w+\Omega e_k)]^T+[\tilde{N}(\bar{z}_w+\bar{\Omega}e_k)]^T](\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z_w+\Omega e_k)} \\ &= e^{i\pi[\hat{N}z_w+\tilde{N}\bar{z}_w]^T e_k} \cdot e^{i\pi[(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z_w+\Omega e_k)} \\ &\quad \cdot f(z_w,\bar{z}_w),\end{aligned} \tag{E.12}$$

$$\begin{aligned}&\hat{\Theta}_{(\alpha^S,\beta^S)}(z_w+\Omega e_k, \bar{z}_w+\bar{\Omega}e_k) \\ &= \sum_m e^{\pi i(m+j^N_{\alpha^S})^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j^N_{\alpha^S})} e^{2\pi i(m+j^N_{\alpha^S})^T[\hat{N}(z_w+\Omega e_k)+\tilde{N}(\bar{z}_w+\bar{\Omega}e_k)-(\beta^S)^T]} \\ &= \sum_m e^{\pi i(m+j^N_{\alpha^S})^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j^N_{\alpha^S})} e^{2\pi i(m+j^N_{\alpha^S})^T[\hat{N}z_w+\tilde{N}\bar{z}_w-(\beta^S)^T]} \\ &\qquad \cdot e^{2\pi i(m+j^N_{\alpha^S})^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k} \\ &= \sum_m e^{\pi i(m+j^N_{\alpha^S}+e_k)^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(m+j^N_{\alpha^S}+e_k)} e^{2\pi i(m+j^N_{\alpha^S}+e_k)^T[\hat{N}z_w+\tilde{N}\bar{z}_w-(\beta^S)^T]} \\ &\qquad \cdot e^{2\pi i(m+j^N_{\alpha^S})^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k} e^{-\pi i e_k^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)e_k} \\ &\qquad \cdot e^{-2\pi i(m+j^N_{\alpha^S})^T(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k} \cdot e^{-2\pi i e_k^T(\hat{N}z_w+\tilde{N}\bar{z}_w-(\beta^S)^T)} \\ &= e^{-\pi i e_k^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)e_k} \cdot e^{-2\pi i e_k^T(\hat{N}z_w+\tilde{N}\bar{z}_w-(\beta^S)^T)} \cdot \hat{\Theta}_{(\alpha^S,\beta^S)}(z_w,\bar{z}_w),\end{aligned} \tag{E.13}$$

where we have used $(N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega)^T = N\mathrm{Re}\Omega + iM\mathrm{Im}\Omega$. We then find

$$
\begin{aligned}
&\psi(z+\Omega e_k) \\
&= e^{i\pi[\hat{N}z_w+\tilde{N}\bar{z}_w]^T e_k} \cdot e^{i\pi[(\hat{N}\Omega+\tilde{N}\bar{\Omega})e_k]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}(z_w+\Omega e_k)} \cdot e^{-\pi i e_k^T(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)e_k} \\
&\cdot e^{-2\pi i e_k^T(\hat{N}z_w+\tilde{N}\bar{z}_w-(\beta^S)^T)} \cdot \psi(z) \\
&= e^{-i\pi(N\mathrm{Re}z_w+iM\mathrm{Im}z_w)e_k} \cdot e^{\pi i[(N\mathrm{Re}\Omega+iM\mathrm{Im}\Omega)(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z_w]e_k} \cdot e^{2\pi i(\beta^S)^T e_k} \cdot \psi(z) \\
&= e^{i\pi e_k^T[N(\mathrm{Re}\Omega(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z_w-\mathrm{Re}z_w)]} \cdot e^{2\pi i(\beta^S)^T e_k} \cdot \psi(z) \\
&= e^{2\pi i(\beta^S)^T e_k} \cdot e^{i\chi_{\Omega e_k}(z)} \cdot \psi(z).
\end{aligned}
\tag{E.14}
$$

Therefore, when we consider the non-vanishing SS phases and the complex Wilson lines, the wave functions satisfy the boundary conditions Eq. (E.6) in magnetized T^6. Similarly, we can verify that the wave functions ψ in Eq. (E.9) are the solutions of the Dirac equation Eq. (3.34). We can verify whether the wave functions in Eq. (E.9) satisfy the Dirac equation by replacing z written in the wave functions Eq. (3.38) with $z_w = z + \eta$.

Appendix F

Relation between z and z' systems

In this paper, we have defined z' system as coordinate in which the fluxes components $F_{z'\bar{z}'}$ are positive definite [81, 82]. Then, by use of the $R = N^{-1}M = PUP^{-1} \in SO(3)$, z' and its complex conjugate $\bar{z}'$ take of the forms

$$\begin{aligned} z' &\equiv \mathrm{Re}z + iR\mathrm{Im}z = \frac{\mathbf{1}_3 + R}{2} z + \frac{\mathbf{1}_3 - R}{2} \bar{z}, \\ \bar{z}' &\equiv \mathrm{Re}z - iR\mathrm{Im}z = \frac{\mathbf{1}_3 - R}{2} z + \frac{\mathbf{1}_3 + R}{2} \bar{z}, \end{aligned} \tag{F.1}$$

which implies that we can derive $(z', \bar{z}')$ basis from $(z, \bar{z})$ basis. Since we obtain $\mathrm{Re}z' = \mathrm{Re}z$ and $\mathrm{Im}z' = R\mathrm{Im}z$, when we consider $(\mathrm{Re}z, \mathrm{Im}z)$ basis, we can find

$$\begin{bmatrix} \mathrm{Re}z \\ \mathrm{Im}z \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{1}_3 & O_3 \\ O_3 & R \end{bmatrix} \begin{bmatrix} \mathrm{Re}z \\ \mathrm{Im}z \end{bmatrix} \equiv \begin{bmatrix} \mathrm{Re}z' \\ \mathrm{Im}z' \end{bmatrix}. \tag{F.2}$$

This is (proper) $SO(6)$ rotation in the $(\mathrm{Re}z, \mathrm{Im}z)$ basis because $R \in SO(3)$. That is, $(\mathrm{Re}z', \mathrm{Im}z')$ basis can be derived from the $SO(6)$ rotation in the $(\mathrm{Re}z, \mathrm{Im}z)$ basis. Similarly, we can define $\Omega' = \mathrm{Re}\Omega + iR\mathrm{Im}\Omega$. On the other hand, we have introduced the following fluxes F,

$$F = F_{z\bar{z}}(idz \wedge d\bar{z}) = F_{z'\bar{z}'}(idz' \wedge d\bar{z}'). \tag{F.3}$$

By use of Eq. (F.1), we can find

$$dz' \wedge d\bar{z}' = R(dz \wedge d\bar{z}). \tag{F.4}$$

Moreover, when we consider the Hermitian Yang-Mills equation in the z system

$$F_{zz} = F_{\bar{z}\bar{z}} = 0, \tag{F.5}$$

we find the corresponding equation in the z' system, that is,

$$F_{z'z'} = F_{\bar{z}'\bar{z}'} = 0, \tag{F.6}$$

where $F_{z'z'}$, $F_{\bar{z}'\bar{z}'}$ denotes components of the $(2,0)$, $(0,2)$-forms, respectively, in the z' system. This means that when we consider the $(1,1)$-form fluxes in the z system, the fluxes also take of the $(1,1)$-form in the z' system. This also leads to the F-term SUSY condition in the z' system, namely $(N\Omega')^T = N\Omega'$. Furthermore, since we have defined the z' system as coordinate realizing $F_{z'\bar{z}'} > 0$, we can automatically derive the Riemann condition $N\Omega' \in \mathcal{H}^3$ if we impose the Hermitian Yang-Mills equation in the $(z, \bar{z})$ basis. Lastly, when we consider the rotation such as $\mathbb{Z}_N$-twist in the $(\mathrm{Re}z, \mathrm{Im}z)$ basis, the corresponding rotation can be derived in the $(\mathrm{Re}z', \mathrm{Im}z')$ basis.

Appendix G

$Sp(2g,\mathbb{Z})$ group and modular transformations

We consider the details of the modular transformations in magnetized T^6. This Appendix is along in Refs. [78,80]. According to Refs. [78,80,81,105], the symplectic modular group $Sp(2g,\mathbb{Z})$ is defined by the following $2g \times 2g$ integer matrices:

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{G.1}$$

satisfying

$$\gamma J \gamma^T = J, \quad J = \begin{bmatrix} O_g & \mathbf{1}_g \\ -\mathbf{1}_g & O_g \end{bmatrix}. \tag{G.2}$$

A, B, C, D denote $g \times g$ integer symmetric matrices, and O_g, $\mathbf{1}_3$ correspond to $g \times g$ zero, unit matrices, respectively.

The generators S and T_i, $(i = 1, 2, \cdots, \frac{g}{2}(g+1))$, are represented by

$$S = \begin{bmatrix} O_g & \mathbf{1}_g \\ -\mathbf{1}_g & O_g \end{bmatrix}, \quad T_i = \begin{bmatrix} \mathbf{1}_g & B_i \\ O_g & \mathbf{1}_g \end{bmatrix}. \tag{G.3}$$

For $g = 2$, we can represent the B_i as

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{G.4}$$

For $g=3$, we can also represent the B_i as

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{G.5}$$

$$B_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, B_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{G.6}$$

The complex coordinates z and complex structure moduli Ω in T^{2g} are transformed as

$$z \to (C\Omega + D)^{-1} z, \tag{G.7}$$

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1}, \tag{G.8}$$

respectively, under the $Sp(2g,\mathbb{Z})$ modular symmetry. In this Appendix, we consider the wave functions with positive chirality in z system, ψ_{+++}. We can also apply the following discussions to the wave functions with various chiralities.

G.1 S-transformation

We represent the S generator by

$$S = \begin{bmatrix} O_3 & \mathbf{1}_3 \\ -\mathbf{1}_3 & O_3 \end{bmatrix}. \tag{G.9}$$

Then, $\vec{z} = \vec{x} + \Omega \vec{y}$ and Ω are transformed as

$$S : (\vec{z}, \Omega) \to (\vec{z}_S, \Omega_S) = (-\Omega^{-1}\vec{z}, -\Omega^{-1}), \tag{G.10}$$

under the S transformation. We define the complex coordinates, the moduli after the S transformation as $\vec{z}_S = -\Omega^{-1}\vec{z} = \vec{x}_S - \Omega^{-1}\vec{y}_S$ and $\Omega_S = -\Omega^{-1}$.

G.1.1 Magnetic fluxes under S-transformation

We describe magnetic flux on T^6

$$\begin{aligned} F &= \frac{1}{2} p_{IJ} dX^I \wedge dX^J \\ &= \frac{1}{2}(p_{xx})_{ij} dx^i \wedge dx^j + \frac{1}{2}(p_{yy})_{ij} dy^i \wedge dy^j + (p_{xy})_{ij} dx^i \wedge dy^j, \end{aligned} \tag{G.11}$$

where $X^I = (x^i, y^i)$, $i = 1, 2, 3$ denotes the real coordinate. Since the magnetic flux F after the S transformation can be given by $\vec{x}_S = -\vec{y}$ and $\vec{y}_S = \vec{x}$

$$
\begin{aligned}
F &= \frac{1}{2}(p^S_{xx})_{ij}dx^i_S \wedge dx^j_S + \frac{1}{2}(p^S_{yy})_{ij}dy^i_S \wedge dy^j_S + (p^S_{xy})_{ij}dx^i_S \wedge dy^j_S \\
&= \frac{1}{2}(p^S_{xx})_{ij}dy^i \wedge dy^j + \frac{1}{2}(p^S_{yy})_{ij}dx^i \wedge dx^j + (p^S_{xy})^T_{ij}dx^i \wedge dy^j,
\end{aligned} \tag{G.12}
$$

we obtain

$$
\begin{aligned}
p^S_{xx} &= p_{yy}, \\
p^S_{yy} &= p_{xx}, \\
p^S_{xy} &= (p_{xy})^T.
\end{aligned} \tag{G.13}
$$

In what follows, we suppose $p_{xx} = p_{yy} = 0$ for simplicity. Then, we find that the flux $N = \frac{p^T_{xy}}{2\pi}$ under S becomes

$$
S : N \to N^T. \tag{G.14}
$$

We can verify that the F-term SUSY condition $(N\Omega)^T = N\Omega$ satisfies under the S transformation and symmetric Ω

$$
\begin{aligned}
(N_S\Omega_S)^T &= (N^T(-\Omega^{-1}))^T \\
&= ((\Omega^T)^{-1}N\Omega(-\Omega^{-1}))^T \\
&= N_S\Omega_S.
\end{aligned} \tag{G.15}
$$

This implies the consistency between the F-term SUSY condition and the S transformation.

G.1.2 S-transformation of zero modes

In the main text, we have introduced the zero-mode wave functions with positive chirality $(+++)$

$$
\psi^{\vec{J},N}(\vec{z},\Omega) = N \cdot e^{i\pi(N\vec{z})^T(Im\Omega)^{-1}\cdot Im(\vec{z})} \cdot \theta \begin{bmatrix} \vec{J}N^{-1} \\ 0 \end{bmatrix} (N\vec{z}, N\Omega). \tag{G.16}
$$

In what follows, we consider the S transformation under the condition $N = N^T$. This is because the magnetized $T^6/\mathbb{Z}_N$ models in the main text implies that we should restrict the symmetric fluxes N. In conclusion, the zero modes $\psi^{\vec{J},N}(\vec{z},\Omega)$ behave as

$$\begin{aligned}\psi_N^{\vec{J}}(-\Omega^{-1}\vec{z},-\Omega^{-1}) &= \sqrt{\det(N^{-1}\Omega/i)}\sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^T N^{-1}\vec{K}}\psi_N^{\vec{K}}(\vec{z},\Omega)\\ &= \frac{\sqrt{\det(-i\Omega)}}{\sqrt{\det N}}\sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^T N^{-1}\vec{K}}\psi_N^{\vec{K}}(\vec{z},\Omega), \end{aligned} \tag{G.17}$$

under the S transformation. Here we denote Λ_N as lattice spanned by $N\vec{e}_i$.

In what follows, we prove Eq. (G.17). According to Ref. [105], we find

$$\theta(-\Omega^{-1}\vec{z},-\Omega^{-1}) = \sqrt{\det(\Omega/i)}e^{\pi i\vec{z}^T\Omega^{-1}\vec{z}}\cdot\theta(\vec{z},\Omega), \tag{G.18}$$

under $\Omega\in\mathcal{H}_3$.

Next, we replace $\vec{z}$ with $\vec{z}+N^{-1}\vec{J}$ in Eq. (G.18). We then rewrite

$$\theta\begin{bmatrix}\vec{J}^T N^{-1}\\ \vec{0}\end{bmatrix}(-\Omega^{-1}\vec{z},-\Omega^{-1}) = \sqrt{\det(\Omega/i)}\cdot e^{i\pi\vec{z}^T\Omega^{-1}\vec{z}}\cdot\theta\begin{bmatrix}\vec{0}\\ \vec{J}^T N^{-1}\end{bmatrix}(\vec{z},\Omega). \tag{G.19}$$

Here, we have used the following relations

$$\begin{aligned}\theta\begin{bmatrix}\vec{a}\\ \vec{b}+\vec{b}'\end{bmatrix}(\vec{z},\Omega) &= \theta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}+\vec{b}',\Omega),\\ \theta\begin{bmatrix}\vec{a}+\vec{a}'\\ \vec{b}\end{bmatrix}(\vec{z},\Omega) &= e^{i\pi\vec{a}'^T\Omega\vec{a}'+2\pi i\vec{a}'^T(\vec{z}+\vec{b})}\theta\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}(\vec{z}+\Omega\vec{a}',\Omega),\end{aligned} \tag{G.20}$$

where $\vec{a}',\vec{b}'\in\mathbb{R}^3$.

Then, we replace $\Omega\in\mathcal{H}_3$ with $N^{-1}\Omega_I$, where N denotes the flux. Then, the Riemann Theta-functions are well-defined. Thus, we can see

$$\begin{aligned}&\theta\begin{bmatrix}\vec{J}^T N^{-1}\\ \vec{0}\end{bmatrix}(-\Omega^{-1}N\vec{z},-\Omega^{-1}N)\\ &= \sqrt{\det(N^{-1}\Omega/i)}\cdot e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}}\cdot\theta\begin{bmatrix}\vec{0}\\ \vec{J}^T N^{-1}\end{bmatrix}(\vec{z},N^{-1}\Omega).\end{aligned} \tag{G.21}$$

The right-hand side of Eq. (G.21) becomes

$$\begin{aligned}\theta\begin{bmatrix}\vec{0}\\ \vec{J}^T N^{-1}\end{bmatrix}(\vec{z}, N^{-1}\Omega) &= \sum_{\vec{\ell}\in\mathbb{Z}^3} e^{\pi i\vec{\ell}^T N^{-1}\Omega\vec{\ell}} e^{2\pi i\vec{\ell}^T(\vec{z}+N^{-1}\vec{J})} \\ &= \sum_{\vec{K}\in\Lambda_N}\sum_{\vec{a}\in\mathbb{Z}^3} e^{\pi i(N\vec{a}+\vec{K})^T N^{-1}\Omega(N\vec{a}+\vec{K})} e^{2\pi i(N\vec{a}+\vec{K})^T(\vec{z}+N^{-1}\vec{J})} \\ &= \sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{K}^T N^{-1}\vec{J}} \sum_{\vec{a}\in\mathbb{Z}^3} e^{\pi i(N\vec{a}+\vec{K})^T N^{-1}\Omega(N\vec{a}+\vec{K})} e^{2\pi i(N\vec{a}+\vec{K})^T\vec{z}} \\ &= \sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^T N^{-1}\vec{K}}\theta\begin{bmatrix}\vec{K}^T N^{-1}\\ \vec{0}\end{bmatrix}(N\vec{z}, N\Omega),\end{aligned} \tag{G.22}$$

where the summation variable $\vec{l}$ can be expressed by two variables $\vec{a}\in\mathbb{Z}^3$ and $\vec{K}\in\mathbb{Z}^3$. That is, $\vec{l} = N\vec{a}+\vec{K}$, $(\vec{K}\in\Lambda_N)$. We note that the N, Ω^{-1} are commutative because of $[N,\Omega]=0$. Then, we can express Eq. (G.21)

$$\begin{aligned}&\theta\begin{bmatrix}\vec{J}^T N^{-1}\\ \vec{0}\end{bmatrix}(N(-\Omega^{-1}\vec{z}), N(-\Omega^{-1})) \\ &= \sqrt{\det(N^{-1}\Omega/i)}\cdot e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^T N^{-1}\vec{K}}\cdot\theta\begin{bmatrix}\vec{K}^T N^{-1}\\ \vec{0}\end{bmatrix}(N\vec{z}, N\Omega).\end{aligned} \tag{G.23}$$

We can calculate the phase of zero modes. That is, we can prove

$$S: e^{i\pi(N\vec{z})^T\cdot(Im\Omega)^{-1}Im(\vec{z})} \to e^{\pi i(-N\Omega^{-1}\vec{z})^T(Im(-\Omega^{-1}))^{-1}Im(-\Omega^{-1}\vec{z})}. \tag{G.24}$$

Therefore we obtain

$$e^{\pi i(-N\Omega^{-1}\vec{z})^T(Im(-\Omega^{-1}))^{-1}Im(-\Omega^{-1}\vec{z})} e^{i\pi\vec{z}^T(N^{-1}\Omega)^{-1}\vec{z}} = e^{i\pi(N\vec{z})^T(Im\Omega)^{-1}Im\vec{z}}. \tag{G.25}$$

This leads to the S transformation of zero modes

$$\psi_N^{\vec{J}}(-\Omega^{-1}\vec{z}, -\Omega^{-1}) = \frac{\sqrt{\det(-i\Omega)}}{\sqrt{\det N}} \sum_{\vec{K}\in\Lambda_N} e^{2\pi i\vec{J}^T N^{-1}\vec{K}}\psi_N^{\vec{K}}(\vec{z},\Omega). \tag{G.26}$$

G.2 T-transformation of zero modes

The T_i transformations correspond to a shift of the lattice vector or complex structure moduli in the $\mathbb{C}^3$. We describe the T_i generator

$$T_i = \begin{bmatrix} \mathbf{1}_3 & B_i \\ O_3 & \mathbf{1}_3 \end{bmatrix}, \quad \text{where} \quad i = 1, \cdots 6. \tag{G.27}$$

Similarly to the S transformations, the complex coordinates $\vec{z}$ and complex structure moduli Ω behave as

$$T : (\vec{z}, \Omega) \to (\vec{z}_T, \Omega_T) = (\vec{z}, \Omega + B_i), \tag{G.28}$$

under the T_i transformations. When we define $\vec{z}_T$ as complex coordinates after T transformation and omit the index i, we obtain

$$\begin{aligned} &\vec{z} = \vec{x} + \Omega\vec{y} \to \vec{z}_T = \vec{x}_T + (\Omega + B)\vec{y}_T = \vec{z} = \vec{x} + \Omega\vec{y} \\ &\leftrightarrow \vec{x}_T = \vec{x} - B\vec{y}, \quad \vec{y}_T = \vec{y}. \end{aligned} \tag{G.29}$$

After the T transformation, the magnetic flux F can be written as

$$\begin{aligned} F &= \frac{1}{2}(p_{xx}^{(T)})_{ij} dx_T^i \wedge dx_T^j + \frac{1}{2}(p_{yy}^{(T)})_{ij} dy_T^i \wedge dy_T^j + (p_{xy}^{(T)})_{ij} dx_T^i \wedge dy_T^j \\ &= \frac{1}{2}(p_{xx}^{(T)})_{ij} dx^i \wedge dx^j + \frac{1}{2}[p_{yy}^{(T)} + Bp_{xx}^{(T)}B - (Bp_{xy}^{(T)}) + (Bp_{xy}^{(T)})^T]_{ij} dy^i \wedge dy^j \\ &+ [(p_{xy}^{(T)}) - (p_{xx}^{(T)}B)]_{ij} dx^i \wedge dy_j. \end{aligned} \tag{G.30}$$

From Eq. (G.30), the components p_{xx}, p_{yy}, p_{xy} should satisfy the following relations

$$\begin{aligned} p_{xx}^{(T)} &= p_{xx}, \\ p_{yy}^{(T)} &= p_{yy} + Bp_{xx}B + (Bp_{xy} - (Bp_{xy}^{(T)})^T), \\ p_{xy}^{(T)} &= p_{xy} - (p_{xx}B). \end{aligned} \tag{G.31}$$

Thus, the condition $p_{xx} = p_{yy} = 0$ leads to the following relation

$$(Bp_{xy})^T = Bp_{xy}. \tag{G.32}$$

This is consistent with the T transformation.

Then, we find

$$(NB)^T = NB. \tag{G.33}$$

This is because we should introduce both the Dirac quantization condition $p_{xy} = 2\pi N^T$ and the symmetric B. We find the F-term SUSY condition after the T transformation

$$(N_T \Omega_T)^T = (\Omega + B)^T N^T = (N\Omega)^T + (NB)^T = N\Omega + NB = N_T \Omega_T, \tag{G.34}$$

which implies that the F-term SUSY condition is consistent with the T transformation.

Next we consider the T transformation of zero modes. Here, all diagonal components of NB should be even because of the consistency with the T symmetry

$$\begin{aligned}
&\theta(N\vec{z}, N(\Omega + B)) \\
&= \sum_{\vec{m} \in \mathbb{Z}^3} \exp(\pi i \vec{m}^T N(\Omega + B)\vec{m} + 2\pi i \vec{m}^T N\vec{z}) \\
&= \sum_{\vec{m} \in \mathbb{Z}^3} e^{\pi i \vec{m}^T N\Omega\vec{m} + 2\pi i \vec{m}^T N\vec{z}} e^{\pi i \vec{m}^T NB\vec{m}} \\
&= \sum_{\vec{m} \in \mathbb{Z}^3} e^{\pi i \vec{m}^T N\Omega\vec{m} + 2\pi i \vec{m}^T N\vec{z}} \cdot 1 \\
&= \theta(N\vec{z}, N\Omega).
\end{aligned} \tag{G.35}$$

By replacing $\vec{z}$ with $\vec{z} + (\Omega + B)N^{-1T}\vec{J}$, we find

$$\theta(N(\vec{z} + (\Omega + B)N^{-1T}\vec{J}), N(\Omega + B)) = \theta(N(\vec{z} + (\Omega + B)N^{-1T}\vec{J}), N\Omega). \tag{G.36}$$

Eq. (G.20) leads to the following formula

$$\begin{aligned}
\theta \begin{bmatrix} \vec{J}^T N^{-1} \\ \vec{0} \end{bmatrix} (N\vec{z}, N(\Omega + B)) &= e^{-\pi i \vec{J}^T N^{-1} B\vec{J}} \theta \begin{bmatrix} \vec{J}^T N^{-1} \\ \vec{0} \end{bmatrix} (N\vec{z} + B\vec{J}, N\Omega) \\
&= e^{-\pi i \vec{J}^T N^{-1} B\vec{J}} \theta \begin{bmatrix} \vec{J}^T N^{-1} \\ \vec{J}^T B \end{bmatrix} (N\vec{z}, N\Omega) \\
&= e^{\pi i \vec{J}^T N^{-1} B\vec{J}} \theta \begin{bmatrix} \vec{J}^T N^{-1} \\ \vec{0} \end{bmatrix} (N\vec{z}, N\Omega).
\end{aligned} \tag{G.37}$$

On the other hands, the phase factor $e^{\pi i (N\vec{z})^T (Im\Omega)^{-1} Im\vec{z}}$ written in the zero modes is invariant under the T transformation. Therefore, the T transformation of zero modes is written by

$$\psi_N^{\vec{J}}(\vec{z}, \Omega + B) = e^{\pi i \vec{J}^T N^{-1} B\vec{J}} \psi_N^{\vec{J}}(\vec{z}, \Omega). \tag{G.38}$$

G.3 A-transformation

We consider a following matrix γ with $A \in GL(3,\mathbb{Z})$ [80]:

$$\gamma = \begin{bmatrix} A & 0 \\ 0 & (A^{-1})^T \end{bmatrix}. \tag{G.39}$$

Transformation using the above γ is called the A-transformation. Under the A-transformation, complex coordinates and complex structure moduli are transformed as

$$\begin{aligned} \gamma &: \vec{z} \to A\vec{z}, \\ \gamma &: \Omega \to A\Omega A^T. \end{aligned} \tag{G.40}$$

In real basis of the complex coordinates,

$$\vec{x}_A = A\vec{x}, \tag{G.41}$$

$$\vec{y}_A = (A^{-1})^T\vec{y}, \tag{G.42}$$

where

$$Az = A(x + \Omega y) \equiv z_A \equiv x_A + A\Omega A^T y_A. \tag{G.43}$$

Regarding the background magnetic fluxes F, we can calculate the A-transformation of the F as well as the S and T transformations

$$\begin{aligned} F &= \frac{1}{2}(p^A_{xx})_{ij} dx^i_A \wedge dx^j_A + \frac{1}{2}(p^A_{yy})_{ij} dy^i_A \wedge dy^j_A + (p^A_{xy})_{ij} dx^i_A \wedge dy^j_A \\ &= \frac{1}{2}(A^T p^A_{xx} A)_{ij} dx^i \wedge dx^j + \frac{1}{2}(A^{-1} p^A_{yy} (A^T)^{-1})_{ij} dy^i \wedge dy^j + (A^T p^A_{xy} (A^T)^{-1})_{ij} dx^i \wedge dy^j \\ &= \frac{1}{2}(p_{xx})_{ij} dx^i \wedge dx^j + \frac{1}{2}(p_{yy})_{ij} dy^i \wedge dy^j + (p_{xy})_{ij} dx^i \wedge dy^j. \end{aligned} \tag{G.44}$$

Then we obtain these components after the A-transformation

$$\begin{aligned} p^A_{xx} &= (A^T)^{-1} p_{xx} A^{-1}, \\ p^A_{yy} &= A p_{yy} A^T, \\ p^A_{xy} &= (A^T)^{-1} p_{xy} A^T. \end{aligned} \tag{G.45}$$

When we suppose $p_{xx} = p_{yy} = 0$, we obtain

$$N_A = ANA^{-1} \tag{G.46}$$

from $N = 2\pi p_{xy}^T$.

The A-transformation is consistent with the F-term SUSY condition, that is,

$$\begin{aligned}
(N_A\Omega_A)^T &= (ANA^{-1}A\Omega A^T)^T \\
&= (AN\Omega A^T)^T \\
&= (A(N\Omega)^T A^T)^T \\
&= AN\Omega A^T \\
&= ANA^{-1}A\Omega A^T \\
&= N_A\Omega_A
\end{aligned} \tag{G.47}$$

G.3.1 A-transformation of zero modes

We consider the wave functions behaviors under the A-transformation. In the following, the integer fluxes N are invariant under the A-transformation , namely $N = ANA^{-1}$. We can find the following transformations of the Riemann theta-functions

$$\begin{aligned}
\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (A\vec{z}, A\Omega A^T) &= \sum_{\vec{m}\in\mathbb{Z}^3} e^{\pi i \vec{m}^T A\Omega A^T \vec{m}} e^{2\pi i \vec{m}^T A\vec{z}} \\
&= \sum_{\vec{M}\in\mathbb{Z}^3} e^{\pi i \vec{M}^T \Omega \vec{M}} e^{2\pi i \vec{M}^T \vec{z}} \\
&= \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\vec{z}, \Omega),
\end{aligned} \tag{G.48}$$

where $M = A^T m \in \mathbb{Z}^3$ and $\Omega \in \mathcal{H}^3$.

When we replace $\vec{z}$ with $\vec{z} + \Omega A^T N^{-1T}\vec{j}$, we can obtain

$$\theta \begin{bmatrix} \vec{j}^T N^{-1} \\ 0 \end{bmatrix} (A\vec{z}, A\Omega A^T) = \theta \begin{bmatrix} \vec{j}^T N^{-1} A \\ 0 \end{bmatrix} (\vec{z}, \Omega) = \theta \begin{bmatrix} \vec{j}^T A N^{-1} \\ 0 \end{bmatrix} (\vec{z}, \Omega), \tag{G.49}$$

where $[N^{-1}, A] = [N, A] = 0$. Also, we replace $\vec{z}$, Ω with $N\vec{z}$, $N\Omega$. Then, we find

$$\begin{aligned}
\theta \begin{bmatrix} \vec{j}^T N^{-1} \\ 0 \end{bmatrix} (AN\vec{z}, AN\Omega A^T) &= \theta \begin{bmatrix} \vec{j}^T N^{-1} \\ 0 \end{bmatrix} (ANA^{-1}A\vec{z}, ANA^{-1}A\Omega A^T) \\
&= \theta \begin{bmatrix} \vec{j}^T A N^{-1} \\ 0 \end{bmatrix} (N\vec{z}, N\Omega).
\end{aligned} \tag{G.50}$$

Regarding the phase factor $e^{\pi i (N\vec{z})^T (\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}$ of the wave functions under the A-transformation, we find

$$\begin{aligned} e^{\pi i (ANA^{-1}A\vec{z})^T (A(\mathrm{Im}\Omega)A^T)^{-1}A\mathrm{Im}\vec{z}} &= e^{\pi i (N\vec{z})^T A^T (A^T)^{-1}(\mathrm{Im}\Omega)^{-1}A^{-1}A\mathrm{Im}\vec{z}} \\ &= e^{\pi i (N\vec{z})^T (\mathrm{Im}\Omega)^{-1}\mathrm{Im}\vec{z}}. \end{aligned} \tag{G.51}$$

Finally, the A-transformation of the wave functions becomes

$$\psi_N^{\vec{j}}(A\vec{z}, A\Omega A^T) = \psi_N^{A^T\vec{j}}(\vec{z},\Omega). \tag{G.52}$$

Appendix H

Wave functions in magnetized $T^6/\mathbb{Z}_7$

In the main text, we have considered the wave functions with positive chirality $(+++)$ in magnetized $T^6/\mathbb{Z}_7$ [78]. Similarly to zero-mode counting in magnetized $T^6/\mathbb{Z}_{12}$ [81, 82], we can also clarify the number of the wave functions with various chiralities $(\underline{+--})$, where the underline denotes the possible permutations. We can derive the w system, that is, the diagonal basis of the magnetic fluxes, by using the diagonalization matrix P. If $P \in SO(3)$, we can then introduce the $R = N^{-1}M = PUP^{-1} \in SO(3)$ which determines signs of the eigenvalues of the fluxes, that is, the chirality. Let us take the $P \in SO(3)$ and obtain the number of zero modes with various chiralities in magnetized $T^6/\mathbb{Z}_7$.

By use of numerical computations, when we consider the eigen vectors of the Ω_7 in Eq. (4.10), we can take the $P \in SO(3)$ as follows,

$$\begin{aligned} P &= \begin{bmatrix} 0.32798528 & 0.73697623 & -0.59100905 \\ 0.59100905 & 0.32798528 & 0.73697623 \\ 0.73697623 & -0.59100905 & -0.32798528 \end{bmatrix} \\ &\equiv \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ -p_{13} & p_{11} & p_{12} \\ p_{12} & p_{13} & -p_{11} \end{bmatrix} \in SO(3). \end{aligned} \tag{H.1}$$

According to numerical computations, we find that the above P can let the fluxes $N^T(\mathrm{Im}\Omega_7)^{-1}$ diagonalize, so we can construct the $R = PUP^{-1} \in SO(3)$.

Similarly to zero-mode counting in magnetized $T^6/\mathbb{Z}_{12}$ written in the subsection 4.2.2, when we take $U_1 = \mathrm{diag}(+,-,-)$, we find the following modular transformation of the wave functions on magnetized T^6 given by the z' system,

$$\begin{aligned} &\psi'^{J}_{+++}(\Omega'_{U_1} z', \Omega'_{U_1}) \\ &= \frac{e^{-\frac{3}{28}\pi i}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B_7 K} \cdot \psi'^{K}_{+++}(z', \Omega'_{U_1}), \end{aligned} \tag{H.2}$$

where $\Omega'_{U_1} = \mathrm{Re}\Omega_7 + iPU_1P^{-1}\mathrm{Im}\Omega_7$, and $B_7 = B_3 + B_4 + B_5$. Then, we perform the numerical computations of the zero-mode counting, however, there is no three-generation model satisfying the F-term SUSY condition.

Similarly, when we take $U_2 = \mathrm{diag}(-,+,-)$, $U_3 = \mathrm{diag}(-,-,+)$, respectively, we find the following modular transformations of the wave functions on magnetized T^6,

$$\begin{aligned} &\psi'^{J}_{+++}(\Omega'_{U_2} z', \Omega'_{U_2}) \\ &= \frac{e^{\frac{9}{28}\pi i}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B_7 K} \cdot \psi'^{K}_{+++}(z', \Omega'_{U_2}), \\ &\psi'^{J}_{+++}(\Omega'_{U_3} z', \Omega'_{U_3}) \\ &= \frac{e^{\frac{\pi}{28} i}}{\sqrt{|\det N|}} \sum_{K \in \Lambda_N} e^{2\pi i J^T N^{-1} K} e^{\pi i K^T N^{-1} B_7 K} \cdot \psi'^{K}_{+++}(z', \Omega'_{U_3}), \end{aligned}$$

where $\Omega'_{U_i} = \mathrm{Re}\Omega_7 + iPU_iP^{-1}\mathrm{Im}\Omega_7$, $(i = 2, 3)$, and $B_7 = B_3 + B_4 + B_5$. We take the fluxes N written in Eq. (4.11) on the main text. Then, we perform the numerical computations of the zero-mode counting in regions $|n_{ii}| < 200$, $(i = 1, 2, 3)$, and $1 < \det N < 500$, however, there is no three-generation model satisfying the F-term SUSY condition. Additionally, in these parameter regions, we do not obtain any generation models satisfying both the F-term SUSY and D-term SUSY conditions. In conclusion, we do not obtain three-generation models in magnetized $T^6/\mathbb{Z}_7$ orbifold when we assume that 6D overall chirality is fixed as positive, and consider the wave functions with various chiralities under the vanishing Scherk-Schwarz (SS) phases. In the future, we would like to analyze the SS phases and verify whether three-generation models satisfying the SUSY conditions can be realized or not.

Appendix I

Calculation for overlap integral

I.1 Calculation for products of wave functions

We introduce the details of the overlap integral leading to Yukawa couplings. This Appendix is along in Ref. [81]. We can calculate the product of the wave functions $\psi^I_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^J_{N_R,M_R}(z'_R,\Omega'_R)$ as follows

$$\begin{aligned}
&\psi^I_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^J_{N_R,M_R}(z'_R,\Omega'_R)\\
&\propto e^{i\pi[N_L z'_L]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z}\cdot e^{i\pi[N_R z'_R]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z}\cdot\Theta^I_{N_L}(z'_L,\Omega'_L)\cdot\Theta^J_{N_R}(z'_R,\Omega'_R)\\
&=e^{i\pi[(N_L+N_R)\mathrm{Re}z+i(M_L+M_R)\mathrm{Im}z]^T(\mathrm{Im}\Omega)^{-1}\mathrm{Im}z}\cdot\Theta^I_{N_L}(z'_L,\Omega'_L)\cdot\Theta^J_{N_R}(z'_R,\Omega'_R)\\
&=e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\cdot\Theta^I_{N_L}(z'_L,\Omega'_L)\cdot\Theta^J_{N_R}(z'_R,\Omega'_R)\\
&=e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\\
&\cdot\sum_{\vec\ell_1,\vec\ell_2\in\mathbb{Z}^g}e^{i\pi(\vec\ell_1+\vec i)^T(N_L\mathrm{Re}\Omega+iM_L\mathrm{Im}\Omega)(\vec\ell_1+\vec i)}\cdot e^{i\pi(\vec\ell_2+\vec j)^T(N_R\mathrm{Re}\Omega+iM_R\mathrm{Im}\Omega)(\vec\ell_2+\vec j)}\\
&\cdot e^{2i\pi(\vec\ell_1+\vec i)^T(N_Lx+(N_L\mathrm{Re}\Omega+iM_L\mathrm{Im}\Omega)y)}\cdot e^{2i\pi(\vec\ell_2+\vec j)^T(N_Rx+(N_R\mathrm{Re}\Omega+iM_R\mathrm{Im}\Omega)y)}\\
&\equiv e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\\
&\cdot\sum_{\vec\ell_1,\vec\ell_2\in\mathbb{Z}^g}e^{i\pi(i)[\vec L^T\hat Q_{N,M}\vec L]}\cdot e^{2\pi i[\vec L^T Q_N\vec X]}\cdot e^{2\pi i(i)[\vec L^T\hat Q_{N,M}\vec Y]}.
\end{aligned}\tag{I.1}$$

In Eqs. (5.11), (5.12) and (5.13), we consider

$$\vec L=\begin{bmatrix}\vec i+\vec\ell_1\\ \vec j+\vec\ell_2\end{bmatrix},\vec X=\begin{bmatrix}\vec x\\ \vec x\end{bmatrix},\vec Y=\begin{bmatrix}\vec y\\ \vec y\end{bmatrix},\tag{I.2}$$

and the following $2g \times 2g$ matrices:

$$
\begin{aligned}
Q_N &= \begin{bmatrix} N_L & O \\ O & N_R \end{bmatrix}, \\
\hat{Q}_{N,M} &= \begin{bmatrix} M_L \mathrm{Im}\Omega - iN_L \mathrm{Re}\Omega & O \\ O & M_R \mathrm{Im}\Omega - iN_R \mathrm{Re}\Omega \end{bmatrix} \equiv \begin{bmatrix} M'_L & O \\ O & M'_R \end{bmatrix}.
\end{aligned} \tag{I.3}
$$

From these definitions, the following quantities can be written by

$$
\begin{aligned}
&\vec{L}^T \hat{Q}_{N,M} \vec{L} = \vec{L}^T ((T')^{-1} T') \hat{Q}_{N,M} ((T')^{-1} T')^T \vec{L} \\
&= (\vec{L}^T (T')^{-1})((T')\hat{Q}_{N,M}(T')^T)(((T')^{-1})^T \vec{L}), \\
&\vec{L}^T Q_N \vec{X} = \vec{L}^T ((T')^{-1} T') Q_N ((T')^{-1} T')^T \vec{X} \\
&= (\vec{L}^T (T')^{-1})(T' Q_N (T')^T)(((T')^{-1})^T \vec{X}), \\
&\vec{L}^T \hat{Q}_{N,M} \vec{Y} = \vec{L}^T ((T')^{-1} T') \hat{Q}_{N,M} ((T')^{-1} T')^T \vec{Y} \\
&= (\vec{L}^T (T')^{-1})(T' \hat{Q}_{N,M} (T')^T)(((T')^{-1})^T \vec{Y}).
\end{aligned} \tag{I.4}
$$

The $\hat{Q}'_{N,M}$, Q'_N are written by

$$
\begin{aligned}
\hat{Q}'_{N,M} &\equiv T' \hat{Q}_{N,M} (T')^T = \begin{bmatrix} M'_L + M'_R & (M'_L N_L^{-1} - M'_R N_R^{-1})\alpha^T, \\ \alpha(N_L^{-1} M'_L - N_R^{-1} M'_R) & \alpha(N_L^{-1} M'_L N_L^{-1} + N_R^{-1} M'_R N_R^{-1})\alpha^T \end{bmatrix}, \\
Q'_N &\equiv T' Q_N (T')^T = \begin{bmatrix} N_L + N_R & O \\ O & \alpha(N_L^{-1} + N_R^{-1})\alpha^T \end{bmatrix},
\end{aligned} \tag{I.5}
$$

and the vectors $\vec{L}^T (T')^{-1}$, $((T')^{-1})^T \vec{X}$, and $((T')^{-1})^T \vec{Y}$:

$$
\vec{L}^T (T')^{-1} = \begin{bmatrix} (\vec{i} + \vec{\ell}_1)(N_L^{-1} + N_R^{-1})^{-1} N_R^{-1} + (\vec{j} + \vec{\ell}_2)(N_L^{-1} + N_R^{-1})^{-1} N_L^{-1} \\ [(\vec{i} + \vec{\ell}_1) - (\vec{j} + \vec{\ell}_2)](N_L^{-1} + N_R^{-1})^{-1} \alpha^{-1} \end{bmatrix}^T, \tag{I.6}
$$

$$
((T')^{-1})^T \vec{L} = \begin{bmatrix} N_R^{-1}(N_L^{-1} + N_R^{-1})^{-1}(\vec{i} + \vec{\ell}_1) + N_L^{-1}(N_L^{-1} + N_R^{-1})^{-1}(\vec{j} + \vec{\ell}_2) \\ (\alpha^{-1})^T (N_L^{-1} + N_R^{-1})^{-1}[(\vec{i} + \vec{\ell}_1) - (\vec{j} + \vec{\ell}_2)] \end{bmatrix}, \tag{I.7}
$$

$$
((T')^{-1})^T \vec{X} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \text{and} \quad ((T')^{-1})^T \vec{Y} = \begin{bmatrix} y \\ 0 \end{bmatrix}. \tag{I.8}
$$

In what follows, we consider the calculations for the overlap integral with the value of $\alpha = |N_L||N_R|1_g$ following Refs. [55, 81]. Then the $\psi^I_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^J_{N_R,M_R}(z'_R,\Omega'_R)$ is expressed as follows

$$
\begin{aligned}
&\psi^I_{N_L,M_L}(z'_L,\Omega'_L)\cdot\psi^J_{N_R,M_R}(z'_R,\Omega'_R)\\
&\ \propto e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\\
&\cdot\sum_{\vec{\ell}_1,\vec{\ell}_2\in\mathbb{Z}^g} e^{i\pi(i)[\vec{L}^T\hat{Q}_{N,M}\vec{L}]}\cdot e^{2\pi i[\vec{L}^TQ_N\vec{X}]}\cdot e^{2\pi i(i)[\vec{L}^T\hat{Q}_{N,M}\vec{Y}]}\\
&\ = e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\cdot\\
&\sum_{\vec{\ell}_1,\vec{\ell}_2\in\mathbb{Z}^g} e^{i\pi(i)[(\vec{L}^TT^{-1})(T\hat{Q}_{N,M}T^T)((T^{-1})^T\vec{L})]}\cdot e^{2\pi i[(\vec{L}^TT^{-1})(TQ_NT^T)((T^{-1})^T\vec{X})]}\\
&\cdot e^{2\pi i(i)[(\vec{L}^TT^{-1})(T\hat{Q}_{N,M}T^T)((T^{-1})^T\vec{Y})]}\\
&\ = e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\cdot\\
&\sum_{\vec{\ell}_1,\vec{\ell}_2\in\mathbb{Z}^g} e^{i\pi(i)[((\vec{i}+\vec{\ell}_1)N_L+(\vec{j}+\vec{\ell}_2)N_R)(N_L+N_R)^{-1}(M'_L+M'_R)(N_L+N_R)^{-1}(N_L(\vec{i}+\vec{\ell}_1)+N_R(\vec{j}+\vec{\ell}_2))]}\cdot\\
&e^{2\pi i[((\vec{i}+\vec{\ell}_1)N_L+(\vec{j}+\vec{\ell}_2)N_R)(N_L+N_R)^{-1}](N_L+N_R)x}\cdot e^{2\pi i(i)[(\vec{i}+\vec{\ell}_1)N_L+(\vec{j}+\vec{\ell}_2)N_R](N_L+N_R)^{-1}(M'_L+M'_R)y}\cdot\\
&e^{2\pi i(i)[(\vec{i}+\vec{\ell}_1)-(\vec{j}+\vec{\ell}_2)](N_L^{-1}+N_R^{-1})^{-1}\alpha^{-1}\alpha(N_L^{-1}M'_L-N_R^{-1}M'_R)y}\cdot\\
&e^{\pi i(i)[((\vec{i}+\vec{\ell}_1)N_L+(\vec{j}+\vec{\ell}_2)N_R)(N_L+N_R)^{-1}(M'_LN_L^{-1}-M'_RN_R^{-1})]\alpha^T(\alpha^{-1})^TN_R(N_L+N_R)^{-1}N_L[(\vec{i}-\vec{j})+(\vec{\ell}_1-\vec{\ell}_2)]}\cdot\\
&e^{\pi i(i)[(\vec{i}-\vec{j})+(\vec{\ell}_1-\vec{\ell}_2)]N_L(N_L+N_R)^{-1}N_R\alpha^{-1}\alpha(N_L^{-1}M'_L-N_R^{-1}M'_R)(N_L+N_R)^{-1}(N_L(\vec{i}+\vec{\ell}_1)+N_R(\vec{j}+\vec{\ell}_2))}\cdot\\
&e^{\pi i(i)[(\vec{i}-\vec{j})+(\vec{\ell}_1-\vec{\ell}_2)](N_L^{-1}+N_R^{-1})^{-1}\alpha^{-1}\alpha(N_L^{-1}M'_LN_L^{-1}+N_R^{-1}M'_RN_R^{-1})\alpha^T(\alpha^{-1})^T(N_L^{-1}+N_R^{-1})^{-1}[(\vec{i}-\vec{j})+(\vec{\ell}_1-\vec{\ell}_2)]}\\
&\ = e^{i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega+i(M_L+M_R)\mathrm{Im}\Omega]y]}\cdot\\
&\sum_{\vec{\ell}_3,\vec{\ell}_4\in\mathbb{Z}^g}\sum_{\vec{m}} e^{i\pi(i)[[(\vec{i}N_L+\vec{j}N_R+\vec{m}N_L)(N_L+N_R)^{-1}+\vec{\ell}_3](M'_L+M'_R)[(N_L+N_R)^{-1}(N_L\vec{i}+N_R\vec{j}+N_L\vec{m})+\vec{\ell}_3]]}\cdot\\
&e^{2\pi i[(\vec{i}N_L+\vec{j}N_R+\vec{m}N_L)(N_L+N_R)^{-1}+\vec{\ell}_3](N_L+N_R)x}\cdot e^{2\pi i(i)[(\vec{i}N_L+\vec{j}N_R+\vec{m}N_L)(N_L+N_R)^{-1}+\vec{\ell}_3](M'_L+M'_R)y}\cdot\\
&e^{2\pi i(i)[(\vec{i}-\vec{j}+\vec{m})\frac{N_L(N_L+N_R)^{-1}N_R}{|N_L||N_R|}+\vec{\ell}_4][(|N_L||N_R|)(N_L^{-1}M'_L-N_R^{-1}M'_R)]y}\cdot\\
&e^{i\pi(i)[[(\vec{i}N_L+\vec{j}N_R+\vec{m}N_L)(N_L+N_R)^{-1}+\vec{\ell}_3][(|N_L||N_R|)(M'_LN_L^{-1}-M'_RN_R^{-1})](\frac{N_R(N_L+N_R)^{-1}N_L}{|N_L||N_R|}(\vec{i}-\vec{j}+\vec{m})+\vec{\ell}_4)}\cdot\\
&e^{i\pi(i)[(\vec{i}-\vec{j}+\vec{m})(\frac{N_L(N_L+N_R)^{-1}N_R}{|N_L||N_R|})+\vec{\ell}_4][(|N_L||N_R|)(N_L^{-1}M'_L-N_R^{-1}M'_R)][(N_L+N_R)^{-1}(N_L\vec{i}+N_R\vec{j}+N_L\vec{m})+\vec{\ell}_3]}\cdot\\
&e^{i\pi(i)[(\vec{i}-\vec{j}+\vec{m})(\frac{(N_L^{-1}+N_R^{-1})^{-1}}{|N_L||N_R|})+\vec{\ell}_4][(|N_L||N_R|)^2(N_L^{-1}M'_LN_L^{-1}-N_R^{-1}M'_RN_R^{-1})][\frac{(N_L^{-1}+N_R^{-1})^{-1}}{|N_L||N_R|}(\vec{i}-\vec{j}+\vec{m})+\vec{\ell}_4]},
\end{aligned}
\tag{I.9}
$$

where the relation $(N_L^{-1}+N_R^{-1})^{-1} = N_L(N_L+N_R)^{-1}N_R = N_R(N_L+N_R)^{-1}N_L$ is satisfied under $[N_L, N_R] = 0$. The wave functions of Higgs sector in magnetized T^6 can be written as

$$\begin{aligned}
&[\psi^{\vec{K}}_{N_H=N_L+N_R,M_H}(z'_H,\Omega'_H)]^* \\
&= \mathcal{N}_k \cdot e^{-i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega-iM_H\mathrm{Im}\Omega]y]} \cdot \Theta^{-K}_{-(N_L+N_R)}(\bar{z}'_H,\bar{\Omega}'_H) \\
&= \mathcal{N}_k \cdot e^{-i\pi[x^T(N_L+N_R)y+y^T[(N_L+N_R)\mathrm{Re}\Omega-iM_H\mathrm{Im}\Omega]y]}\cdot \\
&\sum_{\ell'_3\in\mathbb{Z}^3} e^{-i\pi(\ell'_3+k)^T[(N_L+N_R)\mathrm{Re}\Omega-iM_H\mathrm{Im}\Omega](\ell'_3+k)}\cdot e^{-2\pi i(\ell'_3+k)^T[(N_L+N_R)(x+(\mathrm{Re}\Omega)y)-iM_H(\mathrm{Im}\Omega)y]},
\end{aligned} \tag{I.10}$$

where $k = KN_H^{-1} = (-K)(-N_H^{-1})$, $K \in \Lambda_{N_H}$ with Eq. (5.7), we perform the integration over x in C

$$\begin{aligned}
&\int_0^1 d^3x e^{i\pi x^T[N_L+N_R-(N_L+N_R)]y}\cdot \\
&\sum_{\vec{\ell}_3,\vec{\ell}_4,\vec{\ell'}_3\in\mathbb{Z}^3}\sum_{\vec{m}} e^{2\pi i[(\vec{i}N_L+\vec{j}N_R+\vec{m}N_L)(N_L+N_R)^{-1}+\vec{\ell}_3]^T(N_L+N_R)x} e^{-2\pi i(\vec{\ell'}_3+\vec{k})^T(N_L+N_R)x}
\end{aligned} \tag{I.11}$$

and the following rule

$$\begin{aligned}
&\vec{\ell}_3 = \vec{\ell'}_3, \\
&(\vec{i}N_L+\vec{j}N_R+\vec{m}N_L)(N_L+N_R)^{-1} = \vec{k}.
\end{aligned} \tag{I.12}$$

This can correspond to the selection rule. By use of Eqs. (I.9), (I.10), and (I.12), the C in Eq. (5.4) is calculated as follows

$$\begin{aligned}
C = &|-2i\mathrm{Im}\Omega|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k\cdot \\
&\int_0^1 d^3y[e^{-\pi[y^T[M_L+M_R+M_H](\mathrm{Im}\Omega)y]}\cdot \\
&\sum_{\vec{\ell}_3,\vec{\ell}_4\in\mathbb{Z}^3}\sum_{\vec{m}} e^{i\pi(i)[[\vec{k}+\vec{\ell}_3]^T[M'_L+M'_R+(M'_H)^*][\vec{k}+\vec{\ell}_3]]}\cdot e^{2\pi i(i)[\vec{k}+\vec{\ell}_3]^T[M'_L+M'_R+(M'_H)^*]y}\cdot \\
&e^{2\pi i(i)[\vec{h}+\vec{\ell}_4]^T[(|N_L||N_R|)(N_L^{-1}M'_L-N_R^{-1}M'_R)]y}\cdot \\
&e^{i\pi(i)[[\vec{k}+\vec{\ell}_3]^T[(|N_L||N_R|)(M'_LN_L^{-1}-M'_RN_R^{-1})][\vec{h}+\vec{\ell}_4]}\cdot \\
&e^{i\pi(i)[\vec{h}+\vec{\ell}_4]^T[(|N_L||N_R|)(N_L^{-1}M'_L-N_R^{-1}M'_R)][\vec{k}+\vec{\ell}_3]}\cdot \\
&e^{i\pi(i)[\vec{h}+\vec{\ell}_4]^T[(|N_L||N_R|)^2(N_L^{-1}M'_LN_L^{-1}-N_R^{-1}M'_RN_R^{-1})][\vec{h}+\vec{\ell}_4]}],
\end{aligned} \tag{I.13}$$

where $(M'_H)^* \equiv M_H\mathrm{Im}\Omega + i(N_L+N_R)\mathrm{Re}\Omega$ and $\vec{h} \equiv \frac{N_R(N_L+N_R)^{-1}N_L}{|N_L||N_R|}(\vec{i}-\vec{j}+\vec{m})$.

I.2 Calculation for C

According to Ref. [81], we can derive the overlap integral. For $\alpha = \text{L.C.M.}(|N_L|, |N_R|)1_g \equiv L_R 1_g$ and $g = 3$, we can find the integral part

$$C = |-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i\mathcal{N}_j\mathcal{N}_k \cdot \int_0^1 d^3y \left[e^{-\pi[y^T[M_L+M_R+M_H](\mathrm{Im}\Omega)y]} \cdot \sum_{\vec{\ell}_3,\vec{\ell}_4\in\mathbb{Z}^3} \sum_{\vec{p},\vec{\tilde{p}}} e^{\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T\cdot\hat{\mathbf{Q}}'^{L_R}_{N,M'}\cdot[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]} \cdot e^{2\pi i(i)[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T\vec{\mathbf{Y}}'^{L_R}_{N,M'}} \right],$$

where

$$\vec{\mathbf{L_1}} = \begin{bmatrix} \vec{\ell}_3 \\ \vec{\ell}_4 \end{bmatrix}, \quad \vec{\mathbf{K}} = \begin{bmatrix} \vec{k} \\ \frac{N_R(N_L+N_R)^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}') \end{bmatrix}, \tag{I.14}$$

$$\vec{\mathbf{Y}}'^{L_R}_{N,M'} \equiv \begin{bmatrix} [M'_L+M'_R+(M'_H)^*]y \\ [L_R(M'_LN_L^{-1}-M'_RN_R^{-1})]y \end{bmatrix} = \begin{bmatrix} [(M_L+M_R+M_H)\mathrm{Im}\Omega]y \\ L_R[(R_L-R_R)\mathrm{Im}\Omega]y \end{bmatrix}, \tag{I.15}$$

and

$$\hat{\mathbf{Q}}'^{L_R}_{N,M'} \equiv \begin{bmatrix} M'_L+M'_R+(M'_H)^* & L_R(M'_LN_L^{-1}-M'_RN_R^{-1}) \\ L_R(N_L^{-1}M'_L-N_R^{-1}M'_R) & (L_R)^2(N_L^{-1}M'_LN_L^{-1}+N_R^{-1}M'_RN_R^{-1}) \end{bmatrix}. \tag{I.16}$$

Here, we denote $\mathrm{Re}\hat{\mathbf{Q}}'^{L_R}_{N,M'} > 0$, $M'_L = M_L\mathrm{Im}\Omega - iN_L\mathrm{Re}\Omega$, $M'_R = M_R\mathrm{Im}\Omega - iN_R\mathrm{Re}\Omega$, and $(M'_H)^* = M_H\mathrm{Im}\Omega + i(N_L+N_R)\mathrm{Re}\Omega$. Furthermore, $\vec{\tilde{m}}$ is the solution for $(N_L+N_R)\vec{k} = N_L\vec{i} + N_R\vec{j} + N_L\vec{\tilde{m}}$, and $\vec{p} \in \Lambda_{L_RN_R^{-1}}, \vec{\tilde{p}} \in \Lambda_{L_RN_L^{-1}}$. We can then replace $\vec{m}$ with $\vec{\tilde{m}}' = \vec{\tilde{m}} + \vec{p}(N_L+N_R)N_L^{-1} + \vec{\tilde{p}}(N_L+N_R)N_R^{-1}$.

In the main text, by imposing Eq. (5.22), the C is written by

$$C = \frac{|-2i\mathrm{Im}\Omega|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L+M_R+M_H)\mathrm{Im}\Omega|}} \cdot \sum_{\vec{p},\vec{\tilde{p}}} \theta \begin{bmatrix} \frac{N_R(N_L+N_R)^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}') \\ \vec{0} \end{bmatrix} (\vec{0}, \tilde{\Omega}_Y) \cdot \delta_{(N_L+N_R)\vec{k}, N_L\vec{i}+N_R\vec{j}+N_L\vec{\tilde{m}}}, \tag{I.17}$$

where

$$\tilde{\Omega}_Y \equiv (L_R)^2[(N_L^{-1}+N_R^{-1})\mathrm{Re}\Omega + i[(R_LN_L^{-1}+R_RN_R^{-1}) - (R_L-R_R)(M_L+M_R+M_H)^{-1}(R_L-R_R)]\mathrm{Im}\Omega]. \tag{I.18}$$

To prove this result, We consider the following Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x} + b^T \mathbf{x}} d\mathbf{x} = \sqrt{\frac{(2\pi)^n}{|A|}} \cdot e^{\frac{1}{2} b^T A^{-1} b}, \tag{I.19}$$

where A is a symmetric positive-definite matrix. Then, we would like to introduce the above C. From Eq. (I.14), we get

$$\begin{aligned} C = &|-2i\mathrm{Im}\Omega| \cdot \mathcal{N}_i \mathcal{N}_j \mathcal{N}_k \\ &\cdot \int_0^1 d\mathbf{y} \left[e^{-\pi[\mathbf{y}^T [M_L+M_R+M_H](\mathrm{Im}\Omega)\mathbf{y}]} \sum_{\vec{\ell}_3, \vec{\ell}_4 \in \mathbb{Z}^3} \sum_{\vec{p}, \vec{\tilde{p}}} e^{-\pi[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T \hat{\mathbf{Q}'}^{L_R}_{N,M'} [\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]} \cdot e^{-2\pi[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T \vec{\mathbf{Y}'}^{L_R}_{N,M'}} \right], \end{aligned} \tag{I.20}$$

where

$$\begin{aligned} (\vec{\mathbf{K}} + \vec{\mathbf{L_1}})^T &= \left[(\vec{k} + \vec{\ell}_3)^T, \ \left(\tfrac{N_R(N_L+N_R)^{-1}N_L}{L_R} (\vec{i} - \vec{j} + \vec{\tilde{m}}') + \vec{\ell}_4 \right)^T \right], \\ \vec{\mathbf{K}} + \vec{\mathbf{L_1}} &= \begin{bmatrix} \vec{k} + \vec{\ell}_3 \\ \tfrac{N_R(N_L+N_R)^{-1}N_L}{L_R} (\vec{i} - \vec{j} + \vec{\tilde{m}}') + \vec{\ell}_4 \end{bmatrix}, \end{aligned} \tag{I.21}$$

$$\begin{aligned} &\hat{\mathbf{Q}'}^{L_R}_{N,M'} = \\ &\begin{bmatrix} (M_L + M_R + M_H)\mathrm{Im}\Omega & L_R(R_L - R_R)\mathrm{Im}\Omega \\ L_R(R_L - R_R)\mathrm{Im}\Omega & (L_R)^2[(R_L N_L^{-1} + R_R N_R^{-1})(\mathrm{Im}\Omega) - i(\mathrm{Re}\Omega)(N_L^{-1} + N_R^{-1})] \end{bmatrix}, \end{aligned} \tag{I.22}$$

$$\vec{\mathbf{Y}'}^{L_R}_{N,M'} = \begin{bmatrix} [(M_L + M_R + M_H)\mathrm{Im}\Omega]\mathbf{y} \\ L_R[(R_L - R_R)\mathrm{Im}\Omega]\mathbf{y} \end{bmatrix}. \tag{I.23}$$

In what follows, we can explicitly calculate the values of the C by properties of the Riemann Theta functions

$$\int_0^1 d\mathbf{y}\left[e^{-\pi[\mathbf{y}^T[M_L+M_R+M_H](\mathrm{Im}\Omega)\mathbf{y}]}\sum_{\vec{\ell}_3,\vec{\ell}_4\in\mathbb{Z}^3}\sum_{\vec{p},\vec{\tilde{p}}}e^{-\pi[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T\hat{\mathbf{Q}}'^{L_R}_{N,M'}[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]}\cdot e^{-2\pi[\vec{\mathbf{K}}+\vec{\mathbf{L_1}}]^T\vec{\mathbf{Y}}'^{L_R}_{N,M'}}\right]$$

$$=\sum_{\vec{p},\vec{\tilde{p}}}\sum_{\vec{\ell}_4\in\mathbb{Z}^3}e^{-\pi\left[\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')+\vec{\ell}_4\right]^T(L_R)^2[(R_LN_L^{-1}+R_RN_R^{-1})\mathrm{Im}\Omega-i(N_L^{-1}+N_R^{-1})\mathrm{Re}\Omega]\left[\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')+\vec{\ell}_4\right]}$$

$$\cdot\sum_{\vec{\ell}_3\in\mathbb{Z}^3}\int_0^1 d\mathbf{y}\left[e^{-\pi[\mathbf{y}+\vec{k}+\vec{\ell}_3]^T(M_L+M_R+M_H)(\mathrm{Im}\Omega)[\mathbf{y}+\vec{k}+\vec{\ell}_3]}\right.$$

$$\left.\times\, e^{-2\pi\left[L_R[(R_L-R_R)\mathrm{Im}\Omega]^T\left[\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')+\vec{\ell}_4\right]\right]^T[\mathbf{y}+\vec{k}+\vec{\ell}_3]}\right]$$

$$=\sum_{\vec{p},\vec{\tilde{p}}}\sum_{\vec{\ell}_4\in\mathbb{Z}^3}e^{-\pi\left[\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')+\vec{\ell}_4\right]^T(L_R)^2[(R_LN_L^{-1}+R_RN_R^{-1})\mathrm{Im}\Omega-i(N_L^{-1}+N_R^{-1})\mathrm{Re}\Omega]\left[\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')+\vec{\ell}_4\right]}$$

$$\cdot\int_{-\infty}^{\infty}d\mathbf{y}'\left[e^{-\pi\mathbf{y}'^T(M_L+M_R+M_H)(\mathrm{Im}\Omega)\mathbf{y}'}\cdot e^{-2\pi\left[L_R[(R_L-R_R)\mathrm{Im}\Omega]^T\left[\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')+\vec{\ell}_4\right]\right]^T\mathbf{y}'}\right]$$

$$=\frac{1}{\sqrt{|(M_L+M_R+M_H)\mathrm{Im}\Omega|}}\sum_{\vec{p},\vec{\tilde{p}}}\theta\begin{bmatrix}\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{\Omega}_Y),$$

where $\mathbf{y}'\equiv\mathbf{y}+\vec{k}+\vec{\ell}_3$ and

$$\begin{aligned}\tilde{\Omega}_Y\equiv&(L_R)^2[(N_L^{-1}+N_R^{-1})\mathrm{Re}\Omega\\&+i[(R_LN_L^{-1}+R_RN_R^{-1})-(R_L-R_R)(M_L+M_R+M_H)^{-1}(R_L-R_R)]\mathrm{Im}\Omega].\end{aligned}\tag{I.24}$$

Here, the Gaussian integral in Eq. (I.19) is used. We obtain the C

$$\begin{aligned}C=&\frac{|-2i\mathrm{Im}\Omega|\cdot\mathcal{N}_i\mathcal{N}_j\mathcal{N}_k}{\sqrt{|(M_L+M_R+M_H)\mathrm{Im}\Omega|}}\cdot\sum_{\vec{p},\vec{\tilde{p}}}\theta\begin{bmatrix}\frac{N_RN_H^{-1}N_L}{L_R}(\vec{i}-\vec{j}+\vec{\tilde{m}}')\\ \vec{0}\end{bmatrix}(\vec{0},\tilde{\Omega}_Y)\\&\cdot\delta_{N_H\vec{k},N_L\vec{i}+N_R\vec{j}+N_L\vec{\tilde{m}}}.\end{aligned}\tag{I.25}$$

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