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Symmetry classification of magnetic octupole current based on multipole representation theory

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Magnetic octupole (MO) currents have recently attracted significant attention as a driving force for the Néel vector dynamics in d -wave altermagnets, a new class of antiferromagnets that exhibit nonrelativistic spin-split band structures. From a symmetry perspective, the MO includes an axial-dipole component analogous to that of the spin, making it essential to clarify how MO currents differ from spin currents. We here investigate the correspondence between MO conductivities and electronic multipoles, which provide a unified and powerful framework for symmetry analysis. We derive the multipole representation of the rank-five MO conductivity tensor and classify its symmetry-allowed components for all crystallographic point groups, in direct comparison with spin conductivity. We show that time-reversal-even electric-type multipoles give rise to the dissipationless MO current, whereas time-reversal-odd magnetic-type multipoles generate dissipative MO current under an applied electric field. Complementing this macroscopic analysis, the linear-response calculations for a microscopic tight-binding model demonstrate how MO conductivities are activated by symmetry lowering, exemplified by the symmetry reduction from $m\bar{3}m$ to $m\bar{3}$. Our results elucidate the symmetry distinctions between MO currents and spin currents, and provide insights into their experimental identification.

I. INTRODUCTION

In recent years, antiferromagnetic materials exhibiting nonrelativistic momentum-dependent spin-split band structures have been discovered [1–11]. Such antiferromagnetic materials are referred to as *altermagnets* [12–14], and several intriguing phenomena free from spin-orbit coupling (SOC) have been proposed, such as the spin current generation [4, 15–19] and linear and nonlinear piezo-magnetic effects [20–29]. In particular, d -wave altermagnets, which exhibit nonrelativistic symmetric spin-split band structures characterized by the product of the second order of the wave vector $\mathbf{k}$ and collinear spin moment σ , $k_i k_j \sigma$ for $i, j = x, y, z$, have attracted considerable attention. In these systems, phenomena arising from the magnetic octupole (MO) — the lowest-rank magnetic anisotropy that couples to inhomogeneous magnetic fields [23, 30–33] — have been proposed, such as the nonlinear magnetoelectric effect [34].

It has been revealed that MOs in d -wave altermagnets can couple linearly to the Néel vector and act as secondary order parameters [23]. By focusing on the similarities between this coupling and SOC, the concept of an MO torque induced by MO currents — analogous to the spin-orbit torque induced by spin currents — has been proposed [35]. In parallel, theoretical studies have demonstrated Néel vector switching and domain wall dynamics in d -wave altermagnets driven by such MO torques [36]. These pioneering studies have brought MO currents into focus in the context of Néel-vector dynamics, and stimulated active investigation of their fundamental properties. Furthermore, first-principles calculations have revealed finite MO Hall conductivities in a variety of $4d$ and $5d$ transition metals [37].

MO currents can be distinguished from spin currents in certain symmetry settings, even though both are de-

scribed by axial response tensors [38]. This indicates that, despite sharing the same tensor character, the two currents are subject to different symmetry constraints on their components and can therefore appear differently in real materials. Symmetry analyses of such response tensors are thus crucial not only for proposing experimental setups to identify MO currents — by determining which components are allowed or forbidden under given crystal and magnetic point groups — but also for elucidating the essential microscopic electronic degrees of freedom.

To systematically capture these symmetry constraints, we employ the framework of electronic multipoles [39–43]. Multipoles can be categorized into four types based on their spatial inversion ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) parities [42–44]: electric multipoles (E), corresponding to $\mathcal{T}$ -even polar tensors; magnetic multipoles (M), corresponding to $\mathcal{T}$ -odd axial tensors; magnetic toroidal multipoles (MT), corresponding to $\mathcal{T}$ -odd polar tensors; and electric toroidal multipoles (ET), corresponding to $\mathcal{T}$ -even axial tensors. These multipoles are directly related to response tensors through symmetry considerations [45–47]. The symmetries of the system restrict the activation of multipoles and, in turn, determine which components of the response tensors can remain finite under a given symmetry group. Moreover, symmetries link multipoles to microscopic electronic degrees of freedom [48], so that the multipole framework serves as a powerful tool for exploring and designing unconventional response phenomena involving nontrivial couplings between electronic degrees of freedom. For instance, unconventional longitudinal spin currents originating from an electric ferroaxial moment have been proposed in the context of an ET dipole, where the key microscopic ingredients have been identified [49]. For these reasons, multipole analysis is essential from both macroscopic and microscopic perspectives.

In the present study, we investigate the symmetry con-

ditions of MO currents induced by electric fields. We show the multipole representations of the MO conductivity tensors and clarify their symmetry distinctions from spin conductivity tensors under all crystallographic point groups (CPGs), thereby establishing key factors for experimental identification. We also discuss the role of the $\mathcal{T}$ symmetry of the system on the MO conductivity. We then perform linear response calculations for a tight-binding model to elucidate the microscopic properties of the MO conductivity associated with the symmetry lowering from the point group $m\bar{3}m$ to $m\bar{3}$.

This paper is organized as follows. In Sec. II, we show the relationships between the MO conductivity tensors and the multipoles. We compare the obtained results with those of the spin conductivity tensors (i) for the case where the multipoles activated expressed by cubic harmonics (c -multipoles), which are relevant to the CPG $m\bar{3}m$ and its subgroups, and (ii) for all CPGs. Then, we present the role of $\mathcal{T}$ symmetry on the MO conductivity and show the relevance to multipoles from the viewpoint of the dissipation based on the Kubo formula. In Sec. III, we construct a basic tight-binding model and present numerical results to demonstrate symmetry aspects arising from the microscopic details of the system. Section IV provides a summary of the paper. In Appendix A, we present the angle dependences of c -multipoles and multipoles expressed by tesseral harmonics (t -multipoles) relevant to the hexagonal point group $6/mmm$ and its subgroups. Appendix B presents the relationship between the tensor components of the MO conductivities and relevant multipoles. Appendix C summarizes the same comparison as in Sec. IIB for the t -multipoles. Finally, Appendix D lists the explicit forms of the hopping matrices used in the numerical calculations.

II. SYMMETRY OF MAGNETIC OCTUPOLE CONDUCTIVITY

In this section, we investigate the symmetry of the MO conductivity σ_{ij}^{mna} , which is characterized as a rank-5 axial tensor as follows:

$$J_i^{mna} = \sigma_{ij}^{mna} E_j, \quad (1)$$

where J_i^{mna} and E_j for $i, j, m, n, \alpha = x, y, z$ denote the MO current and the electric field, respectively. Here, J_i^{mna} is defined by using the MO operator $O_{mn}^\alpha = \{L_m, L_n\}_+ S_\alpha / (2\hbar^2)$ as $J_i^{mna} \equiv \{O_{mn}^\alpha, v_i\}_+ / 2$ [35, 37, 38], which is analogous to the spin current [50]. The symbols $\{\cdot, \cdot\}_+$, v_i , L_m , S_α , and $\hbar$ denote the anticommutator, velocity, orbital angular momentum, spin angular momentum, and the reduced Planck constant, respectively.

In the following, we analyze the symmetry of the MO conductivities using multipole representations. First, we discuss the correspondence between the MO conductivities and the multipoles by decomposing the MO conduc-

tivities into the Ohmic and Hall parts in Sec. IIA. We then apply this method to the spin conductivities and compare the results with those of the MO conductivities in Sec. IIB. Moreover, we demonstrate the distinctions among them under the 32 CPGs in Sec. IIC. In Sec. IID, we show that multipoles with different parities under $\mathcal{T}$ symmetry give rise to different contributions with respect to the dissipation based on the Kubo formula.

A. Correspondence to Multipole

We first show the correspondence between the MO conductivity tensors σ_{ij}^{mna} and multipoles. We introduce the symbols X and Y to distinguish polar and axial tensors, respectively. We assume that the $\mathcal{T}$ symmetry is present unless stated otherwise; see Sec. IID for the role of $\mathcal{T}$ symmetry on the MO conductivity. Depending on the crystal system, we use two types of multipole notation: One is the c -multipole for the cubic point group and its subgroups, and the other is the t -multipole for the hexagonal point group and its subgroups. Following Refs. [45, 47], the notations for c - (or t -) multipole are given by X_0 for the rank-0 monopole, $\mathbf{X} = \{X_x, X_y, X_z\}$ for the rank-1 dipole, $X_{ij} = \{X_u, X_v, X_{yz}, X_{zx}, X_{xy}\}$ for the rank-2 quadrupole, $X_{ijk} = \{X_{xyz}, X_x^\alpha, X_y^\alpha, X_z^\alpha, X_x^\beta, X_y^\beta, X_z^\beta\}$ ($= \{X_{xyz}, X_z^\alpha, X_z^\beta, X_{3u}, X_{3v}, X_{3a}, X_{3b}\}$) for the rank-3 octupole, $X_{ijkl} = \{X_4, X_{4u}, X_{4v}, X_{4x}^\alpha, X_{4y}^\alpha, X_{4z}^\alpha, X_{4x}^\beta, X_{4y}^\beta, X_{4z}^\beta\}$ ($= \{X_{40}, X_{4a}, X_{4b}, X_{4u}^\alpha, X_{4v}^\alpha, X_{4u}^\beta, X_{4v}^\beta, X_{4u}^{\beta 2}, X_{4v}^{\beta 2}\}$) for the rank-4 hexadecapole, and $X_{ijklm} = \{X_{5u}, X_{5v}, X_{5x}^\alpha, X_{5y}^\alpha, X_{5z}^\alpha, X_{5x}^\beta, X_{5y}^\beta, X_{5z}^\beta, X_{5x}^{\beta 2}, X_{5y}^{\beta 2}, X_{5z}^{\beta 2}\}$ ($= \{X_{50}, X_{5a}, X_{5b}, X_{5u}^\alpha, X_{5v}^\alpha, X_{5u}^\beta, X_{5v}^\beta, X_{5u}^{\beta 2}, X_{5v}^{\beta 2}\}$) for the rank-5 dotriacontapole. It is noted that t -multipoles from rank 0 to rank 2 are the same as c -multipoles. Angular dependences of multipoles are shown in Table XI in Appendix A.

Starting from the decomposition of the orbital part in O_{mn}^α , the multipole correspondence is given by

$$\{L_m, L_n\}_+ \leftrightarrow X_0 \oplus X_{ij}, \quad (2)$$

where X_0 and X_{ij} corresponds to $\mathbf{L} \cdot \mathbf{L}$ and $L_m L_n + L_n L_m$, respectively. Thus, the quantity $\{L_m, L_n\}_+$ includes the rank-0 and rank-2 multipoles; it is noted that no rank-1 multipoles appear owing to the symmetric tensor property of $\{L_m, L_n\}_+$. Then, since the spin corresponds to a rank-1 axial tensor, the multipole representation of O_{mn}^α is given by

$$O_{mn}^\alpha \leftrightarrow (X_0 \oplus X_{ij}) \otimes \mathbf{Y} = \mathbf{Y} \oplus \mathbf{Y}' \oplus X_{ij} \oplus Y_{ijk}, \quad (3)$$

where $\mathbf{Y}$ and $\mathbf{Y}'$ originate from $X_0 \otimes \mathbf{Y}$ and $X_{ij} \otimes \mathbf{Y}$, respectively, the latter of which is referred to as the anisotropic dipole [51–53]. Thus, the rank-3 tensor generally includes two types of dipoles, one quadrupole, and one octupole [54].

In a similar manner, since both the current J_i (or velocity v_i) and the electric field E_j transform as polar dipoles, their product can be decomposed as

$$J_i E_j \leftrightarrow X_0 \oplus \mathbf{Y} \oplus X_{ij}, \quad (4)$$

where the rank-0 X_0 and rank-2 X_{ij} terms constitute the symmetric components $(J_i E_j + J_j E_i)$, whereas the rank-1 $\mathbf{Y}$ represents the antisymmetric component $(J_i E_j - J_j E_i)$. In the context of electric conductivity, the former is called the Ohmic contribution, while the latter is called the Hall contribution. By combining it with the MO in Eq. (3), the correspondence between rank-5 MO conductivity tensor and multipoles are obtained. The relevant multipoles for the Ohmic contribution of the MO conductivity are given by

$$(J_i E_j + J_j E_i) O_{mn}^\alpha \leftrightarrow X_0 \oplus 6\mathbf{Y} \oplus 5X_{ij} \oplus 5Y_{ijk} \oplus 2X_{ijkl} \oplus Y_{ijklm}, \quad (5)$$

while those for Hall contribution of the MO conductivity are given by

$$(J_i E_j - J_j E_i) O_{mn}^\alpha \leftrightarrow 2X_0 \oplus 3\mathbf{Y} \oplus 4X_{ij} \oplus 2Y_{ijk} \oplus X_{ijkl}, \quad (6)$$

where coefficients of multipoles stand for independent multipole numbers. Thus, the MO conductivity tensor $\sigma_{ij}^{mn\alpha}$ contains polar even-rank and axial odd-rank multipoles up to rank 5. From the $\mathcal{P}$ parity, X corresponds to E (MT) multipoles ($X = Q$ or T), while Y corresponds to ET (M) multipoles ($Y = G$ or M).

Next, we express the tensor components in terms of multipoles. As described above, $\sigma_{ij}^{mn\alpha}$ is divided into the Ohmic part $\sigma_{ij}^{mn\alpha(O)}$ and the Hall part $\sigma_{ij}^{mn\alpha(H)}$ as follows:

$$\sigma_{ij}^{mn\alpha} = \sigma_{ij}^{mn\alpha(O)} + \sigma_{ij}^{mn\alpha(H)}, \quad (7)$$

$$\sigma_{ij}^{mn\alpha(O)} \equiv \frac{1}{2}(\sigma_{ij}^{mn\alpha} + \sigma_{ji}^{mn\alpha}), \quad (8)$$

$$\sigma_{ij}^{mn\alpha(H)} \equiv \frac{1}{2}(\sigma_{ij}^{mn\alpha} - \sigma_{ji}^{mn\alpha}), \quad (9)$$

where the Ohmic and Hall parts are characterized by the symmetric and antisymmetric tensors for the interchange of i and j , respectively, i.e., $\sigma_{ij}^{mn\alpha(O)} = \sigma_{ji}^{mn\alpha(O)}$ and $\sigma_{ij}^{mn\alpha(H)} = -\sigma_{ji}^{mn\alpha(H)}$. The Ohmic conductivity tensor is

represented by the 18×6 matrix as

$$\begin{pmatrix} \sigma_{xx}^{xxx(H)} & \sigma_{yy}^{xxx(H)} & \sigma_{zz}^{xxx(H)} & \sigma_{yz}^{xxx(H)} & \sigma_{zx}^{xxx(H)} & \sigma_{xy}^{xxx(H)} \\ \sigma_{xx}^{yyy(H)} & \sigma_{yy}^{yyy(H)} & \sigma_{zz}^{yyy(H)} & \sigma_{yz}^{yyy(H)} & \sigma_{zx}^{yyy(H)} & \sigma_{xy}^{yyy(H)} \\ \sigma_{xx}^{zzx(H)} & \sigma_{yy}^{zzx(H)} & \sigma_{zz}^{zzx(H)} & \sigma_{yz}^{zzx(H)} & \sigma_{zx}^{zzx(H)} & \sigma_{xy}^{zzx(H)} \\ \sigma_{xx}^{yzy(H)} & \sigma_{yy}^{yzy(H)} & \sigma_{zz}^{yzy(H)} & \sigma_{yz}^{yzy(H)} & \sigma_{zx}^{yzy(H)} & \sigma_{xy}^{yzy(H)} \\ \sigma_{xx}^{zxx(H)} & \sigma_{yy}^{zxx(H)} & \sigma_{zz}^{zxx(H)} & \sigma_{yz}^{zxx(H)} & \sigma_{zx}^{zxx(H)} & \sigma_{xy}^{zxx(H)} \\ \sigma_{xx}^{xyx(H)} & \sigma_{yy}^{xyx(H)} & \sigma_{zz}^{xyx(H)} & \sigma_{yz}^{xyx(H)} & \sigma_{zx}^{xyx(H)} & \sigma_{xy}^{xyx(H)} \\ \sigma_{xx}^{xxy(H)} & \sigma_{yy}^{xxy(H)} & \sigma_{zz}^{xxy(H)} & \sigma_{yz}^{xxy(H)} & \sigma_{zx}^{xxy(H)} & \sigma_{xy}^{xxy(H)} \\ \sigma_{xx}^{yyy(H)} & \sigma_{yy}^{yyy(H)} & \sigma_{zz}^{yyy(H)} & \sigma_{yz}^{yyy(H)} & \sigma_{zx}^{yyy(H)} & \sigma_{xy}^{yyy(H)} \\ \sigma_{xx}^{zzz(H)} & \sigma_{yy}^{zzz(H)} & \sigma_{zz}^{zzz(H)} & \sigma_{yz}^{zzz(H)} & \sigma_{zx}^{zzz(H)} & \sigma_{xy}^{zzz(H)} \\ \sigma_{xx}^{yzy(H)} & \sigma_{yy}^{yzy(H)} & \sigma_{zz}^{yzy(H)} & \sigma_{yz}^{yzy(H)} & \sigma_{zx}^{yzy(H)} & \sigma_{xy}^{yzy(H)} \\ \sigma_{xx}^{zxx(H)} & \sigma_{yy}^{zxx(H)} & \sigma_{zz}^{zxx(H)} & \sigma_{yz}^{zxx(H)} & \sigma_{zx}^{zxx(H)} & \sigma_{xy}^{zxx(H)} \\ \sigma_{xx}^{xyx(H)} & \sigma_{yy}^{xyx(H)} & \sigma_{zz}^{xyx(H)} & \sigma_{yz}^{xyx(H)} & \sigma_{zx}^{xyx(H)} & \sigma_{xy}^{xyx(H)} \\ \sigma_{xx}^{xxy(H)} & \sigma_{yy}^{xxy(H)} & \sigma_{zz}^{xxy(H)} & \sigma_{yz}^{xxy(H)} & \sigma_{zx}^{xxy(H)} & \sigma_{xy}^{xxy(H)} \\ \sigma_{xx}^{yyy(H)} & \sigma_{yy}^{yyy(H)} & \sigma_{zz}^{yyy(H)} & \sigma_{yz}^{yyy(H)} & \sigma_{zx}^{yyy(H)} & \sigma_{xy}^{yyy(H)} \\ \sigma_{xx}^{zzz(H)} & \sigma_{yy}^{zzz(H)} & \sigma_{zz}^{zzz(H)} & \sigma_{yz}^{zzz(H)} & \sigma_{zx}^{zzz(H)} & \sigma_{xy}^{zzz(H)} \\ \sigma_{xx}^{yzy(H)} & \sigma_{yy}^{yzy(H)} & \sigma_{zz}^{yzy(H)} & \sigma_{yz}^{yzy(H)} & \sigma_{zx}^{yzy(H)} & \sigma_{xy}^{yzy(H)} \\ \sigma_{xx}^{zxx(H)} & \sigma_{yy}^{zxx(H)} & \sigma_{zz}^{zxx(H)} & \sigma_{yz}^{zxx(H)} & \sigma_{zx}^{zxx(H)} & \sigma_{xy}^{zxx(H)} \\ \sigma_{xx}^{xyx(H)} & \sigma_{yy}^{xyx(H)} & \sigma_{zz}^{xyx(H)} & \sigma_{yz}^{xyx(H)} & \sigma_{zx}^{xyx(H)} & \sigma_{xy}^{xyx(H)} \\ \sigma_{xx}^{xxy(H)} & \sigma_{yy}^{xxy(H)} & \sigma_{zz}^{xxy(H)} & \sigma_{yz}^{xxy(H)} & \sigma_{zx}^{xxy(H)} & \sigma_{xy}^{xxy(H)} \end{pmatrix}, \quad (10)$$

where the column and row stand for the indices of the MO and $J_i E_j$, respectively. The Hall conductivity tensor is represented by the 18×3 matrix, which is given by

$$\begin{pmatrix} \sigma_{yz}^{xxx(H)} & \sigma_{zx}^{xxx(H)} & \sigma_{xy}^{xxx(H)} \\ \sigma_{yz}^{yyy(H)} & \sigma_{zx}^{yyy(H)} & \sigma_{xy}^{yyy(H)} \\ \sigma_{yz}^{zzx(H)} & \sigma_{zx}^{zzx(H)} & \sigma_{xy}^{zzx(H)} \\ \sigma_{yz}^{yzy(H)} & \sigma_{zx}^{yzy(H)} & \sigma_{xy}^{yzy(H)} \\ \sigma_{yz}^{zxx(H)} & \sigma_{zx}^{zxx(H)} & \sigma_{xy}^{zxx(H)} \\ \sigma_{yz}^{xyx(H)} & \sigma_{zx}^{xyx(H)} & \sigma_{xy}^{xyx(H)} \\ \sigma_{yz}^{xxy(H)} & \sigma_{zx}^{xxy(H)} & \sigma_{xy}^{xxy(H)} \\ \sigma_{yz}^{yyy(H)} & \sigma_{zx}^{yyy(H)} & \sigma_{xy}^{yyy(H)} \\ \sigma_{yz}^{zzz(H)} & \sigma_{zx}^{zzz(H)} & \sigma_{xy}^{zzz(H)} \\ \sigma_{yz}^{yzy(H)} & \sigma_{zx}^{yzy(H)} & \sigma_{xy}^{yzy(H)} \\ \sigma_{yz}^{zxx(H)} & \sigma_{zx}^{zxx(H)} & \sigma_{xy}^{zxx(H)} \\ \sigma_{yz}^{xyx(H)} & \sigma_{zx}^{xyx(H)} & \sigma_{xy}^{xyx(H)} \\ \sigma_{yz}^{xxy(H)} & \sigma_{zx}^{xxy(H)} & \sigma_{xy}^{xxy(H)} \\ \sigma_{yz}^{yyy(H)} & \sigma_{zx}^{yyy(H)} & \sigma_{xy}^{yyy(H)} \\ \sigma_{yz}^{zzz(H)} & \sigma_{zx}^{zzz(H)} & \sigma_{xy}^{zzz(H)} \\ \sigma_{yz}^{yzy(H)} & \sigma_{zx}^{yzy(H)} & \sigma_{xy}^{yzy(H)} \\ \sigma_{yz}^{zxx(H)} & \sigma_{zx}^{zxx(H)} & \sigma_{xy}^{zxx(H)} \\ \sigma_{yz}^{xyx(H)} & \sigma_{zx}^{xyx(H)} & \sigma_{xy}^{xyx(H)} \\ \sigma_{yz}^{xxy(H)} & \sigma_{zx}^{xxy(H)} & \sigma_{xy}^{xxy(H)} \end{pmatrix}, \quad (11)$$

where the column and row stand for the indices of the MO and $(\mathbf{J} \times \mathbf{E})$, respectively.

The correspondence between the MO conductivity and the multipoles is obtained through the tensor-product relation between multipoles:

$$\hat{X}_{lq} = \sum_{q_1, q_2} \langle l_1, l_2; q_1, q_2 | l_1, l_2; l, q \rangle \hat{X}_{l_1 q_1} \hat{X}_{l_2 q_2}, \quad (12)$$

where l stands for the angular momentum quantum number, q represents the magnetic angular momentum quantum number, and $\langle l_1, l_2; q_1, q_2 | l_1, l_2; l, q \rangle$ denotes the Clebsch-Gordan coefficient.

Based on this expression, we sequentially apply tensor products to the quantities associated with the MO conductivity tensor. First, the coupling between L_m and L_n leads to the polar monopole and quadrupole $\hat{X}_{l'q'}$ for $l' = 0, 2$ shown in Eq. (2):

$$\begin{aligned} \{L_m, L_n\}_+ &\leftrightarrow \hat{X}_{l'q'} \\ &= \sum_{q_m, q_n} \sum_{m, n} \langle 1, 1; q_m, q_n | 1, 1; l', q' \rangle \\ &\quad \times C_{mk_m} C_{nk_n} Y_m Y_n, \end{aligned} \quad (13)$$

where we take the linear transformation between the dipole X_a for $a = x, y, z$ in the real-number expression and $\hat{X}_{1q_a}$ in the complex-number expression, which is connected by $\hat{X}_{1q_a} = \sum_a C_{ak_a} X_a$. Then, the multipole decomposition of the MO term $\hat{X}_{l''q''}$ for $l'' = 1, 2, 3$ is derived by combining Eq. (13) with the axial dipole $\hat{Y}_{1q_\alpha}$ as

$$\begin{aligned} O_{mn}^\alpha &\leftrightarrow \hat{X}_{l''q''} \\ &= \sum_{q', q_\alpha} \langle l', 1; q', q_\alpha | l', 1; l'', q'' \rangle \hat{X}_{l'q'} \hat{Y}_{1q_\alpha} \\ &= \sum_{q', q_\alpha} \sum_{q_m, q_n} \sum_{m, n, \alpha} \langle l', 1; q', q_\alpha | l', 1; l'', q'' \rangle \\ &\quad \times \langle 1, 1; q_m, q_n | 1, 1; l', q' \rangle \\ &\quad \times C_{mk_m} C_{nk_n} C_{\alpha k_\alpha} Y_m Y_n Y_\alpha. \end{aligned} \quad (14)$$

Similarly, the combination of J_i and E_j yields the multipoles $\hat{X}_{l'''q'''}$ for $l''' = 0, 1, 2$:

$$\begin{aligned} J_i E_j &\leftrightarrow \hat{X}_{l'''q'''} \\ &= \sum_{q_i, q_j} \sum_{i, j} \langle 1, 1; q_i, q_j | 1, 1; l''', q''' \rangle C_{ik_i} C_{jk_j} X_i X_j, \end{aligned} \quad (15)$$

where the monopoles and quadrupoles (or dipoles) correspond to the MO Ohmic (or Hall) conductivities as mentioned above. Thus, the multipole component $\hat{X}_{l,q}$ of the MO Ohmic (Hall) conductivity is obtained by combining

$\hat{X}_{l''q''}$ and $\hat{X}_{l'''q'''}$ for $l''' = 0, 2$ ($l''' = 1$) in the form:

$$\begin{aligned} \sigma_{ij}^{mna(O/H)} &\leftrightarrow \hat{X}_{l,q} \\ &= \sum_{q'', q'''} \langle l'', l'''; q'', q''' | l'', l'''; l, q \rangle \hat{X}_{l''q''} \hat{X}_{l'''q'''} \\ &= \sum_{q'', q'''} \sum_{q_i, q_j} \sum_{q', q_\alpha} \sum_{q_m, q_n} \sum_{i, j, m, n, \alpha} \\ &\quad \langle l'', l'''; q'', q''' | l'', l'''; l, q \rangle \\ &\quad \times \langle 1, 1; q_i, q_j | 1, 1; l'', q'' \rangle \\ &\quad \times \langle l', 1; q', q_\alpha | l', 1; l'', q'' \rangle \\ &\quad \times \langle 1, 1; q_m, q_n | 1, 1; l', q' \rangle \\ &\quad \times C_{ik_i} C_{jk_j} C_{mk_m} C_{nk_n} C_{\alpha k_\alpha} X_i X_j Y_m Y_n Y_\alpha. \end{aligned} \quad (16)$$

In Appendix B, we present the MO conductivities for all c - and t -multipoles in the forms of Eqs. (10) and (11), given as the coefficients of $X_i X_j Y_m Y_n Y_\alpha$ for each multipole, derived as linear combinations of Eq. (16).

B. Comparison between the magnetic octupole and spin conductivities

We show the differences between $\sigma_{i;j}^{mna}$ and the spin conductivity $\sigma_{i;j}^\alpha$ from a symmetry perspective. The spin conductivity appears in the linear response regime as

$$J_j^\alpha = \sigma_{i;j}^\alpha E_j, \quad (17)$$

where J_j^α is defined as $\{S_\alpha, v_i\}_+/2$ [50]. Since the spin transforms as axial dipole, the multipole correspondence of the spin Ohmic (or Hall) conductivities are obtained by combinations of axial dipole $\mathbf{Y}$ and X_0, X_{ij} (or $\mathbf{Y}$) components in Eq. (4) [45, 47]. The Ohmic part of the spin conductivities is given by rank-1 to rank-3 multipoles:

$$\mathbf{Y} \otimes (X_0 \oplus X_{ij}) = 2\mathbf{Y} \oplus X_{ij} \oplus Y_{ijk}, \quad (18)$$

while the Hall part is given by rank-0 to rank-2 multipoles:

$$\mathbf{Y} \otimes \mathbf{Y} = X_0 \oplus \mathbf{Y} \oplus X_{ij}. \quad (19)$$

We compare the nonvanishing tensor components of the MO and spin conductivities from the viewpoint of active c -multipole. Here, an ‘‘active multipole’’ denotes a multipole component that is symmetry-allowed under a given magnetic point group (MPG) to contribute to the conductivity tensor, and thus can generate nonvanishing MO or spin conductivity components. To clarify the distinction between the two, we show which tensor components of the MO conductivity and the spin conductivity become finite under the corresponding active (finite) multipoles (X or Y), as summarized in the case of Ohmic contributions in Tables I and II and Hall contributions in Tables III and IV. In the tables, we introduce

the following matrix representation of nonzero MO conductivities:

$$\begin{bmatrix} \{O_{mn}^\alpha\}_{xx} & \{O_{mn}^\alpha\}_{xy} & \{O_{mn}^\alpha\}_{xz} \\ \{O_{mn}^\alpha\}_{yx} & \{O_{mn}^\alpha\}_{yy} & \{O_{mn}^\alpha\}_{yz} \\ \{O_{mn}^\alpha\}_{zx} & \{O_{mn}^\alpha\}_{zy} & \{O_{mn}^\alpha\}_{zz} \end{bmatrix}, \quad (20)$$

where $\{O_{mn}^\alpha\}_{ij}$ lists nonzero tensor components of MO Ohmic/Hall conductivities $\sigma_{ij}^{mn\alpha(O/H)}$ for the input j -directional electric field and the output i -directional cur-

rent. Similarly, the nonzero components of the spin conductivities are summarized as

$$\begin{bmatrix} \{S_\alpha\}_{xx} & \{S_\alpha\}_{xy} & \{S_\alpha\}_{xz} \\ \{S_\alpha\}_{yx} & \{S_\alpha\}_{yy} & \{S_\alpha\}_{yz} \\ \{S_\alpha\}_{zx} & \{S_\alpha\}_{zy} & \{S_\alpha\}_{zz} \end{bmatrix}, \quad (21)$$

where $\{S_\alpha\}_{ij}$ list nonzero tensor components of spin Ohmic/Hall conductivities $\sigma_{ij}^{\alpha(O/H)}$. The corresponding tables for the t -multipoles, i.e., Tables XII, XIII, and XIV, are provided in Appendix C.

As shown in Tables I–IV, there are mainly two different points between MO and spin conductivity tensors. The first difference lies in the rank of the active multipoles. In the MO conductivity tensor, multipoles of rank 0 through rank 5 contribute, whereas in the spin conductivity tensor only multipoles of rank 0 through rank 3 are involved. This difference manifests itself in the distinct conditions under which each conductivity tensor component can contribute. As shown in Table II, multipoles of rank 0, 4, and 5 contribute to the Ohmic-type MO conductivity tensor, but do not contribute to the spin conductivity tensor. Likewise, as summarized in Table IV, multipoles of rank 3 and rank 4 contribute only to the Hall-type MO conductivity tensor. Thus, the MO conductivity tensor can appear even in cases where the spin conductivity tensor is forbidden.

The second difference lies in the nonzero tensor components when the input and output fields are applied in the ij plane. For example, in the case of the active dipole Y_z , no spin Hall current occurs in the xy plane, while nonzero MO Hall currents with O_{yz}^x and O_{zx}^y are allowed, as shown in Table III. In addition, the MO current generally contains spin components that are absent in conventional spin currents. For instance, the X_0 component of $\sigma_{ij}^{\alpha(H)}$ satisfies $J_i^\alpha = \epsilon_{ij\alpha} \sigma_{ij}^{\alpha(H)} E_j$ [55] and carries only the spin- α component, as shown in Table III. In contrast, the X_0 component of $\sigma_{ij}^{mn\alpha(H)}$ carries additional spin components that differ from those of the spin current. These distinctions provide a way of an experi-

mental identification of MO currents and differentiation from spin conductivity.

C. Classification under 32 point groups

We classify the active components of MO Ohmic and Hall conductivities as well as spin conductivities in the representation forms of Eq. (20) and (21) under each CPGs except for the triclinic ones ($\bar{1}$ and 1). The results are summarized in Table V. This classification is based on the multipole classification under the CPGs given in Table VI, where the primary, secondary, and tertiary axes listed in Table VII are used. In these tables, the abbreviation as $\mathbf{O}_{mn}^\alpha = \{O_{yz}^x, O_{zx}^y, O_{xy}^z\}$, $\mathbf{O}_{yz} = \{O_{yz}^x, O_{yz}^y, O_{yz}^z\}$, $\mathbf{O}_{zx} = \{O_{zx}^x, O_{zx}^y, O_{zx}^z\}$, $\mathbf{O}_{xy} = \{O_{xy}^x, O_{xy}^y, O_{xy}^z\}$, $\mathbf{O}^i = \{O_{xx}^i, O_{yy}^i, O_{zz}^i\}$ for $i = x, y, z$, all = $\{\mathbf{O}_{yz}, \mathbf{O}_{zx}, \mathbf{O}_{xy}, \mathbf{O}^x, \mathbf{O}^y, \mathbf{O}^z\}$ is used for notational simplicity. It is noted that the leftmost column of Table V represents CPGs in which the same components of the MO and spin conductivities become active, rather than those in which the same multipoles become active as in Table VI.

As shown in Table V, both MO and spin Hall currents in the ij -plane are generated under all CPGs. In contrast, spin Ohmic conductivities are not activated under some CPGs, meaning that the MO Ohmic current can be distinguished from the spin one. For instance, under $m\bar{3}m$, 432 , and $\bar{4}3m$, only the MO Ohmic conductivities are allowed with the active multipoles X_0 and X_4 , as shown in Eqs. (5) and (18).

TABLE I. Symmetry-allowed MO Ohmic conductivity tensors and spin Ohmic conductivity tensors with activations from rank-1 to -3 c-multipoles (c-MP). The matrix elements only the upper-triangular parts are shown, since the lower-triangular components are given by satisfying $\sigma_{ij}^{mna(O)} = \sigma_{ji}^{mna(O)}$, $\sigma_{ij}^{mna(H)} = -\sigma_{ji}^{mna(H)}$, $\sigma_{ij}^{\alpha(O)} = \sigma_{ji}^{\alpha(O)}$, and $\sigma_{ij}^{\alpha(H)} = -\sigma_{ji}^{\alpha(H)}$. The symbol $-$ indicates no symmetry-allowed components. The abbreviation as $\mathbf{O}_{mn}^{\alpha} = \{O_{yz}^x, O_{zx}^y, O_{xy}^z\}$, $\mathbf{O}_{yz} = \{O_{yz}^x, O_{yz}^y, O_{yz}^z\}$, $\mathbf{O}_{zx} = \{O_{zx}^x, O_{zx}^y, O_{zx}^z\}$, $\mathbf{O}_{xy} = \{O_{xy}^x, O_{xy}^y, O_{xy}^z\}$, $\mathbf{O}^i = \{O_{xx}^i, O_{yy}^i, O_{zz}^i\}$ for $i = x, y, z$, all $= \{O_{yz}^x, O_{zx}^y, O_{xy}^z, \mathbf{O}^x, \mathbf{O}^y, \mathbf{O}^z\}$ is used for notational simplicity.

c-MP	MO conductivity	spin conductivity
Y_x	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^{\alpha} \\ \hline & & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \end{bmatrix}$	$\begin{bmatrix} S_x & S_y & S_z \\ \hline & S_x & - \\ \hline & & S_x \end{bmatrix}$
Y_y	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^{\alpha} \\ \hline & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & - \\ \hline & S_y & S_z \\ \hline & & S_y \end{bmatrix}$
Y_z	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & \mathbf{O}_{lm}^{\alpha} & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^y, \mathbf{O}^y \\ \hline & & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x \\ \hline & S_z & S_y \\ \hline & & S_z \end{bmatrix}$
X_u	$\begin{bmatrix} \mathbf{O}_{lm}^{\alpha} & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^y, \mathbf{O}^y \\ \hline & \mathbf{O}_{lm}^{\alpha} & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & & O_{yz}^x, O_{zx}^y \end{bmatrix}$	$\begin{bmatrix} - & - & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
X_v	$\begin{bmatrix} \mathbf{O}_{lm}^{\alpha} & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^y, \mathbf{O}^y \\ \hline & \mathbf{O}_{lm}^{\alpha} & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & & \mathbf{O}_{lm}^{\alpha} \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
X_{yz}	$\begin{bmatrix} O_{xy}^x, O_{zx}^z, O_{xy}^y, O_{zx}^z & O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{zx}^y, O_{xy}^z \\ \hline & & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \end{bmatrix}$	$\begin{bmatrix} - & S_y & S_z \\ \hline & S_x & - \\ \hline & & S_x \end{bmatrix}$
X_{zx}	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{yz}^x, O_{xy}^z \\ \hline & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & - \\ \hline & - & S_z \\ \hline & & S_y \end{bmatrix}$
X_{xy}	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{yz}^x, O_{zx}^y & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^y, \mathbf{O}^y \\ \hline & & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x \\ \hline & S_z & S_y \\ \hline & & - \end{bmatrix}$
Y_{xyz}	$\begin{bmatrix} \mathbf{O}_{lm}^{\alpha} & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^y, O_{yz}^y, \mathbf{O}^y \\ \hline & \mathbf{O}_{lm}^{\alpha} & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & & \mathbf{O}_{lm}^{\alpha} \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
Y_x^{α}	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^{\alpha} \\ \hline & & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \end{bmatrix}$	$\begin{bmatrix} S_x & S_y & S_z \\ \hline & S_x & - \\ \hline & & S_x \end{bmatrix}$
Y_y^{α}	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^{\alpha} \\ \hline & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & - \\ \hline & S_y & S_z \\ \hline & & S_y \end{bmatrix}$
Y_z^{α}	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & \mathbf{O}_{lm}^{\alpha} & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^y, \mathbf{O}^y \\ \hline & & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x \\ \hline & S_z & S_y \\ \hline & & S_z \end{bmatrix}$
Y_x^{β}	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, O_{xy}^x, O_{yz}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{yz}^x, O_{xy}^z \\ \hline & & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \end{bmatrix}$	$\begin{bmatrix} - & S_y & S_z \\ \hline & S_x & - \\ \hline & & S_x \end{bmatrix}$
Y_y^{β}	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{yz}^x, O_{xy}^z \\ \hline & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & - \\ \hline & - & S_z \\ \hline & & S_y \end{bmatrix}$
Y_z^{β}	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{yz}^x, O_{zx}^y & O_{xy}^y, O_{zx}^x, \mathbf{O}^x \\ \hline & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^y, \mathbf{O}^y \\ \hline & & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x \\ \hline & S_z & S_y \\ \hline & & - \end{bmatrix}$

TABLE II. Symmetry-allowed MO Ohmic conductivity tensors with activations of rank-0, -4, and -5 c -multipoles (c -MP). It is noted that a corresponding spin conductivity tensor does not exist.

c-MP	MO conductivity	c-MP	MO conductivity
X_0	$\begin{bmatrix} O^y_{zx}, O^z_{xy} & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} \\ \hline & O^x_{yz}, O^z_{xy} & O^x_{yy}, O^z_{zz}, O^x_{xy}, O^z_{zx} \\ \hline & & O^x_{yz}, O^y_{zx} \end{bmatrix}$	Y_{5u}	$\begin{bmatrix} O^{\alpha}_{lm} & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} & O^x_{xy}, O^y_{xx}, O^y_{yy} \\ \hline & O^{\alpha}_{lm} & O^x_{xx}, O^y_{yy}, O^z_{xy} \\ \hline & & - \end{bmatrix}$
X_4	$\begin{bmatrix} O^y_{zx}, O^z_{xy} & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} \\ \hline & O^x_{yz}, O^z_{xy} & O^x_{yy}, O^z_{zz}, O^x_{xy}, O^z_{zx} \\ \hline & & O^x_{yz}, O^y_{zx} \end{bmatrix}$	Y_{5v}	$\begin{bmatrix} O^{\alpha}_{lm} & O^x_{zx}, O^y_{yz}, O^z_{xx} & O^x_{xy}, O^z_{zx}, O^y_{yy} \\ \hline & O^{\alpha}_{lm} & O^x_{xy}, O^z_{zx}, O^x_{xx} \\ \hline & & O^{\alpha}_{lm} \end{bmatrix}$
X_{4u}	$\begin{bmatrix} O^{\alpha}_{lm} & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} & O^x_{xy}, O^z_{yz}, O^y_{yy} \\ \hline & O^{\alpha}_{lm} & O^y_{xy}, O^z_{zx}, O^x_{xx} \\ \hline & & O^x_{yz}, O^y_{zx} \end{bmatrix}$	$Y_{5x}^{\alpha 1}$	$\begin{bmatrix} O^y_{xy}, O^z_{zx}, O^x_{xx} & O^x_{xy}, O^z_{yz}, O^y_{yy} & O^x_{zx}, O^y_{yz}, O^z_{xx} \\ \hline & O^y_{xy}, O^z_{zx}, O^x_{xx} & O^{\alpha}_{lm} \\ \hline & & O^y_{xy}, O^z_{zx}, O^x_{xx} \end{bmatrix}$
X_{4v}	$\begin{bmatrix} O^{\alpha}_{lm} & O^x_{zx}, O^y_{yz}, O^z_{xx} & O^x_{xy}, O^z_{yz}, O^y_{yy} \\ \hline & O^{\alpha}_{lm} & O^y_{xy}, O^z_{zx}, O^x_{xx} \\ \hline & & O^{\alpha}_{lm} \end{bmatrix}$	$Y_{5y}^{\alpha 1}$	$\begin{bmatrix} O^x_{xy}, O^z_{zx}, O^y_{yy} & O^y_{xy}, O^z_{zx}, O^x_{xx} & O^{\alpha}_{lm} \\ \hline & O^x_{xy}, O^z_{zx}, O^y_{yy} & O^x_{zx}, O^y_{yz}, O^z_{xx} \\ \hline & & O^x_{xy}, O^z_{zx}, O^y_{yy} \end{bmatrix}$
X_{4x}^{α}	$\begin{bmatrix} - & O^y_{yy}, O^z_{yz}, O^z_{yz} & O^y_{yz}, O^z_{yy}, O^z_{zz} \\ \hline & O^x_{yy}, O^z_{zz}, O^y_{xy}, O^z_{zx} & O^{\alpha}_{lm} \\ \hline & & O^x_{yy}, O^z_{zz}, O^y_{xy}, O^z_{zx} \end{bmatrix}$	$Y_{5z}^{\alpha 1}$	$\begin{bmatrix} O^x_{zx}, O^y_{yz}, O^z_{xx} & O^{\alpha}_{lm} & O^y_{xy}, O^z_{zx}, O^x_{xx} \\ \hline & O^x_{zx}, O^y_{yz}, O^z_{xx} & O^x_{xy}, O^z_{zx}, O^y_{yy} \\ \hline & & O^x_{zx}, O^y_{yz}, O^z_{xx} \end{bmatrix}$
X_{4y}^{α}	$\begin{bmatrix} O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} & O^x_{xx}, O^z_{zz}, O^z_{zx} & O^{\alpha}_{lm} \\ \hline & - & O^x_{zx}, O^z_{zz}, O^z_{zx} \\ \hline & & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} \end{bmatrix}$	$Y_{5x}^{\alpha 2}$	$\begin{bmatrix} - & O^y_{yy}, O^z_{yz}, O^z_{yz} & O^y_{yz}, O^z_{yy}, O^z_{zz} \\ \hline & O^x_{yy}, O^z_{zz}, O^y_{xy}, O^z_{zx} & O^{\alpha}_{lm} \\ \hline & & O^x_{yy}, O^z_{zz}, O^y_{xy}, O^z_{zx} \end{bmatrix}$
X_{4z}^{α}	$\begin{bmatrix} O^x_{zx}, O^y_{yz}, O^x_{xx}, O^z_{yy} & O^{\alpha}_{lm} & O^x_{xx}, O^x_{yy}, O^y_{xy} \\ \hline & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} & O^x_{xy}, O^y_{xx}, O^y_{yy} \\ \hline & & - \end{bmatrix}$	$Y_{5y}^{\alpha 2}$	$\begin{bmatrix} O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} & O^x_{xx}, O^z_{zz}, O^z_{zx} & O^{\alpha}_{lm} \\ \hline & - & O^x_{zx}, O^z_{zz}, O^z_{zx} \\ \hline & & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} \end{bmatrix}$
X_{4x}^{β}	$\begin{bmatrix} O^x_{yy}, O^z_{zz}, O^y_{xy}, O^z_{zx} & O^x_{xy}, O^z_{yz}, O^y_{yy} & O^x_{zx}, O^y_{yz}, O^z_{xx} \\ \hline & O^y_{xy}, O^z_{zx}, O^x_{xx} & O^y_{zx}, O^z_{xy} \\ \hline & & O^y_{xy}, O^z_{zx}, O^x_{xx} \end{bmatrix}$	$Y_{5z}^{\alpha 2}$	$\begin{bmatrix} O^x_{zx}, O^y_{yz}, O^x_{xx}, O^z_{yy} & O^{\alpha}_{lm} & O^x_{xx}, O^x_{yy}, O^y_{xy} \\ \hline & O^x_{zx}, O^y_{yz}, O^x_{xx}, O^z_{yy} & O^x_{xy}, O^y_{xx}, O^y_{yy} \\ \hline & & - \end{bmatrix}$
X_{4y}^{β}	$\begin{bmatrix} O^x_{xy}, O^z_{yz}, O^y_{yy} & O^y_{xy}, O^z_{zx}, O^x_{xx} & O^x_{yz}, O^z_{xy} \\ \hline & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} & O^x_{zx}, O^y_{yz}, O^z_{xx} \\ \hline & & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} \end{bmatrix}$	Y_{5x}^{β}	$\begin{bmatrix} O^x_{yy}, O^z_{zz}, O^y_{xy}, O^z_{zx} & O^x_{xy}, O^y_{xx}, O^y_{yy} & O^x_{zx}, O^z_{xx}, O^z_{zz} \\ \hline & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} & - \\ \hline & & O^x_{xx}, O^z_{zz}, O^z_{zx} \end{bmatrix}$
X_{4z}^{β}	$\begin{bmatrix} O^x_{zx}, O^y_{yz}, O^z_{xx} & O^x_{yz}, O^y_{zx} & O^y_{xy}, O^z_{xx}, O^x_{xx} \\ \hline & O^x_{zx}, O^y_{yz}, O^z_{xx} & O^x_{xy}, O^z_{xx}, O^y_{yy} \\ \hline & & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} \end{bmatrix}$	Y_{5y}^{β}	$\begin{bmatrix} O^x_{xy}, O^y_{xx}, O^y_{yy} & O^x_{xx}, O^x_{yy}, O^y_{xy} & - \\ \hline & O^x_{xy}, O^y_{xx}, O^y_{zz}, O^z_{yz} & O^y_{yz}, O^z_{yy}, O^z_{zz} \\ \hline & & O^y_{yy}, O^z_{zz}, O^z_{yz} \end{bmatrix}$
		Y_{5z}^{β}	$\begin{bmatrix} O^x_{zx}, O^z_{xx}, O^z_{zz} & - & O^x_{xx}, O^x_{zz}, O^z_{xx} \\ \hline & O^y_{yz}, O^z_{yy}, O^z_{zz} & O^y_{yy}, O^y_{zz}, O^z_{yz} \\ \hline & & O^x_{zx}, O^y_{yz}, O^z_{xx}, O^z_{yy} \end{bmatrix}$

TABLE III. Symmetry-allowed MO Hall conductivity tensors and spin Hall conductivity tensors with activations from rank-0 to -2 c -multipoles(c -MP).

c -MP	MO conductivity	spin conductivity
X_0	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ \hline & - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
Y_x	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & - & O_{zx}^y, O_{xy}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_y & S_z \\ \hline & - & - \\ \hline & & - \end{bmatrix}$
Y_y	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{yz}^x, O_{xy}^z \\ \hline & - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_x & - \\ \hline & - & S_z \\ \hline & & - \end{bmatrix}$
Y_z	$\begin{bmatrix} - & O_{yz}^x, O_{zx}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & - & S_x \\ \hline & - & S_y \\ \hline & & - \end{bmatrix}$
X_u	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ \hline & - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
X_v	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ \hline & - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & - & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
X_{yz}	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & - & \mathbf{O}_{lm}^\alpha \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & - & S_x \\ \hline & - & S_y \\ \hline & & - \end{bmatrix}$
X_{zx}	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ \hline & - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_y & S_z \\ \hline & - & - \\ \hline & & - \end{bmatrix}$
X_{xy}	$\begin{bmatrix} - & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_x & - \\ \hline & - & S_z \\ \hline & & - \end{bmatrix}$

TABLE IV. Symmetry-allowed MO Hall conductivity tensors with activations of rank-3 and -4 c -multipoles (c -MP). It is noted that a corresponding spin conductivity tensor does not exist.

c -MP	MO conductivity	c -MP	MO conductivity
Y_{xyz}	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{xx}^y, O_{zz}^y, O_{yz}^z \\ - & - & O_{yy}^x, O_{zz}^x, O_{xy}^y, O_{zx}^z \\ - & - & - \end{bmatrix}$	X_4	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ - & - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ - & - & - \end{bmatrix}$
Y_x^α	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ - & - & O_{zx}^y, O_{xy}^z \\ - & - & - \end{bmatrix}$	X_{4u}	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ - & - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ - & - & - \end{bmatrix}$
Y_y^α	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^x, O_{yz}^z \\ - & - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ - & - & - \end{bmatrix}$	X_{4v}	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{yy}^y, O_{yz}^z, O_{zz}^z \\ - & - & O_{xx}^x, O_{zz}^x, O_{zx}^z \\ - & - & - \end{bmatrix}$
Y_z^α	$\begin{bmatrix} - & O_{yz}^x, O_{zx}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ - & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ - & - & - \end{bmatrix}$	X_{4x}^α	$\begin{bmatrix} - & O_{yy}^y, O_{yz}^y, O_{zz}^z & O_{yz}^y, O_{yy}^z, O_{zz}^z \\ - & - & - \\ - & - & - \end{bmatrix}$
Y_x^β	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ - & - & \mathbf{O}_{lm}^\alpha \\ - & - & - \end{bmatrix}$	X_{4y}^α	$\begin{bmatrix} - & O_{xx}^x, O_{zz}^x, O_{zx}^z & - \\ - & - & O_{zx}^x, O_{xx}^z, O_{zz}^z \\ - & - & - \end{bmatrix}$
Y_y^β	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ - & - & O_{zx}^x, O_{yz}^z, \mathbf{O}^z \\ - & - & - \end{bmatrix}$	X_{4z}^α	$\begin{bmatrix} - & - & O_{xx}^x, O_{yy}^y, O_{xy}^y \\ - & - & O_{xy}^x, O_{xx}^z, O_{yy}^z \\ - & - & - \end{bmatrix}$
Y_z^β	$\begin{bmatrix} - & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ - & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ - & - & - \end{bmatrix}$	X_{4x}^β	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ - & - & \mathbf{O}_{lm}^\alpha \\ - & - & - \end{bmatrix}$
		X_{4y}^β	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ - & - & O_{zx}^x, O_{yz}^z, \mathbf{O}^z \\ - & - & - \end{bmatrix}$
		X_{4z}^β	$\begin{bmatrix} - & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ - & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ - & - & - \end{bmatrix}$

TABLE V. Symmetry-allowed MO and spin conductivity tensors under the CPGs except for the triclinic ones. The upper and lower panels represent the results for the Ohmic and Hall conductivities, respectively. In this table, the abbreviation as $\mathbf{S} = \{S_x, S_y, S_z\}$ is used for notational simplicity.

CPGs	MO Ohmic conductivity	spin Ohmic conductivity
$m\bar{3}m, 432, \bar{4}3m$	$\begin{bmatrix} O_{zx}^y, O_{xy}^z & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{xx}^y, O_{zz}^y, O_{yz}^z \\ \hline & O_{yz}^x, O_{xy}^z & O_{yy}^x, O_{zz}^y, O_{xx}^z, O_{yz}^z \\ \hline & & O_{yz}^x, O_{yz}^y \end{bmatrix}$	—
$23, m\bar{3}$ $mmm, 222, mm2$	$\begin{bmatrix} O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z & O_{xy}^x, O_{yz}^z, O_{yy}^x \\ \hline & O_{lm}^\alpha & O_{xy}^y, O_{zz}^y, O_{xx}^z \\ \hline & & O_{lm}^\alpha \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
$4/mmm, 422$ $\bar{4}2m, 4mm$	$\begin{bmatrix} O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^z, O_{yy}^x \\ \hline & O_{lm}^\alpha & O_{xy}^y, O_{zz}^y, O_{xx}^z \\ \hline & & O_{yz}^x, O_{yz}^y \end{bmatrix}$	$\begin{bmatrix} - & - & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
$4/m, 4, \bar{4}$	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z, O_{lm}^\alpha & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{xx}^y, O_{yy}^x \\ \hline & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{lm}^\alpha & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{xx}^y, O_{yy}^x \\ \hline & & O_{yz}^x, O_{xx}^y, O_{yz}^y, O_{xx}^z, O_{xx}^y \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x, S_y \\ \hline & S_z & S_x, S_y \\ \hline & & S_z \end{bmatrix}$
$2/m, 2, m$	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z, O_{xx}^z & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha \\ \hline & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{xx}^y, O_{xx}^z \\ \hline & & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha \end{bmatrix}$	$\begin{bmatrix} S_y & S_x, S_z & S_y \\ \hline & S_y & S_x, S_z \\ \hline & & S_y \end{bmatrix}$
$6/mmm, 622$ $\bar{6}m2, 6mm$	$\begin{bmatrix} O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^z, O_{yy}^x \\ \hline & O_{lm}^\alpha & O_{xy}^y, O_{zz}^y, O_{xx}^z \\ \hline & & O_{yz}^x, O_{yz}^y \end{bmatrix}$	$\begin{bmatrix} - & - & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
$6/m, 6, \bar{6}$	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z, O_{lm}^\alpha & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{xx}^y, O_{yy}^x \\ \hline & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{lm}^\alpha & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{xx}^y, O_{yy}^x \\ \hline & & O_{yz}^x, O_{xx}^y, O_{yz}^y, O_{xx}^z, O_{xx}^y \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x, S_y \\ \hline & S_z & S_x, S_y \\ \hline & & S_z \end{bmatrix}$
$\bar{3}m, 32, 3m$	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z, O_{xx}^z & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha \\ \hline & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{xx}^y, O_{xx}^z \\ \hline & & O_{yz}^x, O_{xx}^y, O_{yz}^y, O_{xx}^z, O_{xx}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & S_y \\ \hline & S_y & S_x \\ \hline & & - \end{bmatrix}$
$\bar{3}, 3$	$\begin{bmatrix} \text{all} & O_{xx}^x, O_{yy}^y, O_{xx}^y, O_{xy}^x, O_{yz}^y, O_{xx}^z & O_{xx}^z, O_{yy}^z, O_{xx}^y, O_{xy}^x, O_{yz}^y, O_{xx}^z \\ \hline & \text{all} & O_{xx}^z, O_{yy}^z, O_{xx}^y, O_{xy}^x, O_{yz}^y, O_{xx}^z \\ \hline & & O_{xx}^x, O_{yy}^y, O_{xx}^z, O_{xy}^x, O_{yz}^y, O_{xx}^z, O_{xy}^y, O_{xx}^z \end{bmatrix}$	$\begin{bmatrix} \mathbf{S} & S_x, S_y & S_x, S_y \\ \hline & \mathbf{S} & S_x, S_y \\ \hline & & S_z \end{bmatrix}$
CPGs	MO Hall conductivity	spin Hall conductivity
$m\bar{3}m, 432, \bar{4}3m$ $23, m\bar{3}$ $4/mmm, 422$ $\bar{4}2m, 4mm$ $mmm, 222, mm2$	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z & O_{xy}^x, O_{yz}^z, O_{yy}^x \\ \hline & - & O_{xy}^y, O_{zz}^y, O_{xx}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
$4/m, 4, \bar{4}$	$\begin{bmatrix} - & O_{yz}^x, O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{xx}^z, O_{yy}^x \\ \hline & - & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{xx}^z, O_{yy}^x \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_x, S_y \\ \hline & - & S_x, S_y \\ \hline & & - \end{bmatrix}$
$2/m, 2, m$	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{xy}^x, O_{yz}^z, O_{xx}^z & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{lm}^\alpha \\ \hline & - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{xx}^y, O_{xx}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_x, S_z & S_y \\ \hline & - & S_x, S_z \\ \hline & & - \end{bmatrix}$
$6/mmm, 622$ $\bar{6}m2, 6mm$	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z & O_{xy}^x, O_{yz}^z, O_{yy}^x \\ \hline & - & O_{xy}^y, O_{zz}^y, O_{xx}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
$6/m, 6, \bar{6}$	$\begin{bmatrix} - & O_{yz}^x, O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{xx}^z, O_{yy}^x \\ \hline & - & O_{xy}^x, O_{yz}^z, O_{xx}^y, O_{xx}^z, O_{yy}^x \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_x, S_y \\ \hline & - & S_x, S_y \\ \hline & & - \end{bmatrix}$
$\bar{3}m, 32, 3m$	$\begin{bmatrix} - & O_{xx}^x, O_{yy}^y, O_{xx}^z, O_{yz}^y, O_{xy}^x, O_{yz}^z & O_{xx}^z, O_{yy}^z, O_{xx}^y, O_{lm}^\alpha \\ \hline & - & O_{xx}^z, O_{yy}^z, O_{xx}^y, O_{xx}^z, O_{yy}^z, O_{xx}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$
$\bar{3}, 3$	$\begin{bmatrix} - & O_{xx}^x, O_{yy}^y, O_{xx}^z, O_{yz}^y, O_{xy}^x, O_{yz}^z, O_{xx}^z & O_{xx}^z, O_{yy}^z, O_{xx}^y, O_{lm}^\alpha \\ \hline & - & O_{xx}^z, O_{yy}^z, O_{xx}^y, O_{xx}^z, O_{yy}^z, O_{xx}^z \\ \hline & & - \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_x, S_y \\ \hline & - & S_x, S_y \\ \hline & & - \end{bmatrix}$

TABLE VII. Primary, secondary, and tertiary axes of seven crystal systems with respect to the symmetry operations in the Cartesian coordinates [47].

	Primary	Secondary	Tertiary
Cubic	$\langle 100 \rangle$	$\langle 111 \rangle$	$\langle 110 \rangle$
Tetragonal	$[001]$	$[100]$	$[110]$
Orthorhombic	$[100]$	$[010]$	$[001]$
Monoclinic	$[010]$		
Triclinic			
Hexagonal	$[001]$	$[100]$	$[010]$
Trigonal	$[001]$	$[010]$	

D. Time-reversal property

We have so far assumed the presence of $\mathcal{T}$ symmetry. Meanwhile, analogously to spin conductivity [15, 56–58], the MO conductivity tensor likewise includes magnetic contributions when $\mathcal{T}$ symmetry is broken. In the following, we investigate the role of $\mathcal{T}$ symmetry on the MO conductivity $\sigma_{i;j}^{mn\alpha}$ by analyzing the Kubo formula within the linear response theory [45, 47, 59]:

$$\sigma_{i;j}^{mn\alpha} = -i \frac{e\hbar}{V} \sum_{\xi_1 \xi_2} \frac{f(\epsilon_{\xi_1}) - f(\epsilon_{\xi_2})}{\epsilon_{\xi_1} - \epsilon_{\xi_2}} \frac{\langle \xi_1 | J_j^{mn\alpha} | \xi_2 \rangle \langle \xi_2 | v_i | \xi_1 \rangle}{\epsilon_{\xi_1} - \epsilon_{\xi_2} + i\delta}, \quad (22)$$

where e , V , and δ denote the elementary charge, the system volume, and the broadening factor, respectively. ξ_i is a set consisting of the band index and wave vector $\mathbf{k}$, and ϵ_{ξ_i} is the band energy with the eigenvector $|\xi_i\rangle$. $f(\epsilon_{\xi_i})$ is the Fermi distribution function,

$$f(\epsilon_{\xi_i}) = \frac{1}{e^{(\epsilon_{\xi_i} - \mu)/k_B T} + 1}, \quad (23)$$

where μ , k_B , and T are the chemical potential, the Boltzmann constant, and the temperature, respectively. The velocity operator v_i is defined as

$$v_i = \frac{1}{\hbar} \frac{\partial H}{\partial k_i}, \quad (24)$$

where H is the tight-binding Hamiltonian.

The expression in Eq. (22) is decomposed into two components according to the $\mathcal{T}$ parity:

$$\sigma_{i;j}^{mn\alpha(J)} = -\frac{e\hbar\delta}{V} \sum_{\xi_1 \xi_2} \frac{f(\epsilon_{\xi_1}) - f(\epsilon_{\xi_2})}{\epsilon_{\xi_1} - \epsilon_{\xi_2}} \times \frac{\langle \xi_1 | J_j^{mn\alpha} | \xi_2 \rangle \langle \xi_2 | v_i | \xi_1 \rangle}{(\epsilon_{\xi_1} - \epsilon_{\xi_2})^2 + \delta^2}, \quad (25)$$

$$\sigma_{i;j}^{mn\alpha(E)} = -i \frac{e\hbar}{V} \sum_{\xi_1 \xi_2}^{\epsilon_{\xi_1} \neq \epsilon_{\xi_2}} \frac{f(\epsilon_{\xi_1}) - f(\epsilon_{\xi_2})}{(\epsilon_{\xi_1} - \epsilon_{\xi_2})^2 + \delta^2} \times \langle \xi_1 | J_j^{mn\alpha} | \xi_2 \rangle \langle \xi_2 | v_i | \xi_1 \rangle, \quad (26)$$

where $\sigma_{i;j}^{mn\alpha(J)}$ generates the dissipative MO current, whose main contribution comes from the $\xi_1 = \xi_2$ term and is proportional to $1/\delta$, while $\sigma_{i;j}^{mn\alpha(E)}$ gives rise to dissipationless one, which is independent of δ in the clean limit of $\delta \rightarrow 0$. Noted that both $\sigma_{i;j}^{mn\alpha(J)}$ and $\sigma_{i;j}^{mn\alpha(E)}$ are real quantities.

By acting the $\mathcal{T}$ operator on $\sigma_{i;j}^{mn\alpha(J/E)}$, one finds that $\sigma_{i;j}^{mn\alpha(J)}$ and $\sigma_{i;j}^{mn\alpha(E)}$ exhibit opposite $\mathcal{T}$ properties:

$$\mathcal{T}(\sigma_{i;j}^{mn\alpha(J)}) = t_{J_i}^{mn\alpha} t_{v_j} \sigma_{i;j}^{mn\alpha(J)} \quad (27)$$

$$\mathcal{T}(\sigma_{i;j}^{mn\alpha(E)}) = -t_{J_i}^{mn\alpha} t_{v_j} \sigma_{i;j}^{mn\alpha(E)}, \quad (28)$$

where t_A denotes the $\mathcal{T}$ parity of the operator A , defined as $\mathcal{T}(A) = t_A A$. Owing to $t_{J_i}^{mn\alpha} = +1$ and $t_{v_j} = -1$, $\sigma_{i;j}^{mn\alpha(J)}$ ($\sigma_{i;j}^{mn\alpha(E)}$) is constructed by M and MT (E and ET) multipoles. Thus, $\sigma_{i;j}^{mn\alpha(J)}$ vanishes in the $\mathcal{T}$ -even system, meaning that dissipative MO currents are not allowed under point groups whose symmetry operations contain $\mathcal{T}$.

Meanwhile, the dissipative MO currents are symmetry-allowed under the point groups which do not contain $\mathcal{T}$. Examples of such point groups include MPGs. MPGs are classified into three classes according to the presence or absence of $\mathcal{T}$ symmetry. One class possesses $\mathcal{T}$ symmetry, while the other two do not. The first class consists of ordinary CPGs, denoted by $M^{(I)}$, which include only unitary symmetry operations G . The second class is the gray MPGs, $M^{(II)}$, defined as

$$M^{(II)} = G + \mathcal{T}G, \quad (29)$$

which explicitly contains $\mathcal{T}$ symmetry. The third class is black-and-white MPGs, $M^{(III)}$, given by

$$M^{(III)} = H + \mathcal{T}(G - H), \quad (30)$$

where H is a halving unitary subgroup of G . From these definitions, M and/or MT multipoles become active under $M^{(I)}$ and $M^{(III)}$. For example, in the case of the black-and-white MPG $m\bar{3}m'$, the M octupole M_{xyz} is activated in addition to E multipoles Q_0 and Q_4 [47]. In this situation, the MO conductivity tensor components associated with Y_{xyz} become nonzero, which arises from the dissipation process.

III. MODEL ANALYSIS

As discussed in Sec. II, the MO conductivity $\sigma_{i;j}^{mn\alpha}$ becomes nonzero once the corresponding multipoles are activated. To demonstrate this, we calculate the MO conductivity tensor in the cubic system, where the axial ET octupole G_{xyz} belonging to the A_{2g} representation under the point group $m\bar{3}m$ in Table VIII becomes active through the symmetry reduction to $m\bar{3}$.

TABLE VIII. The character table of $m\bar{3}m$ and the irreducible representation (IRREP) of multipoles Q_0 , Q_4 , and G_{xyz} .

multipole (IRREP)	E	$8C_3$	$3C_2$	$6C_4$	$6C_2'$	I	$8IC_3$	$3\sigma_h$	$6IC_4$	$6\sigma_d$
Q_0, Q_4 (A_{1g})	1	1	1	1	1	1	1	1	1	1
G_{xyz} (A_{2g})	1	1	1	-1	-1	1	1	1	-1	-1

A. Tight-binding model

We construct a tight-binding Hamiltonian on a three-dimensional simple cubic lattice under the $m\bar{3}m$ symmetry, considering three p orbitals $\{p_x, p_y, p_z\}$ and three f

orbitals $\{f_x^\beta, f_y^\beta, f_z^\beta\}$ at each lattice site. The choice of the p - f orbital model is motivated by symmetry considerations. Specifically, we start from a highly symmetric cubic system in order to impose strong constraints on the allowed tensor components, and then consider the minimal symmetry lowering from $m\bar{3}m$ to $m\bar{3}$. Under this symmetry reduction, the ET octupole G_{xyz} belonging to the A_{2g} representation emerges as the relevant order parameter, as shown in Table VIII. The p - f orbital space therefore provides the minimal microscopic basis that can host and activate this multipole degree of freedom. Although similar MO responses may also occur in s - p or p - d models, the p - f orbital space provides the minimal basis in which the ET octupole G_{xyz} is activated under $m\bar{3}m \rightarrow m\bar{3}$ symmetry lowering.

The Hamiltonian is given by

$$\mathcal{H} = \sum_{\mathbf{k}} \sum_{a,a'} \sum_{\sigma,\sigma'} \langle a\sigma | H(\mathbf{k}) | a'\sigma' \rangle c_{\mathbf{k}a\sigma}^\dagger c_{\mathbf{k}a'\sigma'}, \quad (31)$$

where $c_{\mathbf{k}a\sigma}^\dagger$ and $c_{\mathbf{k}a\sigma}$ are the fermionic creation and annihilation operators of the wave number $\mathbf{k}$, the orbital $a = p_x, p_y, p_z, f_x^\beta, f_y^\beta, f_z^\beta$, and the spin $\sigma = \uparrow, \downarrow$. The Hamiltonian matrix $H(\mathbf{k})$ consists of four parts as follows:

$$H(\mathbf{k}) = H_{\text{hop}}(\mathbf{k}) + H_{\text{SOC}} + \Delta_{pf} + gG_{xyz}. \quad (32)$$

The first term, $H_{\text{hop}}(\mathbf{k})$, represents the symmetry-allowed first- and second-nearest-neighbor hopping Hamiltonian based on the Slater-Koster parameters [60, 61], which include $\{(pp\sigma), (pp\pi)\}$ for p orbitals, $\{(ff\sigma), (ff\pi), (ff\delta), (ff\phi)\}$ for f orbitals, and $\{(pf\sigma), (pf\pi)\}$ for the off-diagonal p - f hybridization. Here, the second-nearest-neighbor hopping parameters are scaled by a factor of 1/2, and the explicit forms of the 6×6 matrices in spinless space are given in Appendix D. The second term, H_{SOC} , represents the spin-orbit coupling (SOC), given by

$$H_{\text{SOC}} = \sum_{\text{orb}=p,f} \sum_{i=x,y,z} \lambda_{\text{orb}} L_i \sigma_i, \quad (33)$$

where λ_{orb} for each p and f orbital (represented by: orb) denotes the strength of the SOC. σ_i represents the Pauli matrices and the angular momentum matrices L_i are defined by

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, L_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, L_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (34)$$

where the basis functions are taken as $\{p_x, p_y, p_z\}$ or $\{f_x^\beta, f_y^\beta, f_z^\beta\}$. The third term represents the energy splitting between p and f orbitals, which is given by

$$\Delta_{pf} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta \end{pmatrix}, \quad (35)$$

where the basis functions are taken as $\{p_x, p_y, p_z, f_x^\beta, f_y^\beta, f_z^\beta\}$. These three terms are symmetry-allowed under the point group $m\bar{3}m$.

To describe the symmetry reduction from $m\bar{3}m$ to $m\bar{3}$, we introduce the symmetry-lowering onsite hybridization as the fourth term in the Hamiltonian, which is given by

$$gG_{xyz} = \begin{pmatrix} 0 & 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & g \\ g & 0 & 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 0 & 0 \end{pmatrix}. \quad (36)$$

Here, G_{xyz} represents the ET octupole, whose operator expression is given by $yzg_3^x + zyg_3^y + xyg_3^z$ with $\mathbf{g}_3 = i\mathbf{L}(\mathbf{L} \cdot \mathbf{L})$ except for the numerical coefficient [62], and g is the coupling constant. When $g \neq 0$, the symmetry of the system reduces to $m\bar{3}$.

B. Numerical Results

We show the numerical results of the MO conductivity for the Hamiltonian in Eq. (31). We especially focus on the Hall contribution $\sigma_{ij}^{mna(H)}$. Under the CPG $m\bar{3}$ in Table VI, Q_0 , G_{xyz} and Q_4 are symmetry allowed; no M and MT multipoles are allowed owing to the presence of the $\mathcal{T}$ symmetry. Thus, we obtain multipole correspondence of $\sigma_{ij}^{mna(H)}$ by combination Eq. (B58), (B81), and (B88)

$$\begin{pmatrix}
2Q_0^{(1)} + 8Q_0^{(2)} + 2Q_4 & 0 & 0 \\
6Q_0^{(2)} - Q_4 + 2G_{xyz}^{(1)} - G_{xyz}^{(2)} & 0 & 0 \\
6Q_0^{(2)} - Q_4 - 2G_{xyz}^{(1)} + G_{xyz}^{(2)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & Q_0^{(1)} + Q_0^{(2)} - Q_4 + G_{xyz}^{(1)} + G_{xyz}^{(2)} & 0 \\
0 & 6Q_0^{(2)} - Q_4 - 2G_{xyz}^{(1)} + G_{xyz}^{(2)} & 0 \\
0 & 2Q_0^{(1)} + 8Q_0^{(2)} + 2Q_4 & 0 \\
0 & 6Q_0^{(2)} - Q_4 + 2G_{xyz}^{(1)} - G_{xyz}^{(2)} & 0 \\
0 & 0 & Q_0^{(1)} + Q_0^{(2)} - Q_4 + G_{xyz}^{(1)} + G_{xyz}^{(2)} \\
0 & 0 & 0 \\
Q_0^{(1)} + Q_0^{(2)} - Q_4 - G_{xyz}^{(1)} - G_{xyz}^{(2)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 6Q_0^{(2)} - Q_4 + 2G_{xyz}^{(1)} - G_{xyz}^{(2)} \\
0 & 0 & 6Q_0^{(2)} - Q_4 - 2G_{xyz}^{(1)} + G_{xyz}^{(2)} \\
0 & 0 & 2Q_0^{(1)} + 8Q_0^{(2)} + 2Q_4 \\
0 & Q_0^{(1)} + Q_0^{(2)} - Q_4 - G_{xyz}^{(1)} - G_{xyz}^{(2)} & 0 \\
Q_0^{(1)} + Q_0^{(2)} - Q_4 + G_{xyz}^{(1)} + G_{xyz}^{(2)} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (37)$$

where the different superscripts (1), (2), ... represent the same type but independent components in each multipole. Among the multipoles, the contributions from Q_0 and Q_4 are present under high symmetric $m\bar{3}m$, while those from G_{xyz} are present only under $m\bar{3}$. Although (Q_0, Q_4) and G_{xyz} appears in the same tensor component, we can separate their contributions by taking a linear combination of two tensor components; for example, E multipole components are given by $\sigma_{xy}^{(H)}(Q) \equiv \sigma_{xy}^{zy(H)} + \sigma_{xy}^{zx(H)} \leftrightarrow Q_0^{(1)} \oplus Q_0^{(2)} \oplus Q_4$, while ET multipole components are given by $\sigma_{xy}^{(H)}(G) \equiv \sigma_{xy}^{zy(H)} - \sigma_{xy}^{zx(H)} \leftrightarrow G_{xyz}^{(1)} \oplus G_{xyz}^{(2)}$ from Eq. (37). It is noted that the dissipative contribution $\sigma_{xy}^{(H/J)}$ vanishes owing to the absence of magnetic-type multipoles. In the following numerical calculations, we set $e = \hbar = k_B = 1$. Other parameters are set as shown in Table IX.

TABLE IX. Parameter settings in Figure 1 and 2.

parameter	value	parameter	value
$(pp\sigma)$	-1	λ_p	2
$(pp\pi)$	0.4	λ_f	0.5
$(pf\sigma)$	-0.5	Δ	20
$(pf\pi)$	0.3	V	2^{15}
$(ff\sigma)$	-0.2	T	0.001
$(ff\pi)$	0.1	μ	-2.5
$(ff\delta)$	-0.05		
$(ff\phi)$	0.01		

First, we calculate the δ dependence of $\sigma_{xy}^{(H)}(Q)$ and $\sigma_{xy}^{(H)}(G)$ as shown in Figure 1. The data shows that $\sigma_{xy}^{(H/J)}(Q, G) = 0$ and $\sigma_{xy}^{(H/E)}(Q, G) \xrightarrow{\delta \rightarrow 0} \text{const}$, which is

consistent with the discussion in Sec. IID. It results in that $\sigma_{xy}^{(H)}(Q)$ and $\sigma_{xy}^{(H)}(G)$ are constructed from $\mathcal{T}$ -even multipoles as discussed in Sec. IID, which has no contradiction with the $\mathcal{T}$ -even Hamiltonian given in Eq. (31).

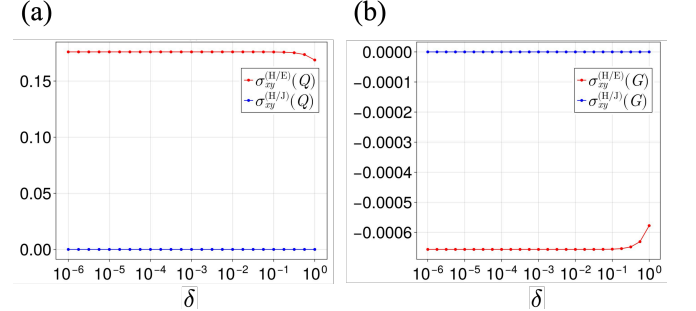


FIG. 1. The broadening factor δ dependence of (a) the electric multipole Q components of MO Hall conductivities $\sigma_{xy}^{(H)}(Q)$ and (b) the electric toroidal multipole G components of MO Hall conductivities $\sigma_{xy}^{(H)}(G)$.

Next, we evaluate the contributions from Q and G multipoles by varying the symmetry-lowering parameter g . Figure 2 shows the g dependence of $\sigma_{xy}^{(H)}(Q)$ and $\sigma_{xy}^{(H)}(G)$, demonstrating the activation of $\sigma_{xy}^{(H)}(G)$ due to symmetry lowering induced by the molecular field. In addition, $\sigma_{xy}^{(H)}(Q)$ expands in even powers of g , whereas $\sigma_{xy}^{(H)}(G)$ expands in odd powers, as shown in Fig. 2. These results reflect the irreducible representations of $Q(=Q_0^{(1)}, Q_0^{(2)}, Q_4)$ and $G(=G_{xyz}^{(1)}, G_{xyz}^{(2)})$ under the CPG $m\bar{3}m$, which belong to A_{1g} and A_{2g} , respectively. This feature is also consistent with our symmetry argument in Sec. IIC.

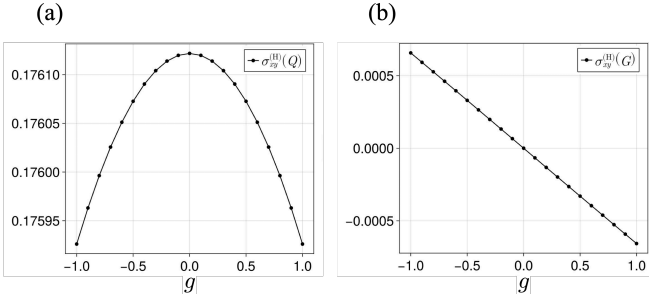


FIG. 2. The strength of the ET octupole g dependence of (a) the electric multipole Q components of MO Hall conductivities $\sigma_{xy}^{(H)}(Q)$ and (b) the electric toroidal multipole G components of MO Hall conductivities $\sigma_{xy}^{(H)}(G)$.

TABLE X. Relevant rank l and multipoles of MO and spin Hall conductivities for the Hamiltonian in Eq. (31).

	MO	spin
rank l	$0 \leq l \leq 4$	$0 \leq l \leq 2$
multipole	Q_0, Q_4, G_{xyz}	Q_0

Finally, we comment on the spin-current response in the present tight-binding model. The spin conductivity is strongly constrained by symmetry and, in particular, only the conventional component associated with the electric monopole sector is allowed, whose tensor component is specifically given as $\sigma_{yz}^{(H)}$, $\sigma_{zx}^{(H)}$, and $\sigma_{xy}^{(H)}$, as shown in Table X. Such a contribution behaves as an even function of the order parameter and therefore does not reflect the activation of the ET octupole G_{xyz} . In contrast, the MO conductivity discussed above is directly induced by the ET octupole order and exhibits distinct symmetry properties. This clearly highlights the symmetry-based distinction between spin and MO current responses in our model.

IV. SUMMARY

In summary, we have derived a multipole representation of MO conductivity and provided its classification under 32 CPGs. We have shown that the MO conductivity tensor components correspond to polar even-rank

and axial odd-rank multipoles, where the multipoles from rank-1 to -5 contribute to the Ohmic contribution, while up to rank 4 contribute to the Hall contribution. In addition, we have presented the symmetry distinctions between MO and spin currents, which provides a key insight for their experimental identification. Furthermore, we have clarified that dissipative MO currents are governed by active M and MT multipoles, whereas nondissipative MO currents are governed by active E and ET multipoles. Finally, we demonstrate nonzero MO current conductivity by performing the microscopic model calculations.

Our multipole formulation naturally extends the polar-type rank-5 response tensor—whose spatial parity differs from that of the MO conductivity—to include, for instance, MT-octupole current conductivity via the exchange $X \leftrightarrow Y$. This extension widens the class of materials capable of exhibiting MO currents, covering not only d -wave altermagnets [38] but also potential f -wave altermagnets and other multipolar systems. We expect that this unified framework will guide the exploration and design of octupole current active in a broad range of materials.

Finally, we discuss possible experimental routes to detect MO currents. First, Néel-vector switching in d -wave altermagnets might support the generation of MO torque induced by MO currents through the coupling between the Néel vector and the MO moment [36]. Second, the observation of MO Hall magnetoresistance [37] and MO accumulation—the MO counterparts of spin Hall magnetoresistance [63] and spin accumulation [64, 65], respectively—can provide indirect evidence of MO currents. The former may manifest itself as a Néel-vector-dependent magnetoresistance in heavy-metal/ d -wave altermagnet bilayers, while the latter may appear in the angular dependence of resonant x-ray scattering [66].

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Appendix A: Angle dependence of multipole

We give angle dependence of c - and t -multipoles in Table XI, where angle dependence of t -multipole up to rank two is the same as c -multipole one.

TABLE XI. Angule dependences of c - and t -multipoles normalized by $\int_S d\Omega X_i^* X_j = \delta_{ij}$ up to rank 5, where S , Ω , and δ_{ij} are the unit sphere, the solid angle, and the Kronecker delta, respectively. $r^2 \equiv x^2 + y^2 + z^2$, and cyclic denotes the cyclic permutation $x \rightarrow y \rightarrow z \rightarrow x$.

rank	c -multipole	angle dependence	rank	t -multipole	angle dependence
0	X_0	$\frac{1}{2\sqrt{\pi}}$	3	X_z^α	$\frac{\sqrt{7}z(-3r^2+5z^2)}{4\sqrt{\pi}r^3}$
1	X_x, X_y, X_z	$\frac{\sqrt{3}x}{2\sqrt{\pi}r}, \frac{\sqrt{3}y}{2\sqrt{\pi}r}, \frac{\sqrt{3}z}{2\sqrt{\pi}r}$		X_{3a}	$\frac{\sqrt{70}x(x^2-3y^2)}{8\sqrt{\pi}r^3}$
2	X_u	$\frac{\sqrt{5}(3z^2-r^2)}{4\sqrt{\pi}r^2}$		X_{3b}	$\frac{\sqrt{70}y(3x^2-y^2)}{8\sqrt{\pi}r^3}$
	X_v	$\frac{\sqrt{15}(x^2-y^2)}{4\sqrt{\pi}r^2}$		X_{3u}, X_{3v}	$\frac{\sqrt{42}x(-r^2+5z^2)}{8\sqrt{\pi}r^3}, \frac{\sqrt{42}y(-r^2+5z^2)}{8\sqrt{\pi}r^3}$
	X_{yz}, X_{zx}, X_{xy}	$\frac{\sqrt{15}yz}{2\sqrt{\pi}r^2}, \text{cyclic}$		X_z^β, X_{xyz}	$\frac{\sqrt{105}z(x^2-y^2)}{4\sqrt{\pi}r^3}, \frac{\sqrt{105}xyz}{2\sqrt{\pi}r^3}$
3	X_{xyz}	$\frac{\sqrt{105}xyz}{2\sqrt{\pi}r^3}$	4	X_{40}	$\frac{3(3r^4-30r^2z^2+35z^4)}{16\sqrt{\pi}r^4}$
	$X_x^\alpha, X_y^\alpha, X_z^\alpha$	$\frac{\sqrt{7}x(5x^2-3r^2)}{4\sqrt{\pi}r^3}, \text{cyclic}$		X_{4a}	$\frac{3\sqrt{70}y(3x^2-y^2)}{8\sqrt{\pi}r^4}$
	$X_x^\beta, X_y^\beta, X_z^\beta$	$\frac{\sqrt{105}x(y^2-z^2)}{4\sqrt{\pi}r^3}, \text{cyclic}$		X_{4b}	$\frac{3\sqrt{70}yz(x^2-3y^2)}{8\sqrt{\pi}r^4}$
4	X_4	$\frac{\sqrt{21}(x^4-3x^2y^2-3x^2z^2+y^4-3y^2z^2+z^4)}{4\sqrt{\pi}r^4}$		$X_{4u}^\alpha, X_{4v}^\alpha$	$\frac{3\sqrt{10}xz(-3r^2+7z^2)}{8\sqrt{\pi}r^4}, \frac{3\sqrt{10}yz(-3r^2+7z^2)}{8\sqrt{\pi}r^4}$
	X_{4u}	$\frac{\sqrt{15}(x^4-12x^2y^2+6x^2z^2+y^4+6y^2z^2-2z^4)}{8\sqrt{\pi}r^4}$		$X_{4u}^{\beta 1}, X_{4v}^{\beta 1}$	$\frac{3\sqrt{35}(x^4-6x^2y^2+y^4)}{16\sqrt{\pi}r^4}, \frac{3\sqrt{35}xy(x^2-y^2)}{4\sqrt{\pi}r^4}$
	X_{4v}	$-\frac{3\sqrt{5}(x-y)(x+y)(x^2+y^2-6z^2)}{8\sqrt{\pi}r^4}$		$X_{4u}^{\beta 2}, X_{4v}^{\beta 2}$	$\frac{3\sqrt{5}(x^2-y^2)(-r^2+7z^2)}{8\sqrt{\pi}r^4}, \frac{3\sqrt{5}xy(-r^2+7z^2)}{4\sqrt{\pi}r^4}$
	$X_{4x}^\alpha, X_{4y}^\alpha, X_{4z}^\alpha$	$\frac{3\sqrt{35}yz(y^2-z^2)}{4\sqrt{\pi}r^4}, \text{cyclic}$	5	X_{50}	$\frac{\sqrt{11}z(15r^4-70r^2z^2+63z^4)}{16\sqrt{\pi}r^5}$
	$X_{4x}^\beta, X_{4y}^\beta, X_{4z}^\beta$	$\frac{3\sqrt{5}yz(7x^2-r^2)}{4\sqrt{\pi}r^4}, \text{cyclic}$		X_{5a}	$\frac{\sqrt{770}x(x^2-3y^2)(-r^2+9z^2)}{32\sqrt{\pi}r^5}$
5	X_{5u}	$\frac{3\sqrt{385}xyz(x-y)(x+y)}{4\sqrt{\pi}r^5}$		X_{5b}	$\frac{\sqrt{770}y(3x^2-y^2)(-r^2+9z^2)}{32\sqrt{\pi}r^5}$
	X_{5v}	$\frac{\sqrt{1155}xyz(-x^2-y^2+2z^2)}{4\sqrt{\pi}r^5}$		$X_{5u}^{\alpha 1}, X_{5v}^{\alpha 1}$	$\frac{3\sqrt{154}x(x^4-10x^2y^2+5y^4)}{32\sqrt{\pi}r^5}, \frac{3\sqrt{154}y(5x^4-10x^2y^2+y^4)}{32\sqrt{\pi}r^5}$
	$X_{5x}^{\alpha 1}, X_{5y}^{\alpha 1}, X_{5z}^{\alpha 1}$	$\frac{\sqrt{11}x(8x^4-40x^2(y^2+z^2)+15(y^2+z^2)^2)}{16\sqrt{\pi}r^5}, \text{cyclic}$		$X_{5u}^{\alpha 2}, X_{5v}^{\alpha 2}$	$\frac{\sqrt{165}x(r^4-14r^2z^2+21z^4)}{16\sqrt{\pi}r^5}, \frac{\sqrt{165}y(r^4-14r^2z^2+21z^4)}{16\sqrt{\pi}r^5}$
	$X_{5x}^{\alpha 2}, X_{5y}^{\alpha 2}, X_{5z}^{\alpha 2}$	$\frac{3\sqrt{385}x(y^4-6y^2z^2+z^4)}{16\sqrt{\pi}r^5}, \text{cyclic}$		$X_{5u}^{\beta 1}, X_{5v}^{\beta 1}$	$\frac{3\sqrt{385}z(x^4-6x^2y^2+y^4)}{16\sqrt{\pi}r^5}, \frac{3\sqrt{385}xyz(x^2-y^2)}{4\sqrt{\pi}r^5}$
	$X_{5x}^\beta, X_{5y}^\beta, X_{5z}^\beta$	$\frac{\sqrt{1155}x(y^2-z^2)(3z^2-r^2)}{8\sqrt{\pi}r^5}, \text{cyclic}$		$X_{5u}^{\beta 2}, X_{5v}^{\beta 2}$	$\frac{\sqrt{1155}z(x^2-y^2)(-r^2+3z^2)}{8\sqrt{\pi}r^5}, \frac{\sqrt{1155}xyz(-r^2+3z^2)}{4\sqrt{\pi}r^5}$

Appendix B: Multipole expressions of MO conductivities

We show multipole expressions of $\sigma_{ij}^{mn\alpha(\text{O,H})}$ up to rank-5 multipoles including t -multipoles as well as c -multipoles, which expressed as the following form:

$$\left(\sigma_{ij}^{mn\alpha(\text{O/L})}\right)^\top = \begin{pmatrix} \sigma_{xx}^{xxx(\text{O})} & \sigma_{xx}^{yyx(\text{O})} & \sigma_{xx}^{zzx(\text{O})} & \sigma_{xx}^{yzx(\text{O})} & \sigma_{xx}^{zxx(\text{O})} & \sigma_{xx}^{xyx(\text{O})} & \sigma_{xx}^{xxy(\text{O})} & \sigma_{xx}^{yyy(\text{O})} & \sigma_{xx}^{zzy(\text{O})} & \sigma_{xx}^{yzy(\text{O})} & \sigma_{xx}^{zxy(\text{O})} & \sigma_{xx}^{xyy(\text{O})} & \sigma_{xx}^{xxz(\text{O})} & \sigma_{xx}^{yyz(\text{O})} & \sigma_{xx}^{zzz(\text{O})} & \sigma_{xx}^{yzz(\text{O})} & \sigma_{xx}^{zxx(\text{O})} & \sigma_{xx}^{xyx(\text{O})} \\ \sigma_{xx}^{xxx(\text{O})} & \sigma_{yy}^{xxx(\text{O})} & \sigma_{yy}^{zzx(\text{O})} & \sigma_{yy}^{yzx(\text{O})} & \sigma_{yy}^{zxx(\text{O})} & \sigma_{yy}^{xyx(\text{O})} & \sigma_{yy}^{xxy(\text{O})} & \sigma_{yy}^{yyy(\text{O})} & \sigma_{yy}^{zzy(\text{O})} & \sigma_{yy}^{yzy(\text{O})} & \sigma_{yy}^{zxy(\text{O})} & \sigma_{yy}^{xyy(\text{O})} & \sigma_{yy}^{xxz(\text{O})} & \sigma_{yy}^{yyz(\text{O})} & \sigma_{yy}^{zzz(\text{O})} & \sigma_{yy}^{yzz(\text{O})} & \sigma_{yy}^{zxx(\text{O})} & \sigma_{yy}^{xyx(\text{O})} \\ \sigma_{yy}^{xxx(\text{O})} & \sigma_{yy}^{yyx(\text{O})} & \sigma_{zz}^{xxx(\text{O})} & \sigma_{zz}^{yzx(\text{O})} & \sigma_{zz}^{zxx(\text{O})} & \sigma_{zz}^{xyx(\text{O})} & \sigma_{zz}^{xxy(\text{O})} & \sigma_{zz}^{yyy(\text{O})} & \sigma_{zz}^{zzy(\text{O})} & \sigma_{zz}^{yzy(\text{O})} & \sigma_{zz}^{zxy(\text{O})} & \sigma_{zz}^{xyy(\text{O})} & \sigma_{zz}^{xxz(\text{O})} & \sigma_{zz}^{yyz(\text{O})} & \sigma_{zz}^{zzz(\text{O})} & \sigma_{zz}^{yzz(\text{O})} & \sigma_{zz}^{zxx(\text{O})} & \sigma_{zz}^{xyx(\text{O})} \\ \sigma_{zz}^{xxx(\text{O})} & \sigma_{zz}^{yyx(\text{O})} & \sigma_{zz}^{zzx(\text{O})} & \sigma_{zz}^{yzx(\text{O})} & \sigma_{zz}^{zxx(\text{O})} & \sigma_{zz}^{xyx(\text{O})} & \sigma_{zz}^{xxy(\text{O})} & \sigma_{zz}^{yyy(\text{O})} & \sigma_{zz}^{zzy(\text{O})} & \sigma_{zz}^{yzy(\text{O})} & \sigma_{zz}^{zxy(\text{O})} & \sigma_{zz}^{xyy(\text{O})} & \sigma_{zz}^{xxz(\text{O})} & \sigma_{zz}^{yyz(\text{O})} & \sigma_{zz}^{zzz(\text{O})} & \sigma_{zz}^{yzz(\text{O})} & \sigma_{zz}^{zxx(\text{O})} & \sigma_{zz}^{xyx(\text{O})} \end{pmatrix}, \quad (\text{B1})$$

$$\left(\sigma_{ij}^{mn\alpha(\text{O/T})}\right)^\top = \begin{pmatrix} \sigma_{yz}^{xxx(\text{O})} & \sigma_{yz}^{yyx(\text{O})} & \sigma_{yz}^{zzx(\text{O})} & \sigma_{yz}^{yzx(\text{O})} & \sigma_{yz}^{zxx(\text{O})} & \sigma_{yz}^{xyx(\text{O})} & \sigma_{yz}^{xxy(\text{O})} & \sigma_{yz}^{yyy(\text{O})} & \sigma_{yz}^{zzy(\text{O})} & \sigma_{yz}^{yzy(\text{O})} & \sigma_{yz}^{zxy(\text{O})} & \sigma_{yz}^{xyy(\text{O})} & \sigma_{yz}^{xxz(\text{O})} & \sigma_{yz}^{yyz(\text{O})} & \sigma_{yz}^{zzz(\text{O})} & \sigma_{yz}^{yzz(\text{O})} & \sigma_{yz}^{zxx(\text{O})} & \sigma_{yz}^{xyx(\text{O})} \\ \sigma_{yz}^{xxx(\text{O})} & \sigma_{yz}^{yyx(\text{O})} & \sigma_{yz}^{zzx(\text{O})} & \sigma_{yz}^{yzx(\text{O})} & \sigma_{yz}^{zxx(\text{O})} & \sigma_{yz}^{xyx(\text{O})} & \sigma_{yz}^{xxy(\text{O})} & \sigma_{yz}^{yyy(\text{O})} & \sigma_{yz}^{zzy(\text{O})} & \sigma_{yz}^{yzy(\text{O})} & \sigma_{yz}^{zxy(\text{O})} & \sigma_{yz}^{xyy(\text{O})} & \sigma_{yz}^{xxz(\text{O})} & \sigma_{yz}^{yyz(\text{O})} & \sigma_{yz}^{zzz(\text{O})} & \sigma_{yz}^{yzz(\text{O})} & \sigma_{yz}^{zxx(\text{O})} & \sigma_{yz}^{xyx(\text{O})} \\ \sigma_{zx}^{xxx(\text{O})} & \sigma_{zx}^{yyx(\text{O})} & \sigma_{zx}^{zzx(\text{O})} & \sigma_{zx}^{yzx(\text{O})} & \sigma_{zx}^{zxx(\text{O})} & \sigma_{zx}^{xyx(\text{O})} & \sigma_{zx}^{xxy(\text{O})} & \sigma_{zx}^{yyy(\text{O})} & \sigma_{zx}^{zzy(\text{O})} & \sigma_{zx}^{yzy(\text{O})} & \sigma_{zx}^{zxy(\text{O})} & \sigma_{zx}^{xyy(\text{O})} & \sigma_{zx}^{xxz(\text{O})} & \sigma_{zx}^{yyz(\text{O})} & \sigma_{zx}^{zzz(\text{O})} & \sigma_{zx}^{yzz(\text{O})} & \sigma_{zx}^{zxx(\text{O})} & \sigma_{zx}^{xyx(\text{O})} \\ \sigma_{xy}^{xxx(\text{O})} & \sigma_{xy}^{yyx(\text{O})} & \sigma_{xy}^{zzx(\text{O})} & \sigma_{xy}^{yzx(\text{O})} & \sigma_{xy}^{zxx(\text{O})} & \sigma_{xy}^{xyx(\text{O})} & \sigma_{xy}^{xxy(\text{O})} & \sigma_{xy}^{yyy(\text{O})} & \sigma_{xy}^{zzy(\text{O})} & \sigma_{xy}^{yzy(\text{O})} & \sigma_{xy}^{zxy(\text{O})} & \sigma_{xy}^{xyy(\text{O})} & \sigma_{xy}^{xxz(\text{O})} & \sigma_{xy}^{yyz(\text{O})} & \sigma_{xy}^{zzz(\text{O})} & \sigma_{xy}^{yzz(\text{O})} & \sigma_{xy}^{zxx(\text{O})} & \sigma_{xy}^{xyx(\text{O})} \end{pmatrix}. \quad (\text{B2})$$

Here, $\sigma_{ij}^{mn\alpha(\text{O})}$ is decomposed into the longitudinal part $\sigma_{ij}^{mn\alpha(\text{O/L})}$ with $i = j$ and the transverse part $\sigma_{ij}^{mn\alpha(\text{O/T})}$ with $i \neq j$, and $\sigma_{ij}^{mn\alpha(\text{H})}$ is shown in Eq. (11). We additionally show the normalized MO conductivity $\tilde{\sigma}_{ij}^{mn\alpha(\text{O,H})}$ for better readability, whose multipole do not obey Eq. (16) but have the same irreducible representations as one of $\sigma_{ij}^{mn\alpha(\text{O,H})}$ in arbitrary point groups. The independent multipole degrees of freedom are distinguished by the superscript (l) for the integer l .

1. rank-0 c -multipoles

$$\left(\sigma_{ij}^{mn\alpha(\text{O/L})}\right)^\top = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -X_0 & 0 & 0 & 0 & 0 & 0 & X_0 \\ 0 & 0 & 0 & X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -X_0 \\ 0 & 0 & 0 & -X_0 & 0 & 0 & 0 & 0 & 0 & 0 & X_0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B3})$$

$$\left(\sigma_{ij}^{mn\alpha(\text{O/T})}\right)^\top = \begin{pmatrix} 0 & -X_0 & X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{X_0}{2} & 0 & 0 & 0 & 0 & -\frac{X_0}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{X_0}{2} & X_0 & 0 & -X_0 & 0 & 0 & 0 & 0 & 0 & \frac{X_0}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{X_0}{2} & 0 & 0 & 0 & 0 & -\frac{X_0}{2} & 0 & 0 & -X_0 & X_0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{B4})$$

$$\begin{aligned}
\sigma_{ij}^{mn\alpha(H)} = & \\
& \begin{pmatrix}
X_0^{(1)} + 4X_0^{(2)} & 0 & 0 \\
3X_0^{(2)} & 0 & 0 \\
3X_0^{(2)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{X_0^{(1)}}{2} + \frac{X_0^{(2)}}{2} \\
0 & \frac{X_0^{(1)}}{2} + \frac{X_0^{(2)}}{2} & 0 \\
0 & 3X_0^{(2)} & 0 \\
0 & X_0^{(1)} + 4X_0^{(2)} & 0 \\
0 & 3X_0^{(2)} & 0 \\
0 & 0 & \frac{X_0^{(1)}}{2} + \frac{X_0^{(2)}}{2} \\
0 & 0 & 0 \\
\frac{X_0^{(1)}}{2} + \frac{X_0^{(2)}}{2} & 0 & 0 \\
0 & 0 & 3X_0^{(2)} \\
0 & 0 & 3X_0^{(2)} \\
0 & 0 & X_0^{(1)} + 4X_0^{(2)} \\
0 & \frac{X_0^{(1)}}{2} + \frac{X_0^{(2)}}{2} & 0 \\
\frac{X_0^{(1)}}{2} + \frac{X_0^{(2)}}{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\end{aligned} \tag{B5}$$

2. rank-1 c -multipoles

$$\sigma_{ij}^{mna(O/L)} = \sigma_1^{(O/L)} + \sigma_2^{(O/L)}, \quad (B6)$$

$$\sigma_1^{(O/L)} = \begin{pmatrix} 12Y_x^{(1)} + 4Y_x^{(3)} & -6Y_x^{(1)} + 4Y_x^{(3)} & -6Y_x^{(1)} + 4Y_x^{(3)} \\ -6Y_x^{(1)} + 3Y_x^{(3)} & 8Y_x^{(1)} - 2Y_x^{(2)} + 3Y_x^{(3)} & -2Y_x^{(1)} + 2Y_x^{(2)} + 3Y_x^{(3)} \\ -6Y_x^{(1)} + 3Y_x^{(3)} & -2Y_x^{(1)} + 2Y_x^{(2)} + 3Y_x^{(3)} & 8Y_x^{(1)} - 2Y_x^{(2)} + 3Y_x^{(3)} \\ 0 & 0 & 0 \\ 8Y_z^{(1)} + Y_z^{(2)} + \frac{Y_z^{(3)}}{2} & -2Y_z^{(1)} - Y_z^{(2)} + \frac{Y_z^{(3)}}{2} & -6Y_z^{(1)} + \frac{Y_z^{(3)}}{2} \\ 8Y_y^{(1)} + Y_y^{(2)} + \frac{Y_y^{(3)}}{2} & -6Y_y^{(1)} + \frac{Y_y^{(3)}}{2} & -2Y_y^{(1)} - Y_y^{(2)} + \frac{Y_y^{(3)}}{2} \\ 8Y_y^{(1)} - 2Y_y^{(2)} + 3Y_y^{(3)} & -6Y_y^{(1)} + 3Y_y^{(3)} & -2Y_y^{(1)} + 2Y_y^{(2)} + 3Y_y^{(3)} \\ -6Y_y^{(1)} + 4Y_y^{(3)} & 12Y_y^{(1)} + 4Y_y^{(3)} & -6Y_y^{(1)} + 4Y_y^{(3)} \\ -2Y_y^{(1)} + 2Y_y^{(2)} + 3Y_y^{(3)} & -6Y_y^{(1)} + 3Y_y^{(3)} & 8Y_y^{(1)} - 2Y_y^{(2)} + 3Y_y^{(3)} \\ -2Y_z^{(1)} - Y_z^{(2)} + \frac{Y_z^{(3)}}{2} & 8Y_z^{(1)} + Y_z^{(2)} + \frac{Y_z^{(3)}}{2} & -6Y_z^{(1)} + \frac{Y_z^{(3)}}{2} \\ 0 & 0 & 0 \\ -6Y_x^{(1)} + \frac{Y_x^{(3)}}{2} & 8Y_x^{(1)} + Y_x^{(2)} + \frac{Y_x^{(3)}}{2} & -2Y_x^{(1)} - Y_x^{(2)} + \frac{Y_x^{(3)}}{2} \\ 8Y_z^{(1)} - 2Y_z^{(2)} + 3Y_z^{(3)} & -2Y_z^{(1)} + 2Y_z^{(2)} + 3Y_z^{(3)} & -6Y_z^{(1)} + 3Y_z^{(3)} \\ -2Y_z^{(1)} + 2Y_z^{(2)} + 3Y_z^{(3)} & 8Y_z^{(1)} - 2Y_z^{(2)} + 3Y_z^{(3)} & -6Y_z^{(1)} + 3Y_z^{(3)} \\ -6Y_z^{(1)} + 4Y_z^{(3)} & -6Y_z^{(1)} + 4Y_z^{(3)} & 12Y_z^{(1)} + 4Y_z^{(3)} \\ -2Y_y^{(1)} - Y_y^{(2)} + \frac{Y_y^{(3)}}{2} & -6Y_y^{(1)} + \frac{Y_y^{(3)}}{2} & 8Y_y^{(1)} + Y_y^{(2)} + \frac{Y_y^{(3)}}{2} \\ -6Y_x^{(1)} + \frac{Y_x^{(3)}}{2} & -2Y_x^{(1)} - Y_x^{(2)} + \frac{Y_x^{(3)}}{2} & 8Y_x^{(1)} + Y_x^{(2)} + \frac{Y_x^{(3)}}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B7)$$

$$\sigma_2^{(O/L)} = \begin{pmatrix} 4Y_x^{(4)} + 16Y_x^{(5)} + Y_x^{(6)} & -2Y_x^{(4)} - 8Y_x^{(5)} + Y_x^{(6)} & -2Y_x^{(4)} - 8Y_x^{(5)} + Y_x^{(6)} \\ 12Y_x^{(5)} & -6Y_x^{(5)} & -6Y_x^{(5)} \\ 12Y_x^{(5)} & -6Y_x^{(5)} & -6Y_x^{(5)} \\ 0 & 0 & 0 \\ -Y_z^{(4)} - Y_z^{(5)} + \frac{Y_z^{(6)}}{2} & -Y_z^{(4)} - Y_z^{(5)} + \frac{Y_z^{(6)}}{2} & 2Y_z^{(4)} + 2Y_z^{(5)} + \frac{Y_z^{(6)}}{2} \\ -Y_y^{(4)} - Y_y^{(5)} + \frac{Y_y^{(6)}}{2} & 2Y_y^{(4)} + 2Y_y^{(5)} + \frac{Y_y^{(6)}}{2} & -Y_y^{(4)} - Y_y^{(5)} + \frac{Y_y^{(6)}}{2} \\ -6Y_y^{(5)} & 12Y_y^{(5)} & -6Y_y^{(5)} \\ -2Y_y^{(4)} - 8Y_y^{(5)} + Y_y^{(6)} & 4Y_y^{(4)} + 16Y_y^{(5)} + Y_y^{(6)} & -2Y_y^{(4)} - 8Y_y^{(5)} + Y_y^{(6)} \\ -6Y_y^{(5)} & 12Y_y^{(5)} & -6Y_y^{(5)} \\ -Y_z^{(4)} - Y_z^{(5)} + \frac{Y_z^{(6)}}{2} & -Y_z^{(4)} - Y_z^{(5)} + \frac{Y_z^{(6)}}{2} & 2Y_z^{(4)} + 2Y_z^{(5)} + \frac{Y_z^{(6)}}{2} \\ 0 & 0 & 0 \\ 2Y_x^{(4)} + 2Y_x^{(5)} + \frac{Y_x^{(6)}}{2} & -Y_x^{(4)} - Y_x^{(5)} + \frac{Y_x^{(6)}}{2} & -Y_x^{(4)} - Y_x^{(5)} + \frac{Y_x^{(6)}}{2} \\ -6Y_z^{(5)} & -6Y_z^{(5)} & 12Y_z^{(5)} \\ -6Y_z^{(5)} & -6Y_z^{(5)} & 12Y_z^{(5)} \\ -2Y_z^{(4)} - 8Y_z^{(5)} + Y_z^{(6)} & -2Y_z^{(4)} - 8Y_z^{(5)} + Y_z^{(6)} & 4Y_z^{(4)} + 16Y_z^{(5)} + Y_z^{(6)} \\ -Y_y^{(4)} - Y_y^{(5)} + \frac{Y_y^{(6)}}{2} & 2Y_y^{(4)} + 2Y_y^{(5)} + \frac{Y_y^{(6)}}{2} & -Y_y^{(4)} - Y_y^{(5)} + \frac{Y_y^{(6)}}{2} \\ 2Y_x^{(4)} + 2Y_x^{(5)} + \frac{Y_x^{(6)}}{2} & -Y_x^{(4)} - Y_x^{(5)} + \frac{Y_x^{(6)}}{2} & -Y_x^{(4)} - Y_x^{(5)} + \frac{Y_x^{(6)}}{2} \\ 0 & 0 & 0 \end{pmatrix}. \quad (B8)$$

$$\sigma_{ij}^{mn\alpha(O/T)} = \sigma_1^{(O/T)} + \sigma_2^{(O/T)}, \quad (B9)$$

$$\sigma_1^{(O/T)} = \begin{pmatrix} 0 & -6Y_z^{(1)} & -6Y_y^{(1)} \\ 0 & -2Y_z^{(1)} - Y_z^{(2)} & 8Y_y^{(1)} + Y_y^{(2)} \\ 0 & 8Y_z^{(1)} + Y_z^{(2)} & -2Y_y^{(1)} - Y_y^{(2)} \\ 5Y_x^{(1)} - 2Y_x^{(2)} & 5Y_y^{(1)} + Y_y^{(2)} & 5Y_z^{(1)} + Y_z^{(2)} \\ -2Y_y^{(1)} + \frac{Y_y^{(2)}}{2} & 8Y_x^{(1)} - \frac{Y_x^{(2)}}{2} & 0 \\ -2Y_z^{(1)} + \frac{Y_z^{(2)}}{2} & 0 & 8Y_x^{(1)} - \frac{Y_x^{(2)}}{2} \\ -2Y_z^{(1)} - Y_z^{(2)} & 0 & 8Y_x^{(1)} + Y_x^{(2)} \\ -6Y_z^{(1)} & 0 & -6Y_x^{(1)} \\ 8Y_z^{(1)} + Y_z^{(2)} & 0 & -2Y_x^{(1)} - Y_x^{(2)} \\ 8Y_y^{(1)} - \frac{Y_y^{(2)}}{2} & -2Y_x^{(1)} + \frac{Y_x^{(2)}}{2} & 0 \\ 5Y_x^{(1)} + Y_x^{(2)} & 5Y_y^{(1)} - 2Y_y^{(2)} & 5Y_z^{(1)} + Y_z^{(2)} \\ 0 & -2Y_z^{(1)} + \frac{Y_z^{(2)}}{2} & 8Y_y^{(1)} - \frac{Y_y^{(2)}}{2} \\ -2Y_y^{(1)} - Y_y^{(2)} & 8Y_x^{(1)} + Y_x^{(2)} & 0 \\ 8Y_y^{(1)} + Y_y^{(2)} & -2Y_x^{(1)} - Y_x^{(2)} & 0 \\ -6Y_y^{(1)} & -6Y_x^{(1)} & 0 \\ 8Y_z^{(1)} - \frac{Y_z^{(2)}}{2} & 0 & -2Y_x^{(1)} + \frac{Y_x^{(2)}}{2} \\ 0 & 8Y_z^{(1)} - \frac{Y_z^{(2)}}{2} & -2Y_y^{(1)} + \frac{Y_y^{(2)}}{2} \\ 5Y_x^{(1)} + Y_x^{(2)} & 5Y_y^{(1)} + Y_y^{(2)} & 5Y_z^{(1)} - 2Y_z^{(2)} \end{pmatrix}, \quad (B10)$$

$$\sigma_2^{(O/T)} = \begin{pmatrix} 0 & 3Y_z^{(4)} + 12Y_z^{(5)} & 3Y_y^{(4)} + 12Y_y^{(5)} \\ 0 & 9Y_z^{(5)} & 9Y_y^{(5)} \\ 0 & 9Y_z^{(5)} & 9Y_y^{(5)} \\ 0 & 0 & 0 \\ \frac{3Y_y^{(4)}}{2} + \frac{3Y_y^{(5)}}{2} & \frac{3Y_x^{(4)}}{2} + \frac{3Y_x^{(5)}}{2} & 0 \\ \frac{3Y_z^{(4)}}{2} + \frac{3Y_z^{(5)}}{2} & 0 & \frac{3Y_x^{(4)}}{2} + \frac{3Y_x^{(5)}}{2} \\ 9Y_z^{(5)} & 0 & 9Y_x^{(5)} \\ 3Y_z^{(4)} + 12Y_z^{(5)} & 0 & 3Y_x^{(4)} + 12Y_x^{(5)} \\ 9Y_z^{(5)} & 0 & 9Y_x^{(5)} \\ \frac{3Y_y^{(4)}}{2} + \frac{3Y_y^{(5)}}{2} & \frac{3Y_x^{(4)}}{2} + \frac{3Y_x^{(5)}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{3Y_z^{(4)}}{2} + \frac{3Y_z^{(5)}}{2} & \frac{3Y_y^{(4)}}{2} + \frac{3Y_y^{(5)}}{2} \\ 9Y_y^{(5)} & 9Y_x^{(5)} & 0 \\ 9Y_y^{(5)} & 9Y_x^{(5)} & 0 \\ 3Y_y^{(4)} + 12Y_y^{(5)} & 3Y_x^{(4)} + 12Y_x^{(5)} & 0 \\ \frac{3Y_z^{(4)}}{2} + \frac{3Y_z^{(5)}}{2} & 0 & \frac{3Y_x^{(4)}}{2} + \frac{3Y_x^{(5)}}{2} \\ 0 & \frac{3Y_z^{(4)}}{2} + \frac{3Y_z^{(5)}}{2} & \frac{3Y_y^{(4)}}{2} + \frac{3Y_y^{(5)}}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B11)$$

where the multipole components with superscript (3) and (6) of $\sigma_{ij}^{mn\alpha(O/T)}$ do not appear.

$$\sigma_{ij}^{mn\alpha(H)} = \begin{pmatrix} 0 & -Y_z^{(1)} - 4Y_z^{(2)} & Y_y^{(1)} + 4Y_y^{(2)} \\ 0 & -3Y_z^{(2)} - Y_z^{(3)} & 3Y_y^{(2)} - Y_y^{(3)} \\ 0 & -3Y_z^{(2)} + Y_z^{(3)} & 3Y_y^{(2)} + Y_y^{(3)} \\ 0 & Y_y^{(3)} & -Y_z^{(3)} \\ -\frac{Y_y^{(1)}}{2} - \frac{Y_y^{(2)}}{2} + \frac{Y_y^{(3)}}{2} & \frac{Y_x^{(1)}}{2} + \frac{Y_x^{(2)}}{2} + \frac{Y_x^{(3)}}{2} & 0 \\ \frac{Y_z^{(1)}}{2} + \frac{Y_z^{(2)}}{2} - \frac{Y_z^{(3)}}{2} & 0 & -\frac{Y_x^{(1)}}{2} - \frac{Y_x^{(2)}}{2} - \frac{Y_x^{(3)}}{2} \\ 3Y_z^{(2)} + Y_z^{(3)} & 0 & -3Y_x^{(2)} + Y_x^{(3)} \\ Y_z^{(1)} + 4Y_z^{(2)} & 0 & -Y_x^{(1)} - 4Y_x^{(2)} \\ 3Y_z^{(2)} - Y_z^{(3)} & 0 & -3Y_x^{(2)} - Y_x^{(3)} \\ -\frac{Y_y^{(1)}}{2} - \frac{Y_y^{(2)}}{2} - \frac{Y_y^{(3)}}{2} & \frac{Y_x^{(1)}}{2} + \frac{Y_x^{(2)}}{2} - \frac{Y_x^{(3)}}{2} & 0 \\ -Y_x^{(3)} & 0 & Y_z^{(3)} \\ 0 & -\frac{Y_z^{(1)}}{2} - \frac{Y_z^{(2)}}{2} + \frac{Y_z^{(3)}}{2} & \frac{Y_y^{(1)}}{2} + \frac{Y_y^{(2)}}{2} + \frac{Y_y^{(3)}}{2} \\ -3Y_y^{(2)} - Y_y^{(3)} & 3Y_x^{(2)} - Y_x^{(3)} & 0 \\ -3Y_y^{(2)} + Y_y^{(3)} & 3Y_x^{(2)} + Y_x^{(3)} & 0 \\ -Y_y^{(1)} - 4Y_y^{(2)} & Y_x^{(1)} + 4Y_x^{(2)} & 0 \\ \frac{Y_z^{(1)}}{2} + \frac{Y_z^{(2)}}{2} + \frac{Y_z^{(3)}}{2} & 0 & -\frac{Y_x^{(1)}}{2} - \frac{Y_x^{(2)}}{2} + \frac{Y_x^{(3)}}{2} \\ 0 & -\frac{Y_z^{(1)}}{2} - \frac{Y_z^{(2)}}{2} - \frac{Y_z^{(3)}}{2} & \frac{Y_y^{(1)}}{2} + \frac{Y_y^{(2)}}{2} - \frac{Y_y^{(3)}}{2} \\ Y_x^{(3)} & -Y_y^{(3)} & 0 \end{pmatrix}. \quad (B12)$$

3. rank-2 c -multipoles

$$\sigma_{ij}^{mna(O/L)} = \sigma_1^{(O/L)} + \sigma_2^{(O/L)}, \quad (B13)$$

$$\sigma_1^{(O/L)} = \begin{pmatrix} 0 & 2\sqrt{3}X_{yz}^{(1)} + 8\sqrt{15}X_{yz}^{(2)} & -2\sqrt{3}X_{yz}^{(1)} - 8\sqrt{15}X_{yz}^{(2)} \\ \sqrt{5}X_{yz}^{(3)} & 6\sqrt{15}X_{yz}^{(2)} + \sqrt{5}X_{yz}^{(3)} & -6\sqrt{15}X_{yz}^{(2)} + \sqrt{5}X_{yz}^{(3)} \\ -\sqrt{5}X_{yz}^{(3)} & 6\sqrt{15}X_{yz}^{(2)} - \sqrt{5}X_{yz}^{(3)} & -6\sqrt{15}X_{yz}^{(2)} - \sqrt{5}X_{yz}^{(3)} \\ \frac{\sqrt{15}X_u^{(3)}}{2} + \frac{\sqrt{5}X_v^{(3)}}{2} & \frac{\sqrt{15}X_u^{(3)}}{2} + \frac{\sqrt{5}X_v^{(3)}}{2} & \frac{\sqrt{15}X_u^{(3)}}{2} + \frac{\sqrt{5}X_v^{(3)}}{2} \\ \sqrt{3}X_{xy}^{(1)} + \sqrt{15}X_{xy}^{(2)} - \frac{\sqrt{5}X_{xy}^{(3)}}{2} & -\sqrt{3}X_{xy}^{(1)} - \sqrt{15}X_{xy}^{(2)} - \frac{\sqrt{5}X_{xy}^{(3)}}{2} & -\frac{\sqrt{5}X_{xy}^{(3)}}{2} \\ -\sqrt{3}X_{zx}^{(1)} - \sqrt{15}X_{zx}^{(2)} + \frac{\sqrt{5}X_{zx}^{(3)}}{2} & \frac{\sqrt{5}X_{zx}^{(3)}}{2} & \sqrt{3}X_{zx}^{(1)} + \sqrt{15}X_{zx}^{(2)} + \frac{\sqrt{5}X_{zx}^{(3)}}{2} \\ -6\sqrt{15}X_{zx}^{(2)} - \sqrt{5}X_{zx}^{(3)} & -\sqrt{5}X_{zx}^{(3)} & 6\sqrt{15}X_{zx}^{(2)} - \sqrt{5}X_{zx}^{(3)} \\ -2\sqrt{3}X_{zx}^{(1)} - 8\sqrt{15}X_{zx}^{(2)} & 0 & 2\sqrt{3}X_{zx}^{(1)} + 8\sqrt{15}X_{zx}^{(2)} \\ -6\sqrt{15}X_{zx}^{(2)} + \sqrt{5}X_{zx}^{(3)} & \sqrt{5}X_{zx}^{(3)} & 6\sqrt{15}X_{zx}^{(2)} + \sqrt{5}X_{zx}^{(3)} \\ \sqrt{3}X_{xy}^{(1)} + \sqrt{15}X_{xy}^{(2)} + \frac{\sqrt{5}X_{xy}^{(3)}}{2} & -\sqrt{3}X_{xy}^{(1)} - \sqrt{15}X_{xy}^{(2)} + \frac{\sqrt{5}X_{xy}^{(3)}}{2} & \frac{\sqrt{5}X_{xy}^{(3)}}{2} \\ -\frac{\sqrt{15}X_u^{(3)}}{2} + \frac{\sqrt{5}X_v^{(3)}}{2} & -\frac{\sqrt{15}X_u^{(3)}}{2} + \frac{\sqrt{5}X_v^{(3)}}{2} & -\frac{\sqrt{15}X_u^{(3)}}{2} + \frac{\sqrt{5}X_v^{(3)}}{2} \\ -\frac{\sqrt{5}X_{yz}^{(3)}}{2} & \sqrt{3}X_{yz}^{(1)} + \sqrt{15}X_{yz}^{(2)} - \frac{\sqrt{5}X_{yz}^{(3)}}{2} & -\sqrt{3}X_{yz}^{(1)} - \sqrt{15}X_{yz}^{(2)} - \frac{\sqrt{5}X_{yz}^{(3)}}{2} \\ 6\sqrt{15}X_{xy}^{(2)} + \sqrt{5}X_{xy}^{(3)} & -6\sqrt{15}X_{xy}^{(2)} + \sqrt{5}X_{xy}^{(3)} & \sqrt{5}X_{xy}^{(3)} \\ 6\sqrt{15}X_{xy}^{(2)} - \sqrt{5}X_{xy}^{(3)} & -6\sqrt{15}X_{xy}^{(2)} - \sqrt{5}X_{xy}^{(3)} & -\sqrt{5}X_{xy}^{(3)} \\ 2\sqrt{3}X_{xy}^{(1)} + 8\sqrt{15}X_{xy}^{(2)} & -2\sqrt{3}X_{xy}^{(1)} - 8\sqrt{15}X_{xy}^{(2)} & 0 \\ -\sqrt{3}X_{zx}^{(1)} - \sqrt{15}X_{zx}^{(2)} - \frac{\sqrt{5}X_{zx}^{(3)}}{2} & -\frac{\sqrt{5}X_{zx}^{(3)}}{2} & \sqrt{3}X_{zx}^{(1)} + \sqrt{15}X_{zx}^{(2)} - \frac{\sqrt{5}X_{zx}^{(3)}}{2} \\ \frac{\sqrt{5}X_{yz}^{(3)}}{2} & \sqrt{3}X_{yz}^{(1)} + \sqrt{15}X_{yz}^{(2)} + \frac{\sqrt{5}X_{yz}^{(3)}}{2} & -\sqrt{3}X_{yz}^{(1)} - \sqrt{15}X_{yz}^{(2)} + \frac{\sqrt{5}X_{yz}^{(3)}}{2} \\ -\sqrt{5}X_v^{(3)} & -\sqrt{5}X_v^{(3)} & -\sqrt{5}X_v^{(3)} \end{pmatrix}, \quad (B14)$$

$$\sigma_2^{(O/L)} = \begin{pmatrix} 0 & -6\sqrt{10}X_{yz}^{(5)} & 6\sqrt{10}X_{yz}^{(5)} \\ -4\sqrt{35}X_{yz}^{(4)} - 10\sqrt{10}X_{yz}^{(5)} & 2\sqrt{35}X_{yz}^{(4)} + 8\sqrt{10}X_{yz}^{(5)} & 2\sqrt{35}X_{yz}^{(4)} + 2\sqrt{10}X_{yz}^{(5)} \\ 4\sqrt{35}X_{yz}^{(4)} + 10\sqrt{10}X_{yz}^{(5)} & -2\sqrt{35}X_{yz}^{(4)} - 2\sqrt{10}X_{yz}^{(5)} & -2\sqrt{35}X_{yz}^{(4)} - 8\sqrt{10}X_{yz}^{(5)} \\ -2\sqrt{105}X_u^{(4)} - 5\sqrt{30}X_u^{(5)} - 2\sqrt{35}X_v^{(4)} - 5\sqrt{10}X_v^{(5)} & 5\sqrt{30}X_u^{(5)} + 4\sqrt{35}X_v^{(4)} - 5\sqrt{10}X_v^{(5)} & 2\sqrt{105}X_u^{(4)} - 2\sqrt{35}X_v^{(4)} + 10\sqrt{10}X_v^{(5)} \\ -\sqrt{35}X_{xy}^{(4)} + 8\sqrt{10}X_{xy}^{(5)} & -\sqrt{35}X_{xy}^{(4)} + 2\sqrt{10}X_{xy}^{(5)} & 2\sqrt{35}X_{xy}^{(4)} - 10\sqrt{10}X_{xy}^{(5)} \\ \sqrt{35}X_{zx}^{(4)} - 8\sqrt{10}X_{zx}^{(5)} & -2\sqrt{35}X_{zx}^{(4)} + 10\sqrt{10}X_{zx}^{(5)} & \sqrt{35}X_{zx}^{(4)} - 2\sqrt{10}X_{zx}^{(5)} \\ -2\sqrt{35}X_{zx}^{(4)} - 8\sqrt{10}X_{zx}^{(5)} & 4\sqrt{35}X_{zx}^{(4)} + 10\sqrt{10}X_{zx}^{(5)} & -2\sqrt{35}X_{zx}^{(4)} - 2\sqrt{10}X_{zx}^{(5)} \\ 6\sqrt{10}X_{zx}^{(5)} & 0 & -6\sqrt{10}X_{zx}^{(5)} \\ 2\sqrt{35}X_{zx}^{(4)} + 2\sqrt{10}X_{zx}^{(5)} & -4\sqrt{35}X_{zx}^{(4)} - 10\sqrt{10}X_{zx}^{(5)} & 2\sqrt{35}X_{zx}^{(4)} + 8\sqrt{10}X_{zx}^{(5)} \\ \sqrt{35}X_{xy}^{(4)} - 2\sqrt{10}X_{xy}^{(5)} & \sqrt{35}X_{xy}^{(4)} - 8\sqrt{10}X_{xy}^{(5)} & -2\sqrt{35}X_{xy}^{(4)} + 10\sqrt{10}X_{xy}^{(5)} \\ -5\sqrt{30}X_u^{(4)} + 4\sqrt{35}X_v^{(4)} - 5\sqrt{10}X_v^{(5)} & 2\sqrt{105}X_u^{(4)} + 5\sqrt{30}X_u^{(5)} - 2\sqrt{35}X_v^{(4)} - 5\sqrt{10}X_v^{(5)} & -2\sqrt{105}X_u^{(4)} - 2\sqrt{35}X_v^{(4)} + 10\sqrt{10}X_v^{(5)} \\ 2\sqrt{35}X_{yz}^{(4)} - 10\sqrt{10}X_{yz}^{(5)} & -\sqrt{35}X_{yz}^{(4)} + 8\sqrt{10}X_{yz}^{(5)} & -\sqrt{35}X_{yz}^{(4)} + 2\sqrt{10}X_{yz}^{(5)} \\ 2\sqrt{35}X_{xy}^{(4)} + 8\sqrt{10}X_{xy}^{(5)} & 2\sqrt{35}X_{xy}^{(4)} + 2\sqrt{10}X_{xy}^{(5)} & -4\sqrt{35}X_{xy}^{(4)} - 10\sqrt{10}X_{xy}^{(5)} \\ -2\sqrt{35}X_{xy}^{(4)} - 2\sqrt{10}X_{xy}^{(5)} & -2\sqrt{35}X_{xy}^{(4)} - 8\sqrt{10}X_{xy}^{(5)} & 4\sqrt{35}X_{xy}^{(4)} + 10\sqrt{10}X_{xy}^{(5)} \\ -6\sqrt{10}X_{xy}^{(5)} & 6\sqrt{10}X_{xy}^{(5)} & 0 \\ -\sqrt{35}X_{zx}^{(4)} + 2\sqrt{10}X_{zx}^{(5)} & 2\sqrt{35}X_{zx}^{(4)} - 10\sqrt{10}X_{zx}^{(5)} & -\sqrt{35}X_{zx}^{(4)} + 8\sqrt{10}X_{zx}^{(5)} \\ -2\sqrt{35}X_{yz}^{(4)} + 10\sqrt{10}X_{yz}^{(5)} & \sqrt{35}X_{yz}^{(4)} - 2\sqrt{10}X_{yz}^{(5)} & \sqrt{35}X_{yz}^{(4)} - 8\sqrt{10}X_{yz}^{(5)} \\ 2\sqrt{105}X_u^{(4)} - 5\sqrt{30}X_u^{(5)} - 2\sqrt{35}X_v^{(4)} - 5\sqrt{10}X_v^{(5)} & -2\sqrt{105}X_u^{(4)} + 5\sqrt{30}X_u^{(5)} - 2\sqrt{35}X_v^{(4)} - 5\sqrt{10}X_v^{(5)} & 4\sqrt{35}X_v^{(4)} + 10\sqrt{10}X_v^{(5)} \end{pmatrix}. \quad (B15)$$

$$\sigma_{ij}^{mn\alpha(O/T)} = \sigma_1^{(O/T)} + \sigma_2^{(O/T)}, \quad (B16)$$

$$\sigma_1^{(O/T)} = \begin{pmatrix} 3X_u^{(1)} + 12\sqrt{5}X_u^{(2)} + \sqrt{3}X_v^{(1)} + 4\sqrt{15}X_v^{(2)} & -\sqrt{3}X_{xy}^{(1)} - 4\sqrt{15}X_{xy}^{(2)} & \sqrt{3}X_{zx}^{(1)} + 4\sqrt{15}X_{zx}^{(2)} \\ 9\sqrt{5}X_u^{(2)} + 3\sqrt{15}X_v^{(2)} & -3\sqrt{15}X_{xy}^{(2)} & 3\sqrt{15}X_{zx}^{(2)} \\ 9\sqrt{5}X_u^{(2)} + 3\sqrt{15}X_v^{(2)} & -3\sqrt{15}X_{xy}^{(2)} & 3\sqrt{15}X_{zx}^{(2)} \\ 0 & 0 & 0 \\ -\frac{\sqrt{3}X_{zx}^{(1)}}{2} - \frac{\sqrt{15}X_{zx}^{(2)}}{2} & \frac{\sqrt{3}X_{yz}^{(1)}}{2} + \frac{\sqrt{15}X_{yz}^{(2)}}{2} & -\sqrt{3}X_v^{(1)} - \sqrt{15}X_v^{(2)} \\ \frac{\sqrt{3}X_{xy}^{(1)}}{2} + \frac{\sqrt{15}X_{xy}^{(2)}}{2} & -\frac{3X_u^{(1)}}{2} - \frac{3\sqrt{5}X_u^{(2)}}{2} + \frac{\sqrt{3}X_v^{(1)}}{2} + \frac{\sqrt{15}X_v^{(2)}}{2} & -\frac{\sqrt{3}X_{yz}^{(1)}}{2} - \frac{\sqrt{15}X_{yz}^{(2)}}{2} \\ 3\sqrt{15}X_{xy}^{(2)} & -9\sqrt{5}X_u^{(2)} + 3\sqrt{15}X_v^{(2)} & -3\sqrt{15}X_{yz}^{(2)} \\ \sqrt{3}X_{xy}^{(1)} + 4\sqrt{15}X_{xy}^{(2)} & -3X_u^{(1)} - 12\sqrt{5}X_u^{(2)} + \sqrt{3}X_v^{(1)} + 4\sqrt{15}X_v^{(2)} & -\sqrt{3}X_{yz}^{(1)} - 4\sqrt{15}X_{yz}^{(2)} \\ 3\sqrt{15}X_{xy}^{(2)} & -9\sqrt{5}X_u^{(2)} + 3\sqrt{15}X_v^{(2)} & -3\sqrt{15}X_{yz}^{(2)} \\ -\frac{\sqrt{3}X_{zx}^{(1)}}{2} - \frac{\sqrt{15}X_{zx}^{(2)}}{2} & \frac{\sqrt{3}X_{yz}^{(1)}}{2} + \frac{\sqrt{15}X_{yz}^{(2)}}{2} & -\sqrt{3}X_v^{(1)} - \sqrt{15}X_v^{(2)} \\ 0 & 0 & 0 \\ \frac{3X_u^{(1)}}{2} + \frac{3\sqrt{5}X_u^{(2)}}{2} + \frac{\sqrt{3}X_v^{(1)}}{2} + \frac{\sqrt{15}X_v^{(2)}}{2} & -\frac{\sqrt{3}X_{yz}^{(1)}}{2} - \frac{\sqrt{15}X_{yz}^{(2)}}{2} & \frac{\sqrt{3}X_{zx}^{(1)}}{2} + \frac{\sqrt{15}X_{zx}^{(2)}}{2} \\ -3\sqrt{15}X_{zx}^{(2)} & 3\sqrt{15}X_{yz}^{(2)} & -6\sqrt{15}X_v^{(2)} \\ -3\sqrt{15}X_{zx}^{(2)} & 3\sqrt{15}X_{yz}^{(2)} & -6\sqrt{15}X_v^{(2)} \\ -\sqrt{3}X_{zx}^{(1)} - 4\sqrt{15}X_{zx}^{(2)} & \sqrt{3}X_{yz}^{(1)} + 4\sqrt{15}X_{yz}^{(2)} & -2\sqrt{3}X_v^{(1)} - 8\sqrt{15}X_v^{(2)} \\ \frac{\sqrt{3}X_{xy}^{(1)}}{2} + \frac{\sqrt{15}X_{xy}^{(2)}}{2} & -\frac{3X_u^{(1)}}{2} - \frac{3\sqrt{5}X_u^{(2)}}{2} + \frac{\sqrt{3}X_v^{(1)}}{2} + \frac{\sqrt{15}X_v^{(2)}}{2} & -\frac{\sqrt{3}X_{yz}^{(1)}}{2} - \frac{\sqrt{15}X_{yz}^{(2)}}{2} \\ \frac{3X_u^{(1)}}{2} + \frac{3\sqrt{5}X_u^{(2)}}{2} + \frac{\sqrt{3}X_v^{(1)}}{2} + \frac{\sqrt{15}X_v^{(2)}}{2} & -\frac{\sqrt{3}X_{xy}^{(1)}}{2} - \frac{\sqrt{15}X_{xy}^{(2)}}{2} & \frac{\sqrt{3}X_{zx}^{(1)}}{2} + \frac{\sqrt{15}X_{zx}^{(2)}}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B17)$$

$$\sigma_2^{(O/T)} = \begin{pmatrix} -3\sqrt{30}X_u^{(5)} - 3\sqrt{10}X_v^{(5)} & -12\sqrt{10}X_{xy}^{(5)} & 12\sqrt{10}X_{zx}^{(5)} \\ \sqrt{105}X_u^{(4)} - \sqrt{30}X_u^{(5)} - 3\sqrt{35}X_v^{(4)} + 9\sqrt{10}X_v^{(5)} & 3\sqrt{35}X_{xy}^{(4)} + 6\sqrt{10}X_{xy}^{(5)} & 3\sqrt{35}X_{zx}^{(4)} - 6\sqrt{10}X_{zx}^{(5)} \\ -\sqrt{105}X_u^{(4)} + 4\sqrt{30}X_u^{(5)} + 3\sqrt{35}X_v^{(4)} - 6\sqrt{10}X_v^{(5)} & -3\sqrt{35}X_{xy}^{(4)} + 6\sqrt{10}X_{xy}^{(5)} & -3\sqrt{35}X_{zx}^{(4)} + 6\sqrt{10}X_{zx}^{(5)} \\ 0 & 3\sqrt{35}X_{zx}^{(4)} & -3\sqrt{35}X_{xy}^{(4)} \\ -\frac{3\sqrt{35}X_{zx}^{(4)}}{2} + 6\sqrt{10}X_{zx}^{(5)} & -\frac{3\sqrt{35}X_{xy}^{(4)}}{2} - 6\sqrt{10}X_{xy}^{(5)} & \sqrt{105}X_u^{(4)} + 5\sqrt{30}X_u^{(5)} - 3\sqrt{10}X_v^{(5)} \\ \frac{3\sqrt{35}X_{zx}^{(4)}}{2} - 6\sqrt{10}X_{zx}^{(5)} & \frac{\sqrt{105}X_u^{(4)}}{2} + \sqrt{30}X_u^{(5)} + \frac{3\sqrt{35}X_v^{(4)}}{2} + 9\sqrt{10}X_v^{(5)} & \frac{3\sqrt{35}X_{yz}^{(4)}}{2} + 6\sqrt{10}X_{yz}^{(5)} \\ -3\sqrt{35}X_{xy}^{(4)} - 6\sqrt{10}X_{xy}^{(5)} & -\sqrt{105}X_u^{(4)} + \sqrt{30}X_u^{(5)} - 3\sqrt{35}X_v^{(4)} + 9\sqrt{10}X_v^{(5)} & -3\sqrt{35}X_{yz}^{(4)} + 6\sqrt{10}X_{yz}^{(5)} \\ 12\sqrt{10}X_{xy}^{(5)} & 3\sqrt{30}X_u^{(5)} - 3\sqrt{10}X_v^{(5)} & -12\sqrt{10}X_{yz}^{(5)} \\ 3\sqrt{35}X_{xy}^{(4)} - 6\sqrt{10}X_{xy}^{(5)} & \sqrt{105}X_u^{(4)} - 4\sqrt{30}X_u^{(5)} + 3\sqrt{35}X_v^{(4)} - 6\sqrt{10}X_v^{(5)} & 3\sqrt{35}X_{yz}^{(4)} + 6\sqrt{10}X_{yz}^{(5)} \\ \frac{3\sqrt{35}X_{zx}^{(4)}}{2} + 6\sqrt{10}X_{zx}^{(5)} & \frac{3\sqrt{35}X_{yz}^{(4)}}{2} - 6\sqrt{10}X_{yz}^{(5)} & -\sqrt{105}X_u^{(4)} - 5\sqrt{30}X_u^{(5)} - 3\sqrt{10}X_v^{(5)} \\ -3\sqrt{35}X_{yz}^{(4)} & 0 & 3\sqrt{35}X_{xy}^{(4)} \\ -\frac{\sqrt{105}X_u^{(4)}}{2} - \sqrt{30}X_u^{(5)} + \frac{3\sqrt{35}X_v^{(4)}}{2} + 9\sqrt{10}X_v^{(5)} & -\frac{3\sqrt{35}X_{zx}^{(4)}}{2} + 6\sqrt{10}X_{zx}^{(5)} & -\frac{3\sqrt{35}X_{xy}^{(4)}}{2} - 6\sqrt{10}X_{xy}^{(5)} \\ 3\sqrt{35}X_{zx}^{(4)} + 6\sqrt{10}X_{zx}^{(5)} & 3\sqrt{35}X_{xy}^{(4)} - 6\sqrt{10}X_{xy}^{(5)} & -2\sqrt{105}X_u^{(4)} + 5\sqrt{30}X_u^{(5)} - 3\sqrt{10}X_v^{(5)} \\ -3\sqrt{35}X_{zx}^{(4)} + 6\sqrt{10}X_{zx}^{(5)} & -3\sqrt{35}X_{yz}^{(4)} - 6\sqrt{10}X_{yz}^{(5)} & 2\sqrt{105}X_u^{(4)} - 5\sqrt{30}X_u^{(5)} - 3\sqrt{10}X_v^{(5)} \\ -12\sqrt{10}X_{xy}^{(5)} & 12\sqrt{10}X_{yz}^{(5)} & 6\sqrt{10}X_{xy}^{(5)} \\ -\frac{3\sqrt{35}X_{xy}^{(4)}}{2} - 6\sqrt{10}X_{xy}^{(5)} & -\frac{\sqrt{105}X_u^{(4)}}{2} - 4\sqrt{30}X_u^{(5)} - \frac{3\sqrt{35}X_v^{(4)}}{2} - 6\sqrt{10}X_v^{(5)} & -\frac{3\sqrt{35}X_{yz}^{(4)}}{2} + 6\sqrt{10}X_{yz}^{(5)} \\ \frac{\sqrt{105}X_u^{(4)}}{2} + 4\sqrt{30}X_u^{(5)} - \frac{3\sqrt{35}X_v^{(4)}}{2} - 6\sqrt{10}X_v^{(5)} & \frac{3\sqrt{35}X_{xy}^{(4)}}{2} + 6\sqrt{10}X_{xy}^{(5)} & \frac{3\sqrt{35}X_{zx}^{(4)}}{2} - 6\sqrt{10}X_{zx}^{(5)} \\ 3\sqrt{35}X_{yz}^{(4)} & -3\sqrt{35}X_{zx}^{(4)} & 0 \end{pmatrix}, \quad (B18)$$

where the multipole component with superscript (3) of $\sigma_{ij}^{mn\alpha(O/T)}$ does not appear.

$$\sigma_{ij}^{mn\alpha(H)} = \sigma_1^{(H)} + \sigma_2^{(H)}, \quad (B19)$$

$$\sigma_1^{(H)} = \begin{pmatrix} \sqrt{3}X_u^{(1)} + 4\sqrt{15}X_u^{(2)} - 3X_v^{(1)} - 12\sqrt{5}X_v^{(2)} & -3X_{xy}^{(1)} - 12\sqrt{5}X_{xy}^{(2)} & -3X_{zx}^{(1)} - 12\sqrt{5}X_{zx}^{(2)} \\ 3\sqrt{15}X_u^{(2)} - 9\sqrt{5}X_v^{(2)} & -9\sqrt{5}X_{xy}^{(2)} & -9\sqrt{5}X_{zx}^{(2)} \\ 3\sqrt{15}X_u^{(2)} - 9\sqrt{5}X_v^{(2)} & -9\sqrt{5}X_{xy}^{(2)} & -9\sqrt{5}X_{zx}^{(2)} \\ 0 & 0 & 0 \\ -\frac{3X_{zx}^{(1)}}{2} - \frac{3\sqrt{5}X_{zx}^{(2)}}{2} & -\frac{3X_{yz}^{(1)}}{2} - \frac{3\sqrt{5}X_{yz}^{(2)}}{2} & -\sqrt{3}X_u^{(1)} - \sqrt{15}X_u^{(2)} \\ -\frac{3X_{xy}^{(1)}}{2} - \frac{3\sqrt{5}X_{xy}^{(2)}}{2} & \frac{\sqrt{3}X_u^{(1)}}{2} + \frac{\sqrt{15}X_u^{(2)}}{2} + \frac{3X_v^{(1)}}{2} + \frac{3\sqrt{5}X_v^{(2)}}{2} & -\frac{3X_{yz}^{(1)}}{2} - \frac{3\sqrt{5}X_{yz}^{(2)}}{2} \\ -9\sqrt{5}X_{xy}^{(2)} & 3\sqrt{15}X_u^{(2)} + 9\sqrt{5}X_v^{(2)} & -9\sqrt{5}X_{yz}^{(2)} \\ -3X_{xy}^{(1)} - 12\sqrt{5}X_{xy}^{(2)} & \sqrt{3}X_u^{(1)} + 4\sqrt{15}X_u^{(2)} + 3X_v^{(1)} + 12\sqrt{5}X_v^{(2)} & -3X_{yz}^{(1)} - 12\sqrt{5}X_{yz}^{(2)} \\ -9\sqrt{5}X_{xy}^{(2)} & 3\sqrt{15}X_u^{(2)} + 9\sqrt{5}X_v^{(2)} & -9\sqrt{5}X_{yz}^{(2)} \\ -\frac{3X_{zx}^{(1)}}{2} - \frac{3\sqrt{5}X_{zx}^{(2)}}{2} & -\frac{3X_{yz}^{(1)}}{2} - \frac{3\sqrt{5}X_{yz}^{(2)}}{2} & -\sqrt{3}X_u^{(1)} - \sqrt{15}X_u^{(2)} \\ 0 & 0 & 0 \\ \frac{\sqrt{3}X_u^{(1)}}{2} + \frac{\sqrt{15}X_u^{(2)}}{2} - \frac{3X_v^{(1)}}{2} - \frac{3\sqrt{5}X_v^{(2)}}{2} & -\frac{3X_{xy}^{(1)}}{2} - \frac{3\sqrt{5}X_{xy}^{(2)}}{2} & -\frac{3X_{zx}^{(1)}}{2} - \frac{3\sqrt{5}X_{zx}^{(2)}}{2} \\ -9\sqrt{5}X_{zx}^{(2)} & -9\sqrt{5}X_{yz}^{(2)} & -6\sqrt{15}X_u^{(2)} \\ -9\sqrt{5}X_{zx}^{(2)} & -9\sqrt{5}X_{yz}^{(2)} & -6\sqrt{15}X_u^{(2)} \\ -3X_{zx}^{(1)} - 12\sqrt{5}X_{zx}^{(2)} & -3X_{yz}^{(1)} - 12\sqrt{5}X_{yz}^{(2)} & -2\sqrt{3}X_u^{(1)} - 8\sqrt{15}X_u^{(2)} \\ -\frac{3X_{xy}^{(1)}}{2} - \frac{3\sqrt{5}X_{xy}^{(2)}}{2} & \frac{\sqrt{3}X_u^{(1)}}{2} + \frac{\sqrt{15}X_u^{(2)}}{2} + \frac{3X_v^{(1)}}{2} + \frac{3\sqrt{5}X_v^{(2)}}{2} & -\frac{3X_{yz}^{(1)}}{2} - \frac{3\sqrt{5}X_{yz}^{(2)}}{2} \\ \frac{\sqrt{3}X_u^{(1)}}{2} + \frac{\sqrt{15}X_u^{(2)}}{2} - \frac{3X_v^{(1)}}{2} - \frac{3\sqrt{5}X_v^{(2)}}{2} & -\frac{3X_{xy}^{(1)}}{2} - \frac{3\sqrt{5}X_{xy}^{(2)}}{2} & -\frac{3X_{zx}^{(1)}}{2} - \frac{3\sqrt{5}X_{zx}^{(2)}}{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B20)$$

$$\sigma_2^{(H)} = \begin{pmatrix} 3\sqrt{15}X_u^{(4)} - 9\sqrt{5}X_v^{(4)} & 6\sqrt{5}X_{xy}^{(4)} & 6\sqrt{5}X_{zx}^{(4)} \\ \sqrt{15}X_u^{(3)} + \sqrt{15}X_u^{(4)} + \sqrt{5}X_v^{(3)} + 7\sqrt{5}X_v^{(4)} & \sqrt{5}X_{xy}^{(3)} - 8\sqrt{5}X_{xy}^{(4)} & -\sqrt{5}X_{zx}^{(3)} + 2\sqrt{5}X_{zx}^{(4)} \\ -\sqrt{15}X_u^{(3)} - 4\sqrt{15}X_u^{(4)} - \sqrt{5}X_v^{(3)} + 2\sqrt{5}X_v^{(4)} & -\sqrt{5}X_{xy}^{(3)} + 2\sqrt{5}X_{xy}^{(4)} & \sqrt{5}X_{zx}^{(3)} - 8\sqrt{5}X_{zx}^{(4)} \\ -2\sqrt{5}X_{yz}^{(3)} - 5\sqrt{5}X_{yz}^{(4)} & \sqrt{5}X_{zx}^{(3)} - 5\sqrt{5}X_{zx}^{(4)} & \sqrt{5}X_{xy}^{(3)} - 5\sqrt{5}X_{xy}^{(4)} \\ -\frac{\sqrt{5}X_{zx}^{(3)}}{2} - 8\sqrt{5}X_{zx}^{(4)} & \frac{\sqrt{5}X_{yz}^{(3)}}{2} + 2\sqrt{5}X_{yz}^{(4)} & 3\sqrt{15}X_u^{(4)} + \sqrt{5}X_v^{(3)} - 5\sqrt{5}X_v^{(4)} \\ -\frac{\sqrt{5}X_{xy}^{(3)}}{2} - 8\sqrt{5}X_{xy}^{(4)} & -\frac{\sqrt{15}X_u^{(3)}}{2} + \sqrt{15}X_u^{(4)} + \frac{\sqrt{5}X_v^{(3)}}{2} - 7\sqrt{5}X_v^{(4)} & \frac{\sqrt{5}X_{yz}^{(3)}}{2} + 2\sqrt{5}X_{yz}^{(4)} \\ \sqrt{5}X_{xy}^{(3)} - 8\sqrt{5}X_{xy}^{(4)} & \sqrt{15}X_u^{(3)} + \sqrt{15}X_u^{(4)} - \sqrt{5}X_v^{(3)} - 7\sqrt{5}X_v^{(4)} & -\sqrt{5}X_{yz}^{(3)} + 2\sqrt{5}X_{yz}^{(4)} \\ 6\sqrt{5}X_{xy}^{(4)} & 3\sqrt{15}X_u^{(4)} + 9\sqrt{5}X_v^{(4)} & 6\sqrt{5}X_{yz}^{(4)} \\ -\sqrt{5}X_{xy}^{(3)} + 2\sqrt{5}X_{xy}^{(4)} & -\sqrt{15}X_u^{(3)} - 4\sqrt{15}X_u^{(4)} + \sqrt{5}X_v^{(3)} - 2\sqrt{5}X_v^{(4)} & \sqrt{5}X_{yz}^{(3)} - 8\sqrt{5}X_{yz}^{(4)} \\ \frac{\sqrt{5}X_{zx}^{(3)}}{2} + 2\sqrt{5}X_{zx}^{(4)} & -\frac{\sqrt{5}X_{yz}^{(3)}}{2} - 8\sqrt{5}X_{yz}^{(4)} & 3\sqrt{15}X_u^{(4)} - \sqrt{5}X_v^{(3)} + 5\sqrt{5}X_v^{(4)} \\ \sqrt{5}X_{yz}^{(3)} - 5\sqrt{5}X_{yz}^{(4)} & -2\sqrt{5}X_{zx}^{(3)} - 5\sqrt{5}X_{zx}^{(4)} & \sqrt{5}X_{xy}^{(3)} - 5\sqrt{5}X_{xy}^{(4)} \\ -\frac{\sqrt{15}X_u^{(3)}}{2} + \sqrt{15}X_u^{(4)} - \frac{\sqrt{5}X_v^{(3)}}{2} + 7\sqrt{5}X_v^{(4)} & -\frac{\sqrt{5}X_{xy}^{(3)}}{2} - 8\sqrt{5}X_{xy}^{(4)} & \frac{\sqrt{5}X_{zx}^{(3)}}{2} + 2\sqrt{5}X_{zx}^{(4)} \\ \sqrt{5}X_{zx}^{(3)} - 8\sqrt{5}X_{zx}^{(4)} & -\sqrt{5}X_{yz}^{(3)} + 2\sqrt{5}X_{yz}^{(4)} & 3\sqrt{15}X_u^{(4)} - 2\sqrt{5}X_v^{(3)} - 5\sqrt{5}X_v^{(4)} \\ -\sqrt{5}X_{zx}^{(3)} + 2\sqrt{5}X_{zx}^{(4)} & \sqrt{5}X_{yz}^{(3)} - 8\sqrt{5}X_{yz}^{(4)} & 3\sqrt{15}X_u^{(4)} + 2\sqrt{5}X_v^{(3)} + 5\sqrt{5}X_v^{(4)} \\ 6\sqrt{5}X_{zx}^{(4)} & 6\sqrt{5}X_{yz}^{(4)} & -6\sqrt{15}X_u^{(4)} \\ \frac{\sqrt{5}X_{xy}^{(3)}}{2} + 2\sqrt{5}X_{xy}^{(4)} & \frac{\sqrt{15}X_u^{(3)}}{2} - 4\sqrt{15}X_u^{(4)} - \frac{\sqrt{5}X_v^{(3)}}{2} - 2\sqrt{5}X_v^{(4)} & -\frac{\sqrt{5}X_{yz}^{(3)}}{2} - 8\sqrt{5}X_{yz}^{(4)} \\ \frac{\sqrt{15}X_u^{(3)}}{2} - 4\sqrt{15}X_u^{(4)} + \frac{\sqrt{5}X_v^{(3)}}{2} + 2\sqrt{5}X_v^{(4)} & \frac{\sqrt{5}X_{xy}^{(3)}}{2} + 2\sqrt{5}X_{xy}^{(4)} & -\frac{\sqrt{5}X_{zx}^{(3)}}{2} - 8\sqrt{5}X_{zx}^{(4)} \\ \sqrt{5}X_{yz}^{(3)} - 5\sqrt{5}X_{yz}^{(4)} & \sqrt{5}X_{zx}^{(3)} - 5\sqrt{5}X_{zx}^{(4)} & -2\sqrt{5}X_{xy}^{(3)} - 5\sqrt{5}X_{xy}^{(4)} \end{pmatrix}. \quad (B21)$$

$$\sigma_3^2(O/L) = \begin{pmatrix} -2\sqrt{210}Y_x^{\alpha(5)} & -2\sqrt{210}Y_x^{\alpha(5)} & -2\sqrt{210}Y_x^{\alpha(5)} \\ \sqrt{210}Y_x^{\alpha(5)} - 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} - 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} - 5\sqrt{14}Y_x^{\beta(5)} \\ \sqrt{210}Y_x^{\alpha(5)} + 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} + 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} + 5\sqrt{14}Y_x^{\beta(5)} \\ -5\sqrt{14}Y_{xy}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} \\ \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} \\ \sqrt{210}Y_y^{\alpha(5)} + 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} + 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} + 5\sqrt{14}Y_y^{\beta(5)} \\ \sqrt{210}Y_y^{\alpha(5)} + 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} + 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} + 5\sqrt{14}Y_y^{\beta(5)} \\ -2\sqrt{210}Y_y^{\alpha(5)} & -2\sqrt{210}Y_y^{\alpha(5)} & -2\sqrt{210}Y_y^{\alpha(5)} \\ \sqrt{210}Y_y^{\alpha(5)} - 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} - 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} - 5\sqrt{14}Y_y^{\beta(5)} \\ \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} \\ -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} \\ \sqrt{210}Y_x^{\alpha(5)} - 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} - 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} - 5\sqrt{14}Y_x^{\beta(5)} \\ \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} \\ \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} \\ -2\sqrt{210}Y_z^{\alpha(5)} & -2\sqrt{210}Y_z^{\alpha(5)} & -2\sqrt{210}Y_z^{\alpha(5)} \\ \sqrt{210}Y_y^{\alpha(5)} - 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} - 5\sqrt{14}Y_y^{\beta(5)} & \sqrt{210}Y_y^{\alpha(5)} - 5\sqrt{14}Y_y^{\beta(5)} \\ \sqrt{210}Y_x^{\alpha(5)} + 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} + 5\sqrt{14}Y_x^{\beta(5)} & \sqrt{210}Y_x^{\alpha(5)} + 5\sqrt{14}Y_x^{\beta(5)} \\ -5\sqrt{14}Y_{xy}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} \end{pmatrix}. \quad (B25)$$

$$\sigma_{ij}^{mn\alpha(O/T)} = \sigma_1^{(O/T)} + \sigma_2^{(O/T)}, \quad (B26)$$

$$\sigma_1^{(O/T)} = \left(\begin{array}{ccc} -5\sqrt{6}Y_{xyz}^{(1)} + 4\sqrt{105}Y_{xyz}^{(2)} & 3\sqrt{10}Y_z^{\alpha(1)} + 3\sqrt{7}Y_z^{\alpha(2)} - 5\sqrt{6}Y_z^{\beta(1)} - \sqrt{105}Y_z^{\beta(2)} & 3\sqrt{10}Y_y^{\alpha(1)} + 3\sqrt{7}Y_y^{\alpha(2)} + 5\sqrt{6}Y_y^{\beta(1)} + \sqrt{105}Y_y^{\beta(2)} \\ -2\sqrt{105}Y_{xyz}^{(2)} & \sqrt{7}Y_z^{\alpha(2)} + 3\sqrt{105}Y_z^{\beta(2)} & -4\sqrt{7}Y_y^{\alpha(2)} + 2\sqrt{105}Y_y^{\beta(2)} \\ -2\sqrt{105}Y_{xz}^{(2)} & -4\sqrt{7}Y_z^{\alpha(2)} - 2\sqrt{105}Y_z^{\beta(2)} & \sqrt{7}Y_y^{\alpha(2)} - 3\sqrt{105}Y_y^{\beta(2)} \\ 10\sqrt{7}Y_x^{\alpha(2)} & 10\sqrt{7}Y_y^{\alpha(2)} & 10\sqrt{7}Y_z^{\alpha(2)} \\ \frac{3\sqrt{10}Y_y^{\alpha(1)}}{2} + \sqrt{7}Y_y^{\alpha(2)} - \frac{5\sqrt{6}Y_y^{\beta(1)}}{2} + 3\sqrt{105}Y_y^{\beta(2)} & \frac{3\sqrt{10}Y_x^{\alpha(1)}}{2} - 4\sqrt{7}Y_x^{\alpha(2)} + \frac{5\sqrt{6}Y_x^{\beta(1)}}{2} + 2\sqrt{105}Y_x^{\beta(2)} & -\frac{5\sqrt{6}Y_x^{\beta(1)}}{2} - 2\sqrt{105}Y_{xz}^{(2)} \\ \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} + \sqrt{7}Y_z^{\alpha(2)} + \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} - 3\sqrt{105}Y_z^{\beta(2)} & -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 2\sqrt{105}Y_{xyz}^{(2)} & \frac{3\sqrt{10}Y_x^{\alpha(1)}}{2} - 4\sqrt{7}Y_x^{\alpha(2)} - \frac{5\sqrt{6}Y_x^{\beta(1)}}{2} - 2\sqrt{105}Y_x^{\beta(2)} \\ \sqrt{7}Y_z^{\alpha(2)} - 3\sqrt{105}Y_z^{\beta(2)} & -2\sqrt{105}Y_{yz}^{(2)} & -4\sqrt{7}Y_x^{\alpha(2)} - 2\sqrt{105}Y_x^{\beta(2)} \\ 3\sqrt{10}Y_z^{\alpha(1)} + 3\sqrt{7}Y_z^{\alpha(2)} + 5\sqrt{6}Y_z^{\beta(1)} + \sqrt{105}Y_z^{\beta(2)} & -5\sqrt{6}Y_{xyz}^{(1)} + 4\sqrt{105}Y_{xyz}^{(2)} & 3\sqrt{10}Y_x^{\alpha(1)} + 3\sqrt{7}Y_x^{\alpha(2)} - 5\sqrt{6}Y_x^{\beta(1)} - \sqrt{105}Y_x^{\beta(2)} \\ -4\sqrt{7}Y_z^{\alpha(2)} + 2\sqrt{105}Y_z^{\beta(2)} & -2\sqrt{105}Y_{xy}^{(2)} & \sqrt{7}Y_x^{\alpha(2)} + 3\sqrt{105}Y_x^{\beta(2)} \\ \frac{3\sqrt{10}Y_y^{\alpha(1)}}{2} - 4\sqrt{7}Y_y^{\alpha(2)} - \frac{5\sqrt{6}Y_y^{\beta(1)}}{2} - 2\sqrt{105}Y_y^{\beta(2)} & \frac{3\sqrt{10}Y_x^{\alpha(1)}}{2} + \sqrt{7}Y_x^{\alpha(2)} + \frac{5\sqrt{6}Y_x^{\beta(1)}}{2} - 3\sqrt{105}Y_x^{\beta(2)} & -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 2\sqrt{105}Y_{xyz}^{(2)} \\ 10\sqrt{7}Y_x^{\alpha(2)} & 10\sqrt{7}Y_y^{\alpha(2)} & 10\sqrt{7}Y_z^{\alpha(2)} \\ -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 2\sqrt{105}Y_{xyz}^{(2)} & \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} + \sqrt{7}Y_z^{\alpha(2)} - \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} + 3\sqrt{105}Y_z^{\beta(2)} & \frac{3\sqrt{10}Y_y^{\alpha(1)}}{2} - 4\sqrt{7}Y_y^{\alpha(2)} + \frac{5\sqrt{6}Y_y^{\beta(1)}}{2} + 2\sqrt{105}Y_y^{\beta(2)} \\ \sqrt{7}Y_y^{\alpha(2)} + 3\sqrt{105}Y_y^{\beta(2)} & -4\sqrt{7}Y_x^{\alpha(2)} + 2\sqrt{105}Y_x^{\beta(2)} & -2\sqrt{105}Y_{yz}^{(2)} \\ -4\sqrt{7}Y_y^{\alpha(2)} - 2\sqrt{105}Y_y^{\beta(2)} & \sqrt{7}Y_x^{\alpha(2)} - 3\sqrt{105}Y_x^{\beta(2)} & -2\sqrt{105}Y_{xz}^{(2)} \\ 3\sqrt{10}Y_y^{\alpha(1)} + 3\sqrt{7}Y_y^{\alpha(2)} - 5\sqrt{6}Y_y^{\beta(1)} - \sqrt{105}Y_y^{\beta(2)} & 3\sqrt{10}Y_x^{\alpha(1)} + 3\sqrt{7}Y_x^{\alpha(2)} + 5\sqrt{6}Y_x^{\beta(1)} + \sqrt{105}Y_x^{\beta(2)} & -5\sqrt{6}Y_{xyz}^{(1)} + 4\sqrt{105}Y_{xyz}^{(2)} \\ \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} - 4\sqrt{7}Y_z^{\alpha(2)} + \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} + 2\sqrt{105}Y_z^{\beta(2)} & -\frac{5\sqrt{6}Y_{yz}^{(1)}}{2} - 2\sqrt{105}Y_{yz}^{(2)} & \frac{3\sqrt{10}Y_x^{\alpha(1)}}{2} + \sqrt{7}Y_x^{\alpha(2)} - \frac{5\sqrt{6}Y_x^{\beta(1)}}{2} + 3\sqrt{105}Y_x^{\beta(2)} \\ -\frac{5\sqrt{6}Y_{xz}^{(1)}}{2} - 2\sqrt{105}Y_{xz}^{(2)} & \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} - 4\sqrt{7}Y_z^{\alpha(2)} - \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} - 2\sqrt{105}Y_z^{\beta(2)} & \frac{3\sqrt{10}Y_y^{\alpha(1)}}{2} + \sqrt{7}Y_y^{\alpha(2)} + \frac{5\sqrt{6}Y_y^{\beta(1)}}{2} - 3\sqrt{105}Y_y^{\beta(2)} \\ 10\sqrt{7}Y_x^{\alpha(2)} & 10\sqrt{7}Y_y^{\alpha(2)} & 10\sqrt{7}Y_z^{\alpha(2)} \end{array} \right), \quad (B27)$$

$$\sigma_2^{(O/T)} = \left(\begin{array}{ccc} -4\sqrt{30}Y_{xyz}^{(4)} & 12\sqrt{2}Y_z^{\alpha(4)} - 4\sqrt{30}Y_z^{\beta(4)} & 12\sqrt{2}Y_y^{\alpha(4)} + 4\sqrt{30}Y_y^{\beta(4)} \\ -3\sqrt{30}Y_{xyz}^{(4)} & 2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} - 3\sqrt{30}Y_z^{\beta(4)} & -2\sqrt{3}Y_y^{\alpha(3)} + 9\sqrt{2}Y_y^{\alpha(4)} + 3\sqrt{30}Y_y^{\beta(4)} \\ -3\sqrt{30}Y_{xz}^{(4)} & -2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} - 3\sqrt{30}Y_z^{\beta(4)} & 2\sqrt{3}Y_y^{\alpha(3)} + 9\sqrt{2}Y_y^{\alpha(4)} + 3\sqrt{30}Y_y^{\beta(4)} \\ -\sqrt{3}Y_x^{\alpha(3)} & \frac{\sqrt{3}Y_y^{\alpha(3)}}{2} - \frac{3\sqrt{5}Y_y^{\beta(3)}}{2} & \frac{\sqrt{3}Y_x^{\alpha(3)}}{2} + \frac{3\sqrt{5}Y_x^{\beta(3)}}{2} \\ -\sqrt{3}Y_y^{\alpha(3)} + \frac{3\sqrt{2}Y_y^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_y^{\beta(4)}}{2} & \sqrt{3}Y_x^{\alpha(3)} + \frac{3\sqrt{2}Y_x^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_x^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} \\ -\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{yz}^{(4)}}{2} & \sqrt{3}Y_x^{\alpha(3)} + \frac{3\sqrt{2}Y_x^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_x^{\beta(4)}}{2} \\ 2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} + 3\sqrt{30}Y_z^{\beta(4)} & -3\sqrt{30}Y_{xyz}^{(4)} & -2\sqrt{3}Y_x^{\alpha(3)} + 9\sqrt{2}Y_x^{\alpha(4)} - 3\sqrt{30}Y_x^{\beta(4)} \\ 12\sqrt{2}Y_z^{\alpha(4)} + 4\sqrt{30}Y_z^{\beta(4)} & -4\sqrt{30}Y_{xy}^{(4)} & 12\sqrt{2}Y_x^{\alpha(4)} - 4\sqrt{30}Y_x^{\beta(4)} \\ -2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} + 3\sqrt{30}Y_z^{\beta(4)} & -3\sqrt{30}Y_{yz}^{(4)} & 2\sqrt{3}Y_x^{\alpha(3)} + 9\sqrt{2}Y_x^{\alpha(4)} - 3\sqrt{30}Y_x^{\beta(4)} \\ \sqrt{3}Y_y^{\alpha(3)} + \frac{3\sqrt{2}Y_y^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_y^{\beta(4)}}{2} & -\sqrt{3}Y_x^{\alpha(3)} + \frac{3\sqrt{2}Y_x^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_x^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{xz}^{(4)}}{2} \\ \frac{\sqrt{3}Y_x^{\alpha(3)}}{2} + \frac{3\sqrt{5}Y_x^{\beta(3)}}{2} & -\sqrt{3}Y_y^{\alpha(3)} & \frac{\sqrt{3}Y_z^{\alpha(3)}}{2} - \frac{3\sqrt{5}Y_z^{\beta(3)}}{2} \\ -\frac{\sqrt{30}Y_{xz}^{(4)}}{2} & -\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & \sqrt{3}Y_y^{\alpha(3)} + \frac{3\sqrt{2}Y_y^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_y^{\beta(4)}}{2} \\ 2\sqrt{3}Y_y^{\alpha(3)} + 9\sqrt{2}Y_y^{\alpha(4)} - 3\sqrt{30}Y_y^{\beta(4)} & -2\sqrt{3}Y_x^{\alpha(3)} + 9\sqrt{2}Y_x^{\alpha(4)} + 3\sqrt{30}Y_x^{\beta(4)} & -3\sqrt{30}Y_{xyz}^{(4)} \\ -2\sqrt{3}Y_y^{\alpha(3)} + 9\sqrt{2}Y_y^{\alpha(4)} - 3\sqrt{30}Y_y^{\beta(4)} & 2\sqrt{3}Y_x^{\alpha(3)} + 9\sqrt{2}Y_x^{\alpha(4)} + 3\sqrt{30}Y_x^{\beta(4)} & -3\sqrt{30}Y_{yz}^{(4)} \\ 12\sqrt{2}Y_y^{\alpha(4)} - 4\sqrt{30}Y_y^{\beta(4)} & 12\sqrt{2}Y_x^{\alpha(4)} + 4\sqrt{30}Y_x^{\beta(4)} & -4\sqrt{30}Y_{xz}^{(4)} \\ \sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{xy}^{(4)}}{2} & -\sqrt{3}Y_x^{\alpha(3)} + \frac{3\sqrt{2}Y_x^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_x^{\beta(4)}}{2} \\ -\frac{\sqrt{30}Y_{yz}^{(4)}}{2} & \sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\sqrt{3}Y_y^{\alpha(3)} + \frac{3\sqrt{2}Y_y^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_y^{\beta(4)}}{2} \\ \frac{\sqrt{3}Y_x^{\alpha(3)}}{2} - \frac{3\sqrt{5}Y_x^{\beta(3)}}{2} & \frac{\sqrt{3}Y_y^{\alpha(3)}}{2} + \frac{3\sqrt{5}Y_y^{\beta(3)}}{2} & -\sqrt{3}Y_z^{\alpha(3)} \end{array} \right), \quad (B28)$$

where the multipole component with superscript (5) of $\sigma_{ij}^{mn\alpha(O/T)}$ does not appear.

$$\sigma_{ij}^{mn\alpha(H)} = \begin{pmatrix} 0 & -3\sqrt{105}Y_z^{\alpha(2)} + 15\sqrt{7}Y_z^{\beta(2)} & 3\sqrt{105}Y_y^{\alpha(2)} + 15\sqrt{7}Y_y^{\beta(2)} \\ \sqrt{10}Y_{xyz}^{(1)} - 10\sqrt{7}Y_{xyz}^{(2)} & -\sqrt{6}Y_x^{\alpha(1)} - \sqrt{105}Y_z^{\alpha(2)} - \sqrt{10}Y_z^{\beta(1)} - 5\sqrt{7}Y_z^{\beta(2)} & -\sqrt{6}Y_y^{\alpha(1)} - 4\sqrt{105}Y_x^{\alpha(2)} + \sqrt{10}Y_y^{\beta(1)} - 10\sqrt{7}Y_y^{\beta(2)} \\ -\sqrt{10}Y_{xyz}^{(1)} + 10\sqrt{7}Y_{xyz}^{(2)} & \sqrt{6}Y_z^{\alpha(1)} + 4\sqrt{105}Y_z^{\alpha(2)} + \sqrt{10}Y_z^{\beta(1)} - 10\sqrt{7}Y_z^{\beta(2)} & \sqrt{6}Y_x^{\alpha(1)} + \sqrt{105}Y_x^{\alpha(2)} - \sqrt{10}Y_x^{\beta(1)} - 5\sqrt{7}Y_x^{\beta(2)} \\ -\sqrt{10}Y_x^{\beta(1)} + 10\sqrt{7}Y_x^{\beta(2)} & -\frac{3\sqrt{6}Y_y^{\alpha(1)}}{2} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} + 10\sqrt{7}Y_y^{\beta(2)} & \frac{3\sqrt{6}Y_y^{\alpha(1)}}{2} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} + 10\sqrt{7}Y_y^{\beta(2)} \\ \frac{\sqrt{6}Y_y^{\alpha(1)}}{2} - \sqrt{105}Y_y^{\alpha(2)} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} - 5\sqrt{7}Y_y^{\beta(2)} & \frac{\sqrt{6}Y_x^{\alpha(1)}}{2} - 4\sqrt{105}Y_x^{\alpha(2)} - \frac{\sqrt{10}Y_x^{\beta(1)}}{2} - 10\sqrt{7}Y_x^{\beta(2)} & -\frac{\sqrt{6}Y_x^{\alpha(1)}}{2} + 4\sqrt{105}Y_x^{\alpha(2)} - \frac{\sqrt{10}Y_x^{\beta(1)}}{2} - 10\sqrt{7}Y_x^{\beta(2)} \\ -\frac{\sqrt{6}Y_z^{\alpha(1)}}{2} + \sqrt{105}Y_z^{\alpha(2)} + \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 5\sqrt{7}Y_z^{\beta(2)} & \frac{\sqrt{10}Y_{xyz}^{(1)}}{2} + 10\sqrt{7}Y_{xyz}^{(2)} & -\frac{\sqrt{6}Y_z^{\alpha(1)}}{2} + 4\sqrt{105}Y_z^{\alpha(2)} + \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 10\sqrt{7}Y_z^{\beta(2)} \\ \sqrt{6}Y_z^{\alpha(1)} + \sqrt{105}Y_z^{\alpha(2)} - \sqrt{10}Y_z^{\beta(1)} - 5\sqrt{7}Y_z^{\beta(2)} & -\sqrt{10}Y_{xyz}^{(1)} + 10\sqrt{7}Y_{xyz}^{(2)} & \sqrt{6}Y_x^{\alpha(1)} + 4\sqrt{105}Y_x^{\alpha(2)} + \sqrt{10}Y_x^{\beta(1)} - 10\sqrt{7}Y_x^{\beta(2)} \\ 3\sqrt{105}Y_z^{\alpha(2)} + 15\sqrt{7}Y_z^{\beta(2)} & 0 & -3\sqrt{105}Y_x^{\alpha(2)} + 15\sqrt{7}Y_x^{\beta(2)} \\ -\sqrt{6}Y_z^{\alpha(1)} - 4\sqrt{105}Y_z^{\alpha(2)} + \sqrt{10}Y_z^{\beta(1)} - 10\sqrt{7}Y_z^{\beta(2)} & \sqrt{10}Y_{xyz}^{(1)} - 10\sqrt{7}Y_{xyz}^{(2)} & -\sqrt{6}Y_x^{\alpha(1)} - \sqrt{105}Y_x^{\alpha(2)} - \sqrt{10}Y_x^{\beta(1)} - 5\sqrt{7}Y_x^{\beta(2)} \\ -\frac{\sqrt{6}Y_y^{\alpha(1)}}{2} + 4\sqrt{105}Y_y^{\alpha(2)} - \frac{\sqrt{10}Y_y^{\beta(1)}}{2} - 10\sqrt{7}Y_y^{\beta(2)} & -\frac{\sqrt{6}Y_x^{\alpha(1)}}{2} + \sqrt{105}Y_x^{\alpha(2)} + \frac{\sqrt{10}Y_x^{\beta(1)}}{2} - 5\sqrt{7}Y_x^{\beta(2)} & \frac{\sqrt{10}Y_{xyz}^{(1)}}{2} + 10\sqrt{7}Y_{xyz}^{(2)} \\ \frac{3\sqrt{6}Y_x^{\alpha(1)}}{2} + \frac{\sqrt{10}Y_x^{\beta(1)}}{2} + 10\sqrt{7}Y_x^{\beta(2)} & -\sqrt{10}Y_y^{\beta(1)} + 10\sqrt{7}Y_y^{\beta(2)} & -\frac{3\sqrt{6}Y_y^{\alpha(1)}}{2} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} + 10\sqrt{7}Y_y^{\beta(2)} \\ -\frac{\sqrt{10}Y_{xyz}^{(1)}}{2} - 10\sqrt{7}Y_{xyz}^{(2)} & \frac{\sqrt{6}Y_z^{\alpha(1)}}{2} - \sqrt{105}Y_z^{\alpha(2)} + \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 5\sqrt{7}Y_z^{\beta(2)} & \frac{\sqrt{6}Y_y^{\alpha(1)}}{2} - \sqrt{105}Y_y^{\alpha(2)} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} - 5\sqrt{7}Y_y^{\beta(2)} \\ -\sqrt{6}Y_y^{\alpha(1)} - \sqrt{105}Y_y^{\alpha(2)} - \sqrt{10}Y_y^{\beta(1)} - 5\sqrt{7}Y_y^{\beta(2)} & -\sqrt{6}Y_x^{\alpha(1)} - 4\sqrt{105}Y_x^{\alpha(2)} - \sqrt{10}Y_x^{\beta(1)} - 10\sqrt{7}Y_x^{\beta(2)} & \sqrt{10}Y_{xyz}^{(1)} - 10\sqrt{7}Y_{xyz}^{(2)} \\ \sqrt{6}Y_y^{\alpha(1)} + 4\sqrt{105}Y_y^{\alpha(2)} + \sqrt{10}Y_y^{\beta(1)} - 10\sqrt{7}Y_y^{\beta(2)} & \sqrt{6}Y_x^{\alpha(1)} + \sqrt{105}Y_x^{\alpha(2)} - \sqrt{10}Y_x^{\beta(1)} - 5\sqrt{7}Y_x^{\beta(2)} & -\sqrt{10}Y_{xyz}^{(1)} + 10\sqrt{7}Y_{xyz}^{(2)} \\ -3\sqrt{105}Y_y^{\alpha(2)} + 15\sqrt{7}Y_y^{\beta(2)} & 3\sqrt{105}Y_x^{\alpha(2)} + 15\sqrt{7}Y_x^{\beta(2)} & 0 \\ \frac{\sqrt{6}Y_z^{\alpha(1)}}{2} - 4\sqrt{105}Y_z^{\alpha(2)} - \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 10\sqrt{7}Y_z^{\beta(2)} & -\frac{\sqrt{6}Y_x^{\alpha(1)}}{2} + 4\sqrt{105}Y_x^{\alpha(2)} - \frac{\sqrt{10}Y_x^{\beta(1)}}{2} - 10\sqrt{7}Y_x^{\beta(2)} & \frac{\sqrt{6}Y_y^{\alpha(1)}}{2} - \sqrt{105}Y_y^{\alpha(2)} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} - 5\sqrt{7}Y_y^{\beta(2)} \\ \frac{\sqrt{10}Y_{xyz}^{(1)}}{2} + 10\sqrt{7}Y_{xyz}^{(2)} & -\frac{\sqrt{6}Y_z^{\alpha(1)}}{2} + 4\sqrt{105}Y_z^{\alpha(2)} + \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 10\sqrt{7}Y_z^{\beta(2)} & -\frac{\sqrt{6}Y_y^{\alpha(1)}}{2} + \sqrt{105}Y_y^{\alpha(2)} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} - 5\sqrt{7}Y_y^{\beta(2)} \\ -\frac{3\sqrt{6}Y_x^{\alpha(1)}}{2} + \frac{\sqrt{10}Y_x^{\beta(1)}}{2} + 10\sqrt{7}Y_x^{\beta(2)} & \frac{3\sqrt{6}Y_y^{\alpha(1)}}{2} + \frac{\sqrt{10}Y_y^{\beta(1)}}{2} + 10\sqrt{7}Y_y^{\beta(2)} & -\sqrt{10}Y_z^{\beta(1)} + 10\sqrt{7}Y_z^{\beta(2)} \end{pmatrix}. \quad (B29)$$

5. rank-4 c -multipoles

$$\sigma_{ij}^{mn\alpha(O/L)} = \sigma_1^{(O/L)} + \sigma_2^{(O/L)}, \quad (B30)$$

$$\sigma_1^{(O/L)} = \begin{pmatrix} 0 & 0 & 0 \\ -2\sqrt{105}X_{4x}^{\beta(1)} & -7\sqrt{15}X_{4x}^{\alpha(1)} + \sqrt{105}X_{4x}^{\beta(1)} & 7\sqrt{15}X_{4x}^{\alpha(1)} + \sqrt{105}X_{4x}^{\beta(1)} \\ 2\sqrt{105}X_{4x}^{\beta(1)} & 7\sqrt{15}X_{4x}^{\alpha(1)} - \sqrt{105}X_{4x}^{\beta(1)} & -7\sqrt{15}X_{4x}^{\alpha(1)} - \sqrt{105}X_{4x}^{\beta(1)} \\ -3\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} & 2\sqrt{105}X_{4v}^{(1)} + 21X_4^{(1)} & 3\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} - 21X_4^{(1)} \\ \frac{7\sqrt{15}X_{4z}^{\alpha(1)}}{2} - \frac{\sqrt{105}X_{4z}^{\beta(1)}}{2} & -\frac{7\sqrt{15}X_{4z}^{\alpha(1)}}{2} - \frac{\sqrt{105}X_{4z}^{\beta(1)}}{2} & \frac{\sqrt{105}X_{4z}^{\beta(1)}}{2} \\ \frac{7\sqrt{15}X_{4y}^{\alpha(1)}}{2} + \frac{\sqrt{105}X_{4y}^{\beta(1)}}{2} & -\sqrt{105}X_{4y}^{\beta(1)} & -\frac{7\sqrt{15}X_{4y}^{\alpha(1)}}{2} + \frac{\sqrt{105}X_{4y}^{\beta(1)}}{2} \\ -7\sqrt{15}X_{4y}^{\alpha(1)} - \sqrt{105}X_{4y}^{\beta(1)} & 2\sqrt{105}X_{4y}^{\beta(1)} & 7\sqrt{15}X_{4y}^{\alpha(1)} - \sqrt{105}X_{4y}^{\beta(1)} \\ 0 & 0 & 0 \\ 7\sqrt{15}X_{4y}^{\alpha(1)} + \sqrt{105}X_{4y}^{\beta(1)} & -2\sqrt{105}X_{4y}^{\beta(1)} & -7\sqrt{15}X_{4y}^{\alpha(1)} + \sqrt{105}X_{4y}^{\beta(1)} \\ -\frac{7\sqrt{15}X_{4z}^{\alpha(1)}}{2} + \frac{\sqrt{105}X_{4z}^{\beta(1)}}{2} & \frac{7\sqrt{15}X_{4z}^{\alpha(1)}}{2} + \frac{\sqrt{105}X_{4z}^{\beta(1)}}{2} & -\sqrt{105}X_{4z}^{\beta(1)} \\ 2\sqrt{105}X_{4v}^{(1)} - 21X_4^{(1)} & 3\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} & -3\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} + 21X_4^{(1)} \\ \sqrt{105}X_{4x}^{\beta(1)} & \frac{7\sqrt{15}X_{4x}^{\alpha(1)}}{2} - \frac{\sqrt{105}X_{4x}^{\beta(1)}}{2} & -\frac{7\sqrt{15}X_{4x}^{\alpha(1)}}{2} - \frac{\sqrt{105}X_{4x}^{\beta(1)}}{2} \\ -7\sqrt{15}X_{4z}^{\alpha(1)} + \sqrt{105}X_{4z}^{\beta(1)} & 7\sqrt{15}X_{4z}^{\alpha(1)} + \sqrt{105}X_{4z}^{\beta(1)} & -2\sqrt{105}X_{4z}^{\beta(1)} \\ 7\sqrt{15}X_{4z}^{\alpha(1)} - \sqrt{105}X_{4z}^{\beta(1)} & -7\sqrt{15}X_{4z}^{\alpha(1)} - \sqrt{105}X_{4z}^{\beta(1)} & 2\sqrt{105}X_{4z}^{\beta(1)} \\ 0 & 0 & 0 \\ -\frac{7\sqrt{15}X_{4y}^{\alpha(1)}}{2} - \frac{\sqrt{105}X_{4y}^{\beta(1)}}{2} & \sqrt{105}X_{4y}^{\beta(1)} & \frac{7\sqrt{15}X_{4y}^{\alpha(1)}}{2} - \frac{\sqrt{105}X_{4y}^{\beta(1)}}{2} \\ -\sqrt{105}X_{4x}^{\beta(1)} & -\frac{7\sqrt{15}X_{4x}^{\alpha(1)}}{2} + \frac{\sqrt{105}X_{4x}^{\beta(1)}}{2} & \frac{7\sqrt{15}X_{4x}^{\alpha(1)}}{2} + \frac{\sqrt{105}X_{4x}^{\beta(1)}}{2} \\ 3\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} + 21X_4^{(1)} & -3\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} - 21X_4^{(1)} & 2\sqrt{105}X_{4v}^{(1)} \end{pmatrix}, \quad (B31)$$

$$\sigma_2^{(O/L)} = \begin{pmatrix} 0 & -12\sqrt{15}X_{4x}^{\beta(2)} & 12\sqrt{15}X_{4x}^{\beta(2)} \\ 8\sqrt{15}X_{4x}^{\beta(2)} & -2\sqrt{105}X_{4x}^{\alpha(2)} + 2\sqrt{15}X_{4x}^{\beta(2)} & 2\sqrt{105}X_{4x}^{\alpha(2)} - 10\sqrt{15}X_{4x}^{\beta(2)} \\ -8\sqrt{15}X_{4x}^{\beta(2)} & 2\sqrt{105}X_{4x}^{\alpha(2)} + 10\sqrt{15}X_{4x}^{\beta(2)} & -2\sqrt{105}X_{4x}^{\alpha(2)} - 2\sqrt{15}X_{4x}^{\beta(2)} \\ 12\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} & -12\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} & -8\sqrt{15}X_{4v}^{(2)} \\ -2\sqrt{105}X_{4z}^{\alpha(2)} + 2\sqrt{15}X_{4z}^{\beta(2)} & 2\sqrt{105}X_{4z}^{\alpha(2)} - 10\sqrt{15}X_{4z}^{\beta(2)} & 8\sqrt{15}X_{4z}^{\beta(2)} \\ -2\sqrt{105}X_{4y}^{\alpha(2)} - 2\sqrt{15}X_{4y}^{\beta(2)} & -8\sqrt{15}X_{4y}^{\beta(2)} & 2\sqrt{105}X_{4y}^{\alpha(2)} + 10\sqrt{15}X_{4y}^{\beta(2)} \\ -2\sqrt{105}X_{4y}^{\alpha(2)} - 2\sqrt{15}X_{4y}^{\beta(2)} & -8\sqrt{15}X_{4y}^{\beta(2)} & 2\sqrt{105}X_{4y}^{\alpha(2)} + 10\sqrt{15}X_{4y}^{\beta(2)} \\ 12\sqrt{15}X_{4y}^{\beta(2)} & 0 & -12\sqrt{15}X_{4y}^{\beta(2)} \\ 2\sqrt{105}X_{4y}^{\alpha(2)} - 10\sqrt{15}X_{4y}^{\beta(2)} & 8\sqrt{15}X_{4y}^{\beta(2)} & -2\sqrt{105}X_{4y}^{\alpha(2)} + 2\sqrt{15}X_{4y}^{\beta(2)} \\ 2\sqrt{105}X_{4z}^{\alpha(2)} + 10\sqrt{15}X_{4z}^{\beta(2)} & -2\sqrt{105}X_{4z}^{\alpha(2)} - 2\sqrt{15}X_{4z}^{\beta(2)} & -8\sqrt{15}X_{4z}^{\beta(2)} \\ 12\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} & -12\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} & -8\sqrt{15}X_{4v}^{(2)} \\ 8\sqrt{15}X_{4x}^{\beta(2)} & -2\sqrt{105}X_{4x}^{\alpha(2)} + 2\sqrt{15}X_{4x}^{\beta(2)} & 2\sqrt{105}X_{4x}^{\alpha(2)} - 10\sqrt{15}X_{4x}^{\beta(2)} \\ -2\sqrt{105}X_{4z}^{\alpha(2)} + 2\sqrt{15}X_{4z}^{\beta(2)} & 2\sqrt{105}X_{4z}^{\alpha(2)} - 10\sqrt{15}X_{4z}^{\beta(2)} & 8\sqrt{15}X_{4z}^{\beta(2)} \\ 2\sqrt{105}X_{4z}^{\alpha(2)} + 10\sqrt{15}X_{4z}^{\beta(2)} & -2\sqrt{105}X_{4z}^{\alpha(2)} - 2\sqrt{15}X_{4z}^{\beta(2)} & -8\sqrt{15}X_{4z}^{\beta(2)} \\ -12\sqrt{15}X_{4z}^{\beta(2)} & 12\sqrt{15}X_{4z}^{\beta(2)} & 0 \\ 2\sqrt{105}X_{4y}^{\alpha(2)} - 10\sqrt{15}X_{4y}^{\beta(2)} & 8\sqrt{15}X_{4y}^{\beta(2)} & -2\sqrt{105}X_{4y}^{\alpha(2)} + 2\sqrt{15}X_{4y}^{\beta(2)} \\ -8\sqrt{15}X_{4x}^{\beta(2)} & 2\sqrt{105}X_{4x}^{\alpha(2)} + 10\sqrt{15}X_{4x}^{\beta(2)} & -2\sqrt{105}X_{4x}^{\alpha(2)} - 2\sqrt{15}X_{4x}^{\beta(2)} \\ 12\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} & -12\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} & -8\sqrt{15}X_{4v}^{(2)} \end{pmatrix}. \quad (B32)$$

$$\sigma_{ij}^{mna(O/T)} = \sigma_1^{(O/T)} + \sigma_2^{(O/T)}, \quad (B33)$$

$$\sigma_1^{(O/T)} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ -2\sqrt{35}X_{4u}^{(1)} + 2\sqrt{105}X_{4v}^{(1)} + 14X_4^{(1)} & -2\sqrt{105}X_{4z}^{\beta(1)} & -2\sqrt{105}X_{4y}^{\beta(1)} \\ 2\sqrt{35}X_{4u}^{(1)} - 2\sqrt{105}X_{4v}^{(1)} - 14X_4^{(1)} & 2\sqrt{105}X_{4z}^{\beta(1)} & 2\sqrt{105}X_{4y}^{\beta(1)} \\ 7\sqrt{15}X_{4x}^{\alpha(1)} & -\frac{7\sqrt{15}X_{4y}^{\alpha(1)}}{2} + \frac{3\sqrt{105}X_{4y}^{\beta(1)}}{2} & -\frac{7\sqrt{15}X_{4z}^{\alpha(1)}}{2} - \frac{3\sqrt{105}X_{4z}^{\beta(1)}}{2} \\ \sqrt{105}X_{4y}^{\beta(1)} & \sqrt{105}X_{4x}^{\beta(1)} & -2\sqrt{35}X_{4u}^{(1)} - 7X_4^{(1)} \\ -\sqrt{105}X_{4z}^{\beta(1)} & -\sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} + 7X_4^{(1)} & -\sqrt{105}X_{4x}^{\beta(1)} \\ 2\sqrt{105}X_{4z}^{\beta(1)} & 2\sqrt{35}X_{4u}^{(1)} + 2\sqrt{105}X_{4v}^{(1)} - 14X_4^{(1)} & 2\sqrt{105}X_{4x}^{\beta(1)} \\ 0 & 0 & 0 \\ -2\sqrt{105}X_{4z}^{\beta(1)} & -2\sqrt{35}X_{4u}^{(1)} - 2\sqrt{105}X_{4v}^{(1)} + 14X_4^{(1)} & -2\sqrt{105}X_{4x}^{\beta(1)} \\ -\sqrt{105}X_{4y}^{\beta(1)} & -\sqrt{105}X_{4x}^{\beta(1)} & 2\sqrt{35}X_{4u}^{(1)} + 7X_4^{(1)} \\ -\frac{7\sqrt{15}X_{4x}^{\alpha(1)}}{2} - \frac{3\sqrt{105}X_{4x}^{\beta(1)}}{2} & 7\sqrt{15}X_{4y}^{\alpha(1)} & -\frac{7\sqrt{15}X_{4z}^{\alpha(1)}}{2} + \frac{3\sqrt{105}X_{4z}^{\beta(1)}}{2} \\ \sqrt{35}X_{4u}^{(1)} - \sqrt{105}X_{4v}^{(1)} - 7X_4^{(1)} & \sqrt{105}X_{4z}^{\beta(1)} & \sqrt{105}X_{4y}^{\beta(1)} \\ -2\sqrt{105}X_{4y}^{\beta(1)} & -2\sqrt{105}X_{4x}^{\beta(1)} & 4\sqrt{35}X_{4u}^{(1)} + 14X_4^{(1)} \\ 2\sqrt{105}X_{4y}^{\beta(1)} & 2\sqrt{105}X_{4x}^{\beta(1)} & -4\sqrt{35}X_{4u}^{(1)} - 14X_4^{(1)} \\ 0 & 0 & 0 \\ \sqrt{105}X_{4z}^{\beta(1)} & \sqrt{35}X_{4u}^{(1)} + \sqrt{105}X_{4v}^{(1)} - 7X_4^{(1)} & \sqrt{105}X_{4x}^{\beta(1)} \\ -\sqrt{35}X_{4u}^{(1)} + \sqrt{105}X_{4v}^{(1)} + 7X_4^{(1)} & -\sqrt{105}X_{4z}^{\beta(1)} & -\sqrt{105}X_{4y}^{\beta(1)} \\ -\frac{7\sqrt{15}X_{4x}^{\alpha(1)}}{2} + \frac{3\sqrt{105}X_{4x}^{\beta(1)}}{2} & -\frac{7\sqrt{15}X_{4y}^{\alpha(1)}}{2} - \frac{3\sqrt{105}X_{4y}^{\beta(1)}}{2} & 7\sqrt{15}X_{4z}^{\alpha(1)} \end{pmatrix}, \quad (B34)$$

$$\sigma_2^{(O/T)} =$$

$$\begin{pmatrix} -18\sqrt{5}X_{4u}^{(2)} - 6\sqrt{15}X_{4v}^{(2)} & 3\sqrt{105}X_{4z}^{\alpha(2)} - 3\sqrt{15}X_{4z}^{\beta(2)} & 3\sqrt{105}X_{4y}^{\alpha(2)} + 3\sqrt{15}X_{4y}^{\beta(2)} \\ 8\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} + 10\sqrt{7}X_4^{(2)} & -3\sqrt{105}X_{4z}^{\alpha(2)} + 5\sqrt{15}X_{4z}^{\beta(2)} & 2\sqrt{15}X_{4y}^{\beta(2)} \\ 10\sqrt{5}X_{4u}^{(2)} + 2\sqrt{15}X_{4v}^{(2)} - 10\sqrt{7}X_4^{(2)} & -2\sqrt{15}X_{4z}^{\beta(2)} & -3\sqrt{105}X_{4y}^{\alpha(2)} - 5\sqrt{15}X_{4y}^{\beta(2)} \\ 2\sqrt{105}X_{4x}^{\alpha(2)} & 2\sqrt{105}X_{4y}^{\alpha(2)} & 2\sqrt{105}X_{4z}^{\alpha(2)} \\ -3\sqrt{105}X_{4y}^{\alpha(2)} + 5\sqrt{15}X_{4y}^{\beta(2)} & 2\sqrt{15}X_{4x}^{\beta(2)} & 2\sqrt{5}X_{4u}^{(2)} - 6\sqrt{15}X_{4v}^{(2)} + 10\sqrt{7}X_4^{(2)} \\ -3\sqrt{105}X_{4z}^{\alpha(2)} - 5\sqrt{15}X_{4z}^{\beta(2)} & -8\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} - 10\sqrt{7}X_4^{(2)} & -2\sqrt{15}X_{4x}^{\beta(2)} \\ -3\sqrt{105}X_{4z}^{\alpha(2)} - 5\sqrt{15}X_{4z}^{\beta(2)} & -8\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} - 10\sqrt{7}X_4^{(2)} & -2\sqrt{15}X_{4x}^{\beta(2)} \\ 3\sqrt{105}X_{4z}^{\alpha(2)} + 3\sqrt{15}X_{4z}^{\beta(2)} & 18\sqrt{5}X_{4u}^{(2)} - 6\sqrt{15}X_{4v}^{(2)} & 3\sqrt{105}X_{4x}^{\alpha(2)} - 3\sqrt{15}X_{4x}^{\beta(2)} \\ 2\sqrt{15}X_{4z}^{\beta(2)} & -10\sqrt{5}X_{4u}^{(2)} + 2\sqrt{15}X_{4v}^{(2)} + 10\sqrt{7}X_4^{(2)} & -3\sqrt{105}X_{4x}^{\alpha(2)} + 5\sqrt{15}X_{4x}^{\beta(2)} \\ -2\sqrt{15}X_{4y}^{\beta(2)} & -3\sqrt{105}X_{4x}^{\alpha(2)} - 5\sqrt{15}X_{4x}^{\beta(2)} & -2\sqrt{5}X_{4u}^{(2)} - 6\sqrt{15}X_{4v}^{(2)} - 10\sqrt{7}X_4^{(2)} \\ 2\sqrt{105}X_{4x}^{\alpha(2)} & 2\sqrt{105}X_{4y}^{\alpha(2)} & 2\sqrt{105}X_{4z}^{\alpha(2)} \\ 8\sqrt{5}X_{4u}^{(2)} + 4\sqrt{15}X_{4v}^{(2)} + 10\sqrt{7}X_4^{(2)} & -3\sqrt{105}X_{4z}^{\alpha(2)} + 5\sqrt{15}X_{4z}^{\beta(2)} & 2\sqrt{15}X_{4y}^{\beta(2)} \\ -3\sqrt{105}X_{4y}^{\alpha(2)} + 5\sqrt{15}X_{4y}^{\beta(2)} & 2\sqrt{15}X_{4x}^{\beta(2)} & 2\sqrt{5}X_{4u}^{(2)} - 6\sqrt{15}X_{4v}^{(2)} + 10\sqrt{7}X_4^{(2)} \\ -2\sqrt{15}X_{4y}^{\beta(2)} & -3\sqrt{105}X_{4x}^{\alpha(2)} - 5\sqrt{15}X_{4x}^{\beta(2)} & -2\sqrt{5}X_{4u}^{(2)} - 6\sqrt{15}X_{4v}^{(2)} - 10\sqrt{7}X_4^{(2)} \\ 3\sqrt{105}X_{4y}^{\alpha(2)} - 3\sqrt{15}X_{4y}^{\beta(2)} & 3\sqrt{105}X_{4x}^{\alpha(2)} + 3\sqrt{15}X_{4x}^{\beta(2)} & 12\sqrt{15}X_{4v}^{(2)} \\ 2\sqrt{15}X_{4z}^{\beta(2)} & -10\sqrt{5}X_{4u}^{(2)} + 2\sqrt{15}X_{4v}^{(2)} + 10\sqrt{7}X_4^{(2)} & -3\sqrt{105}X_{4x}^{\alpha(2)} + 5\sqrt{15}X_{4x}^{\beta(2)} \\ 10\sqrt{5}X_{4u}^{(2)} + 2\sqrt{15}X_{4v}^{(2)} - 10\sqrt{7}X_4^{(2)} & -2\sqrt{15}X_{4z}^{\beta(2)} & -3\sqrt{105}X_{4y}^{\alpha(2)} - 5\sqrt{15}X_{4y}^{\beta(2)} \\ 2\sqrt{105}X_{4x}^{\alpha(2)} & 2\sqrt{105}X_{4y}^{\alpha(2)} & 2\sqrt{105}X_{4z}^{\alpha(2)} \end{pmatrix}. \quad (B35)$$

$$\sigma_{ij}^{mn\alpha(H)} = \begin{pmatrix} 10\sqrt{3}X_{4u} - 30X_{4v} + 4\sqrt{105}X_4 & 15\sqrt{7}X_{4z}^\alpha - 15X_{4z}^\beta & -15\sqrt{7}X_{4y}^\alpha - 15X_{4y}^\beta \\ -20\sqrt{3}X_{4u} - 2\sqrt{105}X_4 & -15\sqrt{7}X_{4z}^\alpha - 15X_{4z}^\beta & 30X_{4y}^\beta \\ 10\sqrt{3}X_{4u} + 30X_{4v} - 2\sqrt{105}X_4 & 30X_{4z}^\beta & 15\sqrt{7}X_{4y}^\alpha - 15X_{4y}^\beta \\ 30X_{4x}^\beta & 30X_{4y}^\beta & 30X_{4z}^\beta \\ -15\sqrt{7}X_{4y}^\alpha - 15X_{4y}^\beta & 30X_{4x}^\beta & 10\sqrt{3}X_{4u} + 30X_{4v} - 2\sqrt{105}X_4 \\ 15\sqrt{7}X_{4z}^\alpha - 15X_{4z}^\beta & -20\sqrt{3}X_{4u} - 2\sqrt{105}X_4 & 30X_{4x}^\beta \\ 15\sqrt{7}X_{4z}^\alpha - 15X_{4z}^\beta & -20\sqrt{3}X_{4u} - 2\sqrt{105}X_4 & 30X_{4x}^\beta \\ -15\sqrt{7}X_{4z}^\alpha - 15X_{4z}^\beta & 10\sqrt{3}X_{4u} + 30X_{4v} + 4\sqrt{105}X_4 & 15\sqrt{7}X_{4x}^\alpha - 15X_{4x}^\beta \\ 30X_{4z}^\beta & 10\sqrt{3}X_{4u} - 30X_{4v} - 2\sqrt{105}X_4 & -15\sqrt{7}X_{4x}^\alpha - 15X_{4x}^\beta \\ 30X_{4y}^\beta & 15\sqrt{7}X_{4x}^\alpha - 15X_{4x}^\beta & 10\sqrt{3}X_{4u} - 30X_{4v} - 2\sqrt{105}X_4 \\ 30X_{4x}^\beta & 30X_{4y}^\beta & 30X_{4z}^\beta \\ -20\sqrt{3}X_{4u} - 2\sqrt{105}X_4 & -15\sqrt{7}X_{4z}^\alpha - 15X_{4z}^\beta & 30X_{4y}^\beta \\ -15\sqrt{7}X_{4y}^\alpha - 15X_{4y}^\beta & 30X_{4x}^\beta & 10\sqrt{3}X_{4u} + 30X_{4v} - 2\sqrt{105}X_4 \\ 30X_{4y}^\beta & 15\sqrt{7}X_{4x}^\alpha - 15X_{4x}^\beta & 10\sqrt{3}X_{4u} - 30X_{4v} - 2\sqrt{105}X_4 \\ 15\sqrt{7}X_{4y}^\alpha - 15X_{4y}^\beta & -15\sqrt{7}X_{4x}^\alpha - 15X_{4x}^\beta & -20\sqrt{3}X_{4u} + 4\sqrt{105}X_4 \\ 30X_{4z}^\beta & 10\sqrt{3}X_{4u} - 30X_{4v} - 2\sqrt{105}X_4 & -15\sqrt{7}X_{4x}^\alpha - 15X_{4x}^\beta \\ 10\sqrt{3}X_{4u} + 30X_{4v} - 2\sqrt{105}X_4 & 30X_{4z}^\beta & 15\sqrt{7}X_{4y}^\alpha - 15X_{4y}^\beta \\ 30X_{4x}^\beta & 30X_{4y}^\beta & 30X_{4z}^\beta \end{pmatrix}. \quad (B36)$$

6. rank-5 c-multipoles

$$\sigma_{ij}^{mn\alpha(O/L)} = \begin{pmatrix} 40\sqrt{2}Y_{5x}^{\alpha 1} & -20\sqrt{2}Y_{5x}^{\alpha 1} + 2\sqrt{210}Y_{5x}^{\beta} & -20\sqrt{2}Y_{5x}^{\alpha 1} - 2\sqrt{210}Y_{5x}^{\beta} \\ -20\sqrt{2}Y_{5x}^{\alpha 1} + 2\sqrt{210}Y_{5x}^{\beta} & 15\sqrt{2}Y_{5x}^{\alpha 1} + 3\sqrt{70}Y_{5x}^{\alpha 2} - 2\sqrt{210}Y_{5x}^{\beta} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} \\ -20\sqrt{2}Y_{5x}^{\alpha 1} - 2\sqrt{210}Y_{5x}^{\beta} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & 15\sqrt{2}Y_{5x}^{\alpha 1} + 3\sqrt{70}Y_{5x}^{\alpha 2} + 2\sqrt{210}Y_{5x}^{\beta} \\ 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 2\sqrt{210}Y_{5v} \\ 15\sqrt{2}Y_{5z}^{\alpha 1} + 3\sqrt{70}Y_{5z}^{\alpha 2} - 2\sqrt{210}Y_{5z}^{\beta} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & -20\sqrt{2}Y_{5z}^{\alpha 1} + 2\sqrt{210}Y_{5z}^{\beta} \\ 15\sqrt{2}Y_{5y}^{\alpha 1} + 3\sqrt{70}Y_{5y}^{\alpha 2} + 2\sqrt{210}Y_{5y}^{\beta} & -20\sqrt{2}Y_{5y}^{\alpha 1} - 2\sqrt{210}Y_{5y}^{\beta} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} \\ 15\sqrt{2}Y_{5y}^{\alpha 1} + 3\sqrt{70}Y_{5y}^{\alpha 2} + 2\sqrt{210}Y_{5y}^{\beta} & -20\sqrt{2}Y_{5y}^{\alpha 1} - 2\sqrt{210}Y_{5y}^{\beta} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} \\ -20\sqrt{2}Y_{5y}^{\alpha 1} - 2\sqrt{210}Y_{5y}^{\beta} & 40\sqrt{2}Y_{5y}^{\alpha 1} & -20\sqrt{2}Y_{5y}^{\alpha 1} + 2\sqrt{210}Y_{5y}^{\beta} \\ 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & -20\sqrt{2}Y_{5y}^{\alpha 1} + 2\sqrt{210}Y_{5y}^{\beta} & 15\sqrt{2}Y_{5y}^{\alpha 1} + 3\sqrt{70}Y_{5y}^{\alpha 2} - 2\sqrt{210}Y_{5y}^{\beta} \\ 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & 15\sqrt{2}Y_{5z}^{\alpha 1} + 3\sqrt{70}Y_{5z}^{\alpha 2} + 2\sqrt{210}Y_{5z}^{\beta} & -20\sqrt{2}Y_{5z}^{\alpha 1} - 2\sqrt{210}Y_{5z}^{\beta} \\ 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 2\sqrt{210}Y_{5v} \\ -20\sqrt{2}Y_{5x}^{\alpha 1} + 2\sqrt{210}Y_{5x}^{\beta} & 15\sqrt{2}Y_{5x}^{\alpha 1} + 3\sqrt{70}Y_{5x}^{\alpha 2} - 2\sqrt{210}Y_{5x}^{\beta} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} \\ 15\sqrt{2}Y_{5z}^{\alpha 1} + 3\sqrt{70}Y_{5z}^{\alpha 2} - 2\sqrt{210}Y_{5z}^{\beta} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & -20\sqrt{2}Y_{5z}^{\alpha 1} + 2\sqrt{210}Y_{5z}^{\beta} \\ 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & 15\sqrt{2}Y_{5z}^{\alpha 1} + 3\sqrt{70}Y_{5z}^{\alpha 2} + 2\sqrt{210}Y_{5z}^{\beta} & -20\sqrt{2}Y_{5z}^{\alpha 1} - 2\sqrt{210}Y_{5z}^{\beta} \\ -20\sqrt{2}Y_{5y}^{\alpha 1} + 2\sqrt{210}Y_{5y}^{\beta} & -20\sqrt{2}Y_{5y}^{\alpha 1} - 2\sqrt{210}Y_{5y}^{\beta} & 40\sqrt{2}Y_{5z}^{\alpha 1} \\ 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & -20\sqrt{2}Y_{5y}^{\alpha 1} + 2\sqrt{210}Y_{5y}^{\beta} & 15\sqrt{2}Y_{5y}^{\alpha 1} + 3\sqrt{70}Y_{5y}^{\alpha 2} - 2\sqrt{210}Y_{5y}^{\beta} \\ -20\sqrt{2}Y_{5x}^{\alpha 1} - 2\sqrt{210}Y_{5x}^{\beta} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & 15\sqrt{2}Y_{5x}^{\alpha 1} + 3\sqrt{70}Y_{5x}^{\alpha 2} + 2\sqrt{210}Y_{5x}^{\beta} \\ 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 2\sqrt{210}Y_{5v} \end{pmatrix}. \quad (B37)$$

$\sigma_{ij}^{mn\alpha(O/T)}$

$$\left(\begin{array}{ccc}
 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 15\sqrt{2}Y_{5z}^{\alpha 1} + 3\sqrt{70}Y_{5z}^{\alpha 2} - 2\sqrt{210}Y_{5z}^{\beta} & 15\sqrt{2}Y_{5y}^{\alpha 1} + 3\sqrt{70}Y_{5y}^{\alpha 2} + 2\sqrt{210}Y_{5y}^{\beta} \\
 -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & -20\sqrt{2}Y_{5y}^{\alpha 1} - 2\sqrt{210}Y_{5y}^{\beta} \\
 2\sqrt{210}Y_{5v} & -20\sqrt{2}Y_{5z}^{\alpha 1} + 2\sqrt{210}Y_{5z}^{\beta} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} \\
 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} \\
 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & -20\sqrt{2}Y_{5x}^{\alpha 1} - 2\sqrt{210}Y_{5x}^{\beta} & 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} \\
 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & -20\sqrt{2}Y_{5x}^{\alpha 1} + 2\sqrt{210}Y_{5x}^{\beta} \\
 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & -20\sqrt{2}Y_{5x}^{\alpha 1} + 2\sqrt{210}Y_{5x}^{\beta} \\
 15\sqrt{2}Y_{5z}^{\alpha 1} + 3\sqrt{70}Y_{5z}^{\alpha 2} + 2\sqrt{210}Y_{5z}^{\beta} & -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 15\sqrt{2}Y_{5x}^{\alpha 1} + 3\sqrt{70}Y_{5x}^{\alpha 2} - 2\sqrt{210}Y_{5x}^{\beta} \\
 -20\sqrt{2}Y_{5z}^{\alpha 1} - 2\sqrt{210}Y_{5z}^{\beta} & 2\sqrt{210}Y_{5v} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} \\
 -20\sqrt{2}Y_{5y}^{\alpha 1} + 2\sqrt{210}Y_{5y}^{\beta} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} \\
 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} \\
 -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2} & -20\sqrt{2}Y_{5y}^{\alpha 1} - 2\sqrt{210}Y_{5y}^{\beta} \\
 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & -20\sqrt{2}Y_{5x}^{\alpha 1} - 2\sqrt{210}Y_{5x}^{\beta} & 3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} \\
 -20\sqrt{2}Y_{5y}^{\alpha 1} + 2\sqrt{210}Y_{5y}^{\beta} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & -3\sqrt{70}Y_{5u} - \sqrt{210}Y_{5v} \\
 15\sqrt{2}Y_{5y}^{\alpha 1} + 3\sqrt{70}Y_{5y}^{\alpha 2} - 2\sqrt{210}Y_{5y}^{\beta} & 15\sqrt{2}Y_{5x}^{\alpha 1} + 3\sqrt{70}Y_{5x}^{\alpha 2} + 2\sqrt{210}Y_{5x}^{\beta} & 2\sqrt{210}Y_{5v} \\
 -20\sqrt{2}Y_{5z}^{\alpha 1} - 2\sqrt{210}Y_{5z}^{\beta} & 2\sqrt{210}Y_{5v} & 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} \\
 2\sqrt{210}Y_{5v} & -20\sqrt{2}Y_{5z}^{\alpha 1} + 2\sqrt{210}Y_{5z}^{\beta} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} \\
 5\sqrt{2}Y_{5x}^{\alpha 1} - 3\sqrt{70}Y_{5x}^{\alpha 2} & 5\sqrt{2}Y_{5y}^{\alpha 1} - 3\sqrt{70}Y_{5y}^{\alpha 2} & 5\sqrt{2}Y_{5z}^{\alpha 1} - 3\sqrt{70}Y_{5z}^{\alpha 2}
 \end{array} \right) \quad (B38)$$

7. rank-3 t -multipoles

$$\sigma_{ij}^{mna(O/L)} = \sigma_1^{(O/L)} + \sigma_2^{(O/L)} + \sigma_3^{(O/L)}, \quad (B39)$$

$$\sigma_1^{(O/L)} = \begin{pmatrix} -15Y_{3a}^{(1)} - 8\sqrt{70}Y_{3a}^{(2)} + 3\sqrt{15}Y_{3u}^{(1)} + 8\sqrt{42}Y_{3u}^{(2)} & 15Y_{3a}^{(1)} - 2\sqrt{70}Y_{3a}^{(2)} + \sqrt{15}Y_{3u}^{(1)} - 14\sqrt{42}Y_{3u}^{(2)} & 10\sqrt{70}Y_{3a}^{(2)} - 4\sqrt{15}Y_{3u}^{(1)} + 6\sqrt{42}Y_{3u}^{(2)} \\ 4\sqrt{70}Y_{3a}^{(2)} - 4\sqrt{42}Y_{3u}^{(2)} & 6\sqrt{70}Y_{3a}^{(2)} + 2\sqrt{42}Y_{3u}^{(2)} & -10\sqrt{70}Y_{3a}^{(2)} + 2\sqrt{42}Y_{3u}^{(2)} \\ 4\sqrt{70}Y_{3a}^{(2)} - 4\sqrt{42}Y_{3u}^{(2)} & -4\sqrt{70}Y_{3a}^{(2)} + 12\sqrt{42}Y_{3u}^{(2)} & -8\sqrt{42}Y_{3u}^{(2)} \\ 0 & 0 & 0 \\ \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} + 12\sqrt{7}Y_z^{\alpha(2)} - \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} - 4\sqrt{105}Y_z^{\beta(2)} & \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} - 28\sqrt{7}Y_z^{\alpha(2)} + \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} + 4\sqrt{105}Y_z^{\beta(2)} & -3\sqrt{10}Y_z^{\alpha(1)} + 16\sqrt{7}Y_z^{\alpha(2)} \\ -\frac{15Y_{3b}^{(1)}}{2} - 6\sqrt{70}Y_{3b}^{(2)} + \frac{\sqrt{15}Y_{3v}^{(1)}}{2} + 2\sqrt{42}Y_{3v}^{(2)} & -\frac{15Y_{3b}^{(1)}}{2} - 4\sqrt{70}Y_{3b}^{(2)} + \frac{3\sqrt{15}Y_{3v}^{(1)}}{2} - 4\sqrt{42}Y_{3v}^{(2)} & 10\sqrt{70}Y_{3b}^{(2)} - 2\sqrt{15}Y_{3v}^{(1)} + 2\sqrt{42}Y_{3v}^{(2)} \\ -6\sqrt{70}Y_{3b}^{(2)} + 2\sqrt{42}Y_{3v}^{(2)} & -4\sqrt{70}Y_{3b}^{(2)} - 4\sqrt{42}Y_{3v}^{(2)} & 10\sqrt{70}Y_{3b}^{(2)} + 2\sqrt{42}Y_{3v}^{(2)} \\ -15Y_{3b}^{(1)} + 2\sqrt{70}Y_{3b}^{(2)} + \sqrt{15}Y_{3v}^{(1)} - 14\sqrt{42}Y_{3v}^{(2)} & 15Y_{3b}^{(1)} + 8\sqrt{70}Y_{3b}^{(2)} + 3\sqrt{15}Y_{3v}^{(1)} + 8\sqrt{42}Y_{3v}^{(2)} & -10\sqrt{70}Y_{3b}^{(2)} - 4\sqrt{15}Y_{3v}^{(1)} + 6\sqrt{42}Y_{3v}^{(2)} \\ 4\sqrt{70}Y_{3b}^{(2)} + 12\sqrt{42}Y_{3v}^{(2)} & -4\sqrt{70}Y_{3b}^{(2)} - 4\sqrt{42}Y_{3v}^{(2)} & -8\sqrt{42}Y_{3v}^{(2)} \\ \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} - 28\sqrt{7}Y_z^{\alpha(2)} - \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} - 4\sqrt{105}Y_z^{\beta(2)} & \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} + 12\sqrt{7}Y_z^{\alpha(2)} + \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} + 4\sqrt{105}Y_z^{\beta(2)} & -3\sqrt{10}Y_z^{\alpha(1)} + 16\sqrt{7}Y_z^{\alpha(2)} \\ 0 & 0 & 0 \\ -\frac{15Y_{3a}^{(1)}}{2} + 4\sqrt{70}Y_{3a}^{(2)} + \frac{3\sqrt{15}Y_{3u}^{(1)}}{2} - 4\sqrt{42}Y_{3u}^{(2)} & \frac{15Y_{3a}^{(1)}}{2} + 6\sqrt{70}Y_{3a}^{(2)} + \frac{\sqrt{15}Y_{3u}^{(1)}}{2} + 2\sqrt{42}Y_{3u}^{(2)} & -10\sqrt{70}Y_{3a}^{(2)} - 2\sqrt{15}Y_{3u}^{(1)} + 2\sqrt{42}Y_{3u}^{(2)} \\ 12\sqrt{7}Y_z^{\alpha(2)} - 4\sqrt{105}Y_z^{\beta(2)} & -28\sqrt{7}Y_z^{\alpha(2)} + 4\sqrt{105}Y_z^{\beta(2)} & 16\sqrt{7}Y_z^{\alpha(2)} \\ -28\sqrt{7}Y_z^{\alpha(2)} - 4\sqrt{105}Y_z^{\beta(2)} & 12\sqrt{7}Y_z^{\alpha(2)} + 4\sqrt{105}Y_z^{\beta(2)} & 16\sqrt{7}Y_z^{\alpha(2)} \\ 3\sqrt{10}Y_z^{\alpha(1)} + 16\sqrt{7}Y_z^{\alpha(2)} - 5\sqrt{6}Y_z^{\beta(1)} + 8\sqrt{105}Y_z^{\beta(2)} & 3\sqrt{10}Y_z^{\alpha(1)} + 16\sqrt{7}Y_z^{\alpha(2)} + 5\sqrt{6}Y_z^{\beta(1)} - 8\sqrt{105}Y_z^{\beta(2)} & -6\sqrt{10}Y_z^{\alpha(1)} - 32\sqrt{7}Y_z^{\alpha(2)} \\ -\frac{15Y_{3b}^{(1)}}{2} + 4\sqrt{70}Y_{3b}^{(2)} + \frac{\sqrt{15}Y_{3v}^{(1)}}{2} + 12\sqrt{42}Y_{3v}^{(2)} & -\frac{15Y_{3b}^{(1)}}{2} - 4\sqrt{70}Y_{3b}^{(2)} + \frac{3\sqrt{15}Y_{3v}^{(1)}}{2} - 4\sqrt{42}Y_{3v}^{(2)} & -2\sqrt{15}Y_{3v}^{(1)} - 8\sqrt{42}Y_{3v}^{(2)} \\ -\frac{15Y_{3a}^{(1)}}{2} + 4\sqrt{70}Y_{3a}^{(2)} + \frac{3\sqrt{15}Y_{3u}^{(1)}}{2} - 4\sqrt{42}Y_{3u}^{(2)} & \frac{15Y_{3a}^{(1)}}{2} - 4\sqrt{70}Y_{3a}^{(2)} + \frac{\sqrt{15}Y_{3u}^{(1)}}{2} + 12\sqrt{42}Y_{3u}^{(2)} & -2\sqrt{15}Y_{3u}^{(1)} - 8\sqrt{42}Y_{3u}^{(2)} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B40)$$

$$\sigma_2^{(O/L)} = \begin{pmatrix} -12\sqrt{5}Y_{3a}^{(4)} + 12\sqrt{3}Y_{3u}^{(4)} & 12\sqrt{5}Y_{3a}^{(4)} + 4\sqrt{3}Y_{3u}^{(4)} & -16\sqrt{3}Y_{3u}^{(4)} \\ \sqrt{30}Y_{3a}^{(3)} - 9\sqrt{5}Y_{3a}^{(4)} + 5\sqrt{2}Y_{3u}^{(3)} + 9\sqrt{3}Y_{3u}^{(4)} & -\sqrt{30}Y_{3a}^{(3)} + 9\sqrt{5}Y_{3a}^{(4)} - \sqrt{2}Y_{3u}^{(3)} + 3\sqrt{3}Y_{3u}^{(4)} & -4\sqrt{2}Y_{3u}^{(3)} - 12\sqrt{3}Y_{3u}^{(4)} \\ -\sqrt{30}Y_{3a}^{(3)} - 9\sqrt{5}Y_{3a}^{(4)} - 5\sqrt{2}Y_{3u}^{(3)} + 9\sqrt{3}Y_{3u}^{(4)} & \sqrt{30}Y_{3a}^{(3)} + 9\sqrt{5}Y_{3a}^{(4)} + \sqrt{2}Y_{3u}^{(3)} + 3\sqrt{3}Y_{3u}^{(4)} & 4\sqrt{2}Y_{3u}^{(3)} - 12\sqrt{3}Y_{3u}^{(4)} \\ -4\sqrt{5}Y_{xyz}^{(3)} & 2\sqrt{5}Y_{xyz}^{(3)} & 2\sqrt{5}Y_{xyz}^{(3)} \\ \sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} - \sqrt{5}Y_z^{\beta(3)} - \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} - \sqrt{5}Y_z^{\beta(3)} + \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -3\sqrt{2}Y_z^{\alpha(4)} + 2\sqrt{5}Y_z^{\beta(3)} \\ -\frac{\sqrt{30}Y_{3b}^{(3)}}{2} - \frac{3\sqrt{5}Y_{3b}^{(4)}}{2} + \frac{\sqrt{2}Y_{3v}^{(3)}}{2} + \frac{\sqrt{3}Y_{3v}^{(4)}}{2} & \frac{\sqrt{30}Y_{3b}^{(3)}}{2} + \frac{3\sqrt{5}Y_{3b}^{(4)}}{2} - \frac{5\sqrt{2}Y_{3v}^{(3)}}{2} + \frac{3\sqrt{3}Y_{3v}^{(4)}}{2} & 2\sqrt{2}Y_{3v}^{(3)} - 2\sqrt{3}Y_{3v}^{(4)} \\ \sqrt{30}Y_{3b}^{(3)} - 9\sqrt{5}Y_{3b}^{(4)} - \sqrt{2}Y_{3v}^{(3)} + 3\sqrt{3}Y_{3v}^{(4)} & -\sqrt{30}Y_{3b}^{(3)} + 9\sqrt{5}Y_{3b}^{(4)} + 5\sqrt{2}Y_{3v}^{(3)} + 9\sqrt{3}Y_{3v}^{(4)} & -4\sqrt{2}Y_{3v}^{(3)} - 12\sqrt{3}Y_{3v}^{(4)} \\ -12\sqrt{5}Y_{3b}^{(4)} + 4\sqrt{3}Y_{3v}^{(4)} & 12\sqrt{5}Y_{3b}^{(4)} + 12\sqrt{3}Y_{3v}^{(4)} & -16\sqrt{3}Y_{3v}^{(4)} \\ -\sqrt{30}Y_{3b}^{(3)} - 9\sqrt{5}Y_{3b}^{(4)} + \sqrt{2}Y_{3v}^{(3)} + 3\sqrt{3}Y_{3v}^{(4)} & \sqrt{30}Y_{3b}^{(3)} + 9\sqrt{5}Y_{3b}^{(4)} - 5\sqrt{2}Y_{3v}^{(3)} + 9\sqrt{3}Y_{3v}^{(4)} & 4\sqrt{2}Y_{3v}^{(3)} - 12\sqrt{3}Y_{3v}^{(4)} \\ -\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} + \sqrt{5}Y_z^{\beta(3)} - \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & \sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} + \sqrt{5}Y_z^{\beta(3)} + \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -3\sqrt{2}Y_z^{\alpha(4)} - 2\sqrt{5}Y_z^{\beta(3)} \\ 2\sqrt{5}Y_{xyz}^{(3)} & -4\sqrt{5}Y_{xyz}^{(3)} & 2\sqrt{5}Y_{xyz}^{(3)} \\ -\frac{\sqrt{30}Y_{3a}^{(3)}}{2} - \frac{3\sqrt{5}Y_{3a}^{(4)}}{2} - \frac{5\sqrt{2}Y_{3u}^{(3)}}{2} + \frac{3\sqrt{3}Y_{3u}^{(4)}}{2} & \frac{\sqrt{30}Y_{3a}^{(3)}}{2} + \frac{3\sqrt{5}Y_{3a}^{(4)}}{2} + \frac{\sqrt{2}Y_{3u}^{(3)}}{2} + \frac{\sqrt{3}Y_{3u}^{(4)}}{2} & 2\sqrt{2}Y_{3u}^{(3)} - 2\sqrt{3}Y_{3u}^{(4)} \\ -2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} + 2\sqrt{5}Y_z^{\beta(3)} - 3\sqrt{30}Y_z^{\beta(4)} & 2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} + 2\sqrt{5}Y_z^{\beta(3)} + 3\sqrt{30}Y_z^{\beta(4)} & -18\sqrt{2}Y_z^{\alpha(4)} - 4\sqrt{5}Y_z^{\beta(3)} \\ 2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} - 2\sqrt{5}Y_z^{\beta(3)} - 3\sqrt{30}Y_z^{\beta(4)} & -2\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} - 2\sqrt{5}Y_z^{\beta(3)} + 3\sqrt{30}Y_z^{\beta(4)} & -18\sqrt{2}Y_z^{\alpha(4)} + 4\sqrt{5}Y_z^{\beta(3)} \\ \frac{\sqrt{30}Y_{3b}^{(3)}}{2} - \frac{3\sqrt{5}Y_{3b}^{(4)}}{2} - \frac{\sqrt{2}Y_{3v}^{(3)}}{2} + \frac{\sqrt{3}Y_{3v}^{(4)}}{2} & -\frac{\sqrt{30}Y_{3b}^{(3)}}{2} + \frac{3\sqrt{5}Y_{3b}^{(4)}}{2} + \frac{5\sqrt{2}Y_{3v}^{(3)}}{2} + \frac{3\sqrt{3}Y_{3v}^{(4)}}{2} & -2\sqrt{2}Y_{3v}^{(3)} - 2\sqrt{3}Y_{3v}^{(4)} \\ \frac{\sqrt{30}Y_{3a}^{(3)}}{2} - \frac{3\sqrt{5}Y_{3a}^{(4)}}{2} + \frac{5\sqrt{2}Y_{3u}^{(3)}}{2} + \frac{3\sqrt{3}Y_{3u}^{(4)}}{2} & -\frac{\sqrt{30}Y_{3a}^{(3)}}{2} + \frac{3\sqrt{5}Y_{3a}^{(4)}}{2} - \frac{\sqrt{2}Y_{3u}^{(3)}}{2} + \frac{\sqrt{3}Y_{3u}^{(4)}}{2} & -2\sqrt{2}Y_{3u}^{(3)} - 2\sqrt{3}Y_{3u}^{(4)} \\ 2\sqrt{5}Y_{xyz}^{(3)} & 2\sqrt{5}Y_{xyz}^{(3)} & -4\sqrt{5}Y_{xyz}^{(3)} \end{pmatrix}, \quad (B41)$$

$$\sigma_3^{(O/L)} = \begin{pmatrix} -5\sqrt{21}Y_{3a}^{(5)} + 3\sqrt{35}Y_{3u}^{(5)} & -5\sqrt{21}Y_{3a}^{(5)} + 3\sqrt{35}Y_{3u}^{(5)} & -5\sqrt{21}Y_{3a}^{(5)} + 3\sqrt{35}Y_{3u}^{(5)} \\ 5\sqrt{21}Y_{3a}^{(5)} + \sqrt{35}Y_{3u}^{(5)} & 5\sqrt{21}Y_{3a}^{(5)} + \sqrt{35}Y_{3u}^{(5)} & 5\sqrt{21}Y_{3a}^{(5)} + \sqrt{35}Y_{3u}^{(5)} \\ -4\sqrt{35}Y_{3u}^{(5)} & -4\sqrt{35}Y_{3u}^{(5)} & -4\sqrt{35}Y_{3u}^{(5)} \\ -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} \\ \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} \\ -5\sqrt{21}Y_{3b}^{(5)} + \sqrt{35}Y_{3v}^{(5)} & -5\sqrt{21}Y_{3b}^{(5)} + \sqrt{35}Y_{3v}^{(5)} & -5\sqrt{21}Y_{3b}^{(5)} + \sqrt{35}Y_{3v}^{(5)} \\ -5\sqrt{21}Y_{3b}^{(5)} + \sqrt{35}Y_{3v}^{(5)} & -5\sqrt{21}Y_{3b}^{(5)} + \sqrt{35}Y_{3v}^{(5)} & -5\sqrt{21}Y_{3b}^{(5)} + \sqrt{35}Y_{3v}^{(5)} \\ 5\sqrt{21}Y_{3b}^{(5)} + 3\sqrt{35}Y_{3v}^{(5)} & 5\sqrt{21}Y_{3b}^{(5)} + 3\sqrt{35}Y_{3v}^{(5)} & 5\sqrt{21}Y_{3b}^{(5)} + 3\sqrt{35}Y_{3v}^{(5)} \\ -4\sqrt{35}Y_{3v}^{(5)} & -4\sqrt{35}Y_{3v}^{(5)} & -4\sqrt{35}Y_{3v}^{(5)} \\ \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} \\ -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} \\ 5\sqrt{21}Y_{3a}^{(5)} + \sqrt{35}Y_{3u}^{(5)} & 5\sqrt{21}Y_{3a}^{(5)} + \sqrt{35}Y_{3u}^{(5)} & 5\sqrt{21}Y_{3a}^{(5)} + \sqrt{35}Y_{3u}^{(5)} \\ \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} - 5\sqrt{14}Y_z^{\beta(5)} \\ \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} & \sqrt{210}Y_z^{\alpha(5)} + 5\sqrt{14}Y_z^{\beta(5)} \\ -2\sqrt{210}Y_z^{\alpha(5)} & -2\sqrt{210}Y_z^{\alpha(5)} & -2\sqrt{210}Y_z^{\alpha(5)} \\ -4\sqrt{35}Y_{3v}^{(5)} & -4\sqrt{35}Y_{3v}^{(5)} & -4\sqrt{35}Y_{3v}^{(5)} \\ -4\sqrt{35}Y_{3u}^{(5)} & -4\sqrt{35}Y_{3u}^{(5)} & -4\sqrt{35}Y_{3u}^{(5)} \\ -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} & -5\sqrt{14}Y_{xyz}^{(5)} \end{pmatrix}. \quad (B42)$$

$$\sigma_{ij}^{mn\alpha(O/L)} = \sigma_1^{(O/L)} + \sigma_2^{(O/L)} + \sigma_3^{(O/L)}, \quad (B43)$$

$$\sigma_1^{(O/L)} =$$

$$\begin{pmatrix} -5\sqrt{6}Y_{xyz}^{(1)} + 8\sqrt{105}Y_{xyz}^{(2)} & 3\sqrt{10}Y_z^{\alpha(1)} + 6\sqrt{7}Y_z^{\alpha(2)} - 5\sqrt{6}Y_z^{\beta(1)} - 2\sqrt{105}Y_z^{\beta(2)} & -15Y_{3b}^{(1)} - 3\sqrt{70}Y_{3b}^{(2)} + \sqrt{15}Y_{3v}^{(1)} + \sqrt{42}Y_{3v}^{(2)} \\ -4\sqrt{105}Y_{xyz}^{(2)} & 2\sqrt{7}Y_z^{\alpha(2)} + 6\sqrt{105}Y_z^{\beta(2)} & -\sqrt{70}Y_{3b}^{(2)} + 7\sqrt{42}Y_{3v}^{(2)} \\ -4\sqrt{105}Y_{xyz}^{(2)} & -8\sqrt{7}Y_z^{\alpha(2)} - 4\sqrt{105}Y_z^{\beta(2)} & 4\sqrt{70}Y_{3b}^{(2)} - 8\sqrt{42}Y_{3v}^{(2)} \\ 5\sqrt{70}Y_{3a}^{(2)} - 5\sqrt{42}Y_{3u}^{(2)} & -5\sqrt{70}Y_{3b}^{(2)} - 5\sqrt{42}Y_{3v}^{(2)} & 20\sqrt{7}Y_z^{\alpha(2)} \\ -5\sqrt{70}Y_{3b}^{(2)} - 2\sqrt{15}Y_{3v}^{(1)} + 7\sqrt{42}Y_{3v}^{(2)} & -5\sqrt{70}Y_{3a}^{(2)} - 2\sqrt{15}Y_{3u}^{(1)} - 3\sqrt{42}Y_{3u}^{(2)} & -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 4\sqrt{105}Y_{xyz}^{(2)} \\ \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} + 2\sqrt{7}Y_z^{\alpha(2)} + \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} - 6\sqrt{105}Y_z^{\beta(2)} & -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 4\sqrt{105}Y_{xyz}^{(2)} & \frac{15Y_{3a}^{(1)}}{2} + \sqrt{70}Y_{3a}^{(2)} + \frac{\sqrt{15}Y_{3u}^{(1)}}{2} + 7\sqrt{42}Y_{3u}^{(2)} \\ 2\sqrt{7}Y_z^{\alpha(2)} - 6\sqrt{105}Y_z^{\beta(2)} & -4\sqrt{105}Y_{xyz}^{(2)} & \sqrt{70}Y_{3a}^{(2)} + 7\sqrt{42}Y_{3u}^{(2)} \\ 3\sqrt{10}Y_z^{\alpha(1)} + 6\sqrt{7}Y_z^{\alpha(2)} + 5\sqrt{6}Y_z^{\beta(1)} + 2\sqrt{105}Y_z^{\beta(2)} & -5\sqrt{6}Y_{xyz}^{(1)} + 8\sqrt{105}Y_{xyz}^{(2)} & 15Y_{3a}^{(1)} + 3\sqrt{70}Y_{3a}^{(2)} + \sqrt{15}Y_{3u}^{(1)} + \sqrt{42}Y_{3u}^{(2)} \\ -8\sqrt{7}Y_z^{\alpha(2)} + 4\sqrt{105}Y_z^{\beta(2)} & -4\sqrt{105}Y_{xyz}^{(2)} & -4\sqrt{70}Y_{3a}^{(2)} - 8\sqrt{42}Y_{3u}^{(2)} \\ 5\sqrt{70}Y_{3b}^{(2)} - 2\sqrt{15}Y_{3v}^{(1)} - 3\sqrt{42}Y_{3v}^{(2)} & 5\sqrt{70}Y_{3a}^{(2)} - 2\sqrt{15}Y_{3u}^{(1)} + 7\sqrt{42}Y_{3u}^{(2)} & -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 4\sqrt{105}Y_{xyz}^{(2)} \\ 5\sqrt{70}Y_{3a}^{(2)} - 5\sqrt{42}Y_{3u}^{(2)} & -5\sqrt{70}Y_{3b}^{(2)} - 5\sqrt{42}Y_{3v}^{(2)} & 20\sqrt{7}Y_z^{\alpha(2)} \\ -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 4\sqrt{105}Y_{xyz}^{(2)} & \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} + 2\sqrt{7}Y_z^{\alpha(2)} - \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} + 6\sqrt{105}Y_z^{\beta(2)} & -\frac{15Y_{3b}^{(1)}}{2} - \sqrt{70}Y_{3b}^{(2)} + \frac{\sqrt{15}Y_{3v}^{(1)}}{2} + 7\sqrt{42}Y_{3v}^{(2)} \\ -5\sqrt{70}Y_{3b}^{(2)} + 7\sqrt{42}Y_{3v}^{(2)} & -5\sqrt{70}Y_{3a}^{(2)} - 3\sqrt{42}Y_{3u}^{(2)} & -4\sqrt{105}Y_{xyz}^{(2)} \\ 5\sqrt{70}Y_{3b}^{(2)} - 3\sqrt{42}Y_{3v}^{(2)} & 5\sqrt{70}Y_{3a}^{(2)} + 7\sqrt{42}Y_{3u}^{(2)} & -4\sqrt{105}Y_{xyz}^{(2)} \\ -4\sqrt{15}Y_{3v}^{(1)} - 4\sqrt{42}Y_{3v}^{(2)} & -4\sqrt{15}Y_{3u}^{(1)} - 4\sqrt{42}Y_{3u}^{(2)} & -5\sqrt{6}Y_{xyz}^{(1)} + 8\sqrt{105}Y_{xyz}^{(2)} \\ \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} - 8\sqrt{7}Y_z^{\alpha(2)} + \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} + 4\sqrt{105}Y_z^{\beta(2)} & -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 4\sqrt{105}Y_{xyz}^{(2)} & \frac{15Y_{3a}^{(1)}}{2} - 4\sqrt{70}Y_{3a}^{(2)} + \frac{\sqrt{15}Y_{3u}^{(1)}}{2} - 8\sqrt{42}Y_{3u}^{(2)} \\ -\frac{5\sqrt{6}Y_{xyz}^{(1)}}{2} - 4\sqrt{105}Y_{xyz}^{(2)} & \frac{3\sqrt{10}Y_z^{\alpha(1)}}{2} - 8\sqrt{7}Y_z^{\alpha(2)} - \frac{5\sqrt{6}Y_z^{\beta(1)}}{2} - 4\sqrt{105}Y_z^{\beta(2)} & -\frac{15Y_{3b}^{(1)}}{2} + 4\sqrt{70}Y_{3b}^{(2)} + \frac{\sqrt{15}Y_{3v}^{(1)}}{2} - 8\sqrt{42}Y_{3v}^{(2)} \\ 5\sqrt{70}Y_{3a}^{(2)} - 5\sqrt{42}Y_{3u}^{(2)} & -5\sqrt{70}Y_{3b}^{(2)} - 5\sqrt{42}Y_{3v}^{(2)} & 20\sqrt{7}Y_z^{\alpha(2)} \end{pmatrix}, \quad (B44)$$

$$\sigma_2^{(O/L)} =$$

$$\begin{pmatrix} -4\sqrt{30}Y_{xyz}^{(4)} & 12\sqrt{2}Y_z^{\alpha(4)} - 4\sqrt{30}Y_z^{\beta(4)} & -12\sqrt{5}Y_{3b}^{(4)} + 4\sqrt{3}Y_{3v}^{(4)} \\ -3\sqrt{30}Y_{xyz}^{(4)} & 4\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} - 3\sqrt{30}Y_z^{\beta(4)} & \sqrt{30}Y_{3b}^{(3)} - 9\sqrt{5}Y_{3b}^{(4)} + 3\sqrt{2}Y_{3v}^{(3)} + 3\sqrt{3}Y_{3v}^{(4)} \\ -3\sqrt{30}Y_{xyz}^{(4)} & -4\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} - 3\sqrt{30}Y_z^{\beta(4)} & -\sqrt{30}Y_{3b}^{(3)} - 9\sqrt{5}Y_{3b}^{(4)} - 3\sqrt{2}Y_{3v}^{(3)} + 3\sqrt{3}Y_{3v}^{(4)} \\ -\frac{\sqrt{30}Y_{3a}^{(3)}}{2} + \frac{3\sqrt{2}Y_{3u}^{(3)}}{2} & \frac{\sqrt{30}Y_{3b}^{(3)}}{2} - \frac{9\sqrt{2}Y_{3v}^{(3)}}{2} & \sqrt{3}Y_z^{\alpha(3)} + 3\sqrt{5}Y_z^{\beta(3)} \\ \frac{\sqrt{30}Y_{3b}^{(3)}}{2} + \frac{3\sqrt{2}Y_{3v}^{(3)}}{2} - 2\sqrt{3}Y_{3u}^{(4)} & \frac{\sqrt{30}Y_{3a}^{(3)}}{2} - \frac{3\sqrt{2}Y_{3u}^{(3)}}{2} - 2\sqrt{3}Y_{3u}^{(4)} & -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} \\ -2\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} & \frac{\sqrt{30}Y_{3a}^{(3)}}{2} + \frac{3\sqrt{5}Y_{3a}^{(4)}}{2} - \frac{3\sqrt{2}Y_{3u}^{(3)}}{2} + \frac{\sqrt{3}Y_{3u}^{(4)}}{2} \\ 4\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} + 3\sqrt{30}Y_z^{\beta(4)} & -3\sqrt{30}Y_{xyz}^{(4)} & -\sqrt{30}Y_{3a}^{(3)} + 9\sqrt{5}Y_{3a}^{(4)} + 3\sqrt{2}Y_{3u}^{(3)} + 3\sqrt{3}Y_{3u}^{(4)} \\ 12\sqrt{2}Y_z^{\alpha(4)} + 4\sqrt{30}Y_z^{\beta(4)} & -4\sqrt{30}Y_{xyz}^{(4)} & 12\sqrt{5}Y_{3a}^{(4)} + 4\sqrt{3}Y_{3u}^{(4)} \\ -4\sqrt{3}Y_z^{\alpha(3)} + 9\sqrt{2}Y_z^{\alpha(4)} + 3\sqrt{30}Y_z^{\beta(4)} & -3\sqrt{30}Y_{xyz}^{(4)} & \sqrt{30}Y_{3a}^{(3)} + 9\sqrt{5}Y_{3a}^{(4)} - 3\sqrt{2}Y_{3u}^{(3)} + 3\sqrt{3}Y_{3u}^{(4)} \\ -\frac{\sqrt{30}Y_{3b}^{(3)}}{2} - \frac{3\sqrt{2}Y_{3v}^{(3)}}{2} - 2\sqrt{3}Y_{3v}^{(4)} & -\frac{\sqrt{30}Y_{3a}^{(3)}}{2} + \frac{3\sqrt{2}Y_{3u}^{(3)}}{2} - 2\sqrt{3}Y_{3u}^{(4)} & -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} \\ -\frac{\sqrt{30}Y_{3a}^{(3)}}{2} - \frac{9\sqrt{2}Y_{3u}^{(3)}}{2} & \frac{\sqrt{30}Y_{3b}^{(3)}}{2} + \frac{3\sqrt{2}Y_{3v}^{(3)}}{2} & \sqrt{3}Y_z^{\alpha(3)} - 3\sqrt{5}Y_z^{\beta(3)} \\ -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} & -2\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{3b}^{(3)}}{2} - \frac{3\sqrt{5}Y_{3b}^{(4)}}{2} - \frac{3\sqrt{2}Y_{3v}^{(3)}}{2} + \frac{\sqrt{3}Y_{3v}^{(4)}}{2} \\ -\sqrt{30}Y_{3b}^{(3)} - 3\sqrt{2}Y_{3v}^{(3)} - 12\sqrt{3}Y_{3v}^{(4)} & -\sqrt{30}Y_{3a}^{(3)} + 3\sqrt{2}Y_{3u}^{(3)} - 12\sqrt{3}Y_{3u}^{(4)} & -3\sqrt{30}Y_{xyz}^{(4)} \\ \sqrt{30}Y_{3b}^{(3)} + 3\sqrt{2}Y_{3v}^{(3)} - 12\sqrt{3}Y_{3v}^{(4)} & \sqrt{30}Y_{3a}^{(3)} - 3\sqrt{2}Y_{3u}^{(3)} - 12\sqrt{3}Y_{3u}^{(4)} & -3\sqrt{30}Y_{xyz}^{(4)} \\ -16\sqrt{3}Y_{3v}^{(4)} & -16\sqrt{3}Y_{3u}^{(4)} & -4\sqrt{30}Y_{xyz}^{(4)} \\ 2\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} + \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} & -\frac{\sqrt{30}Y_{3a}^{(3)}}{2} + \frac{3\sqrt{5}Y_{3a}^{(4)}}{2} + \frac{3\sqrt{2}Y_{3u}^{(3)}}{2} + \frac{\sqrt{3}Y_{3u}^{(4)}}{2} \\ -\frac{\sqrt{30}Y_{xyz}^{(4)}}{2} & 2\sqrt{3}Y_z^{\alpha(3)} + \frac{3\sqrt{2}Y_z^{\alpha(4)}}{2} - \frac{\sqrt{30}Y_z^{\beta(4)}}{2} & \frac{\sqrt{30}Y_{3b}^{(3)}}{2} - \frac{3\sqrt{5}Y_{3b}^{(4)}}{2} + \frac{3\sqrt{2}Y_{3v}^{(3)}}{2} + \frac{\sqrt{3}Y_{3v}^{(4)}}{2} \\ \sqrt{30}Y_{3a}^{(3)} + 3\sqrt{2}Y_{3u}^{(3)} & -\sqrt{30}Y_{3b}^{(3)} + 3\sqrt{2}Y_{3v}^{(3)} & -2\sqrt{3}Y_z^{\alpha(3)} \end{pmatrix}, \quad (B45)$$

where the multipole component with superscript (5) of $\sigma_{ij}^{mn\alpha(\text{O/T})}$ does not appear.

$$\sigma_{ij}^{mn\alpha(\text{H})} = \left(\begin{array}{ccc} 0 & -6\sqrt{105}Y_z^{\alpha(2)} + 30\sqrt{7}Y_z^{\beta(2)} & -15\sqrt{42}Y_{3b}^{(2)} + 3\sqrt{70}Y_{3v}^{(2)} \\ \sqrt{10}Y_{xyz}^{(1)} - 20\sqrt{7}Y_{xyz}^{(2)} & -\sqrt{6}Y_z^{\alpha(1)} - 2\sqrt{105}Y_z^{\alpha(2)} - \sqrt{10}Y_z^{\beta(1)} - 10\sqrt{7}Y_z^{\beta(2)} & 15\sqrt{42}Y_{3b}^{(2)} + 4Y_{3v}^{(1)} + \sqrt{70}Y_{3v}^{(2)} \\ -\sqrt{10}Y_{xyz}^{(1)} + 20\sqrt{7}Y_{xyz}^{(2)} & \sqrt{6}Y_z^{\alpha(1)} + 8\sqrt{105}Y_z^{\alpha(2)} + \sqrt{10}Y_z^{\beta(1)} - 20\sqrt{7}Y_z^{\beta(2)} & -4Y_{3v}^{(1)} - 4\sqrt{70}Y_{3v}^{(2)} \\ \frac{\sqrt{15}Y_{3a}^{(1)}}{2} - 5\sqrt{42}Y_{3a}^{(2)} + \frac{Y_{3u}^{(1)}}{2} - 5\sqrt{70}Y_{3u}^{(2)} & \frac{\sqrt{15}Y_{3b}^{(1)}}{2} - 5\sqrt{42}Y_{3b}^{(2)} + \frac{Y_{3v}^{(1)}}{2} + 5\sqrt{70}Y_{3v}^{(2)} & \frac{3\sqrt{6}Y_{3a}^{(1)}}{2} + \frac{\sqrt{10}Y_{3a}^{\beta(1)}}{2} + 20\sqrt{7}Y_z^{\beta(2)} \\ -\frac{\sqrt{15}Y_{3b}^{(1)}}{2} + 5\sqrt{42}Y_{3b}^{(2)} + \frac{Y_{3v}^{(1)}}{2} - \sqrt{70}Y_{3v}^{(2)} & \frac{\sqrt{15}Y_{3a}^{(1)}}{2} - 5\sqrt{42}Y_{3a}^{(2)} + \frac{Y_{3u}^{(1)}}{2} + 11\sqrt{70}Y_{3u}^{(2)} & -\frac{\sqrt{10}Y_{xyz}^{(1)}}{2} - 20\sqrt{7}Y_{xyz}^{(2)} \\ -\frac{\sqrt{6}Y_z^{\alpha(1)}}{2} + 2\sqrt{105}Y_z^{\alpha(2)} + \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 10\sqrt{7}Y_z^{\beta(2)} & \frac{\sqrt{10}Y_{xyz}^{(1)}}{2} + 20\sqrt{7}Y_{xyz}^{(2)} & 15\sqrt{42}Y_{3a}^{(2)} + 2Y_{3u}^{(1)} - \sqrt{70}Y_{3u}^{(2)} \\ \sqrt{6}Y_z^{\alpha(1)} + 2\sqrt{105}Y_z^{\alpha(2)} - \sqrt{10}Y_z^{\beta(1)} - 10\sqrt{7}Y_z^{\beta(2)} & -\sqrt{10}Y_{xyz}^{(1)} + 20\sqrt{7}Y_{xyz}^{(2)} & 15\sqrt{42}Y_{3a}^{(2)} - 4Y_{3u}^{(1)} - \sqrt{70}Y_{3u}^{(2)} \\ 6\sqrt{105}Y_z^{\alpha(2)} + 30\sqrt{7}Y_z^{\beta(2)} & 0 & -15\sqrt{42}Y_{3a}^{(2)} - 3\sqrt{70}Y_{3u}^{(2)} \\ -\sqrt{6}Y_z^{\alpha(1)} - 8\sqrt{105}Y_z^{\alpha(2)} + \sqrt{10}Y_z^{\beta(1)} - 20\sqrt{7}Y_z^{\beta(2)} & \sqrt{10}Y_{xyz}^{(1)} - 20\sqrt{7}Y_{xyz}^{(2)} & 4Y_{3u}^{(1)} + 4\sqrt{70}Y_{3u}^{(2)} \\ \frac{\sqrt{15}Y_{3b}^{(1)}}{2} - 5\sqrt{42}Y_{3b}^{(2)} - \frac{Y_{3v}^{(1)}}{2} - 11\sqrt{70}Y_{3v}^{(2)} & -\frac{\sqrt{15}Y_{3a}^{(1)}}{2} + 5\sqrt{42}Y_{3a}^{(2)} - \frac{Y_{3u}^{(1)}}{2} + \sqrt{70}Y_{3u}^{(2)} & \frac{\sqrt{10}Y_{xyz}^{(1)}}{2} + 20\sqrt{7}Y_{xyz}^{(2)} \\ \frac{\sqrt{15}Y_{3a}^{(1)}}{2} - 5\sqrt{42}Y_{3a}^{(2)} - \frac{Y_{3u}^{(1)}}{2} - 5\sqrt{70}Y_{3u}^{(2)} & \frac{\sqrt{15}Y_{3b}^{(1)}}{2} - 5\sqrt{42}Y_{3b}^{(2)} - \frac{Y_{3v}^{(1)}}{2} + 5\sqrt{70}Y_{3v}^{(2)} & -\frac{3\sqrt{6}Y_{3a}^{(1)}}{2} + \frac{\sqrt{10}Y_{3a}^{\beta(1)}}{2} + 20\sqrt{7}Y_z^{\beta(2)} \\ -\frac{\sqrt{10}Y_{xyz}^{(1)}}{2} - 20\sqrt{7}Y_{xyz}^{(2)} & \frac{\sqrt{6}Y_z^{\alpha(1)}}{2} - 2\sqrt{105}Y_z^{\alpha(2)} + \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 10\sqrt{7}Y_z^{\beta(2)} & 15\sqrt{42}Y_{3b}^{(2)} - 2Y_{3v}^{(1)} + \sqrt{70}Y_{3v}^{(2)} \\ \sqrt{15}Y_{3b}^{(1)} + 5\sqrt{42}Y_{3b}^{(2)} - Y_{3v}^{(1)} - \sqrt{70}Y_{3v}^{(2)} & -\sqrt{15}Y_{3a}^{(1)} - 5\sqrt{42}Y_{3a}^{(2)} - Y_{3u}^{(1)} + 11\sqrt{70}Y_{3u}^{(2)} & \sqrt{10}Y_{xyz}^{(1)} - 20\sqrt{7}Y_{xyz}^{(2)} \\ -\sqrt{15}Y_{3b}^{(1)} - 5\sqrt{42}Y_{3b}^{(2)} + Y_{3v}^{(1)} - 11\sqrt{70}Y_{3v}^{(2)} & \sqrt{15}Y_{3a}^{(1)} + 5\sqrt{42}Y_{3a}^{(2)} + Y_{3u}^{(1)} + \sqrt{70}Y_{3u}^{(2)} & -\sqrt{10}Y_{xyz}^{(1)} + 20\sqrt{7}Y_{xyz}^{(2)} \\ 12\sqrt{70}Y_{3v}^{(2)} & -12\sqrt{70}Y_{3u}^{(2)} & 0 \\ \frac{\sqrt{6}Y_z^{\alpha(1)}}{2} - 8\sqrt{105}Y_z^{\alpha(2)} - \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 20\sqrt{7}Y_z^{\beta(2)} & -\frac{\sqrt{10}Y_{xyz}^{(1)}}{2} - 20\sqrt{7}Y_{xyz}^{(2)} & -2Y_{3u}^{(1)} + 4\sqrt{70}Y_{3u}^{(2)} \\ \frac{\sqrt{10}Y_{xyz}^{(1)}}{2} + 20\sqrt{7}Y_{xyz}^{(2)} & -\frac{\sqrt{6}Y_z^{\alpha(1)}}{2} + 8\sqrt{105}Y_z^{\alpha(2)} - \frac{\sqrt{10}Y_z^{\beta(1)}}{2} - 20\sqrt{7}Y_z^{\beta(2)} & 2Y_{3v}^{(1)} - 4\sqrt{70}Y_{3v}^{(2)} \\ -\sqrt{15}Y_{3a}^{(1)} - 5\sqrt{42}Y_{3a}^{(2)} + Y_{3u}^{(1)} - 5\sqrt{70}Y_{3u}^{(2)} & -\sqrt{15}Y_{3b}^{(1)} - 5\sqrt{42}Y_{3b}^{(2)} - Y_{3v}^{(1)} + 5\sqrt{70}Y_{3v}^{(2)} & -\sqrt{10}Y_z^{\beta(1)} + 20\sqrt{7}Y_z^{\beta(2)} \end{array} \right). \quad (\text{B46})$$

8. rank-4 t -multipoles

$$\sigma_{ij}^{mn\alpha(\text{O/L})} = \sigma_1^{(\text{O/L})} + \sigma_2^{(\text{O/L})}, \quad (\text{B47})$$

$$\sigma_1^{(\text{O/L})} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{210}X_{4v}^{\alpha(1)} - 7\sqrt{30}X_{4a}^{(1)} & 3\sqrt{210}X_{4v}^{\alpha(1)} + 7\sqrt{30}X_{4a}^{(1)} & -4\sqrt{210}X_{4v}^{\alpha(1)} \\ -\sqrt{210}X_{4v}^{\alpha(1)} + 7\sqrt{30}X_{4a}^{(1)} & -3\sqrt{210}X_{4v}^{\alpha(1)} - 7\sqrt{30}X_{4a}^{(1)} & 4\sqrt{210}X_{4v}^{\alpha(1)} \\ -7\sqrt{15}X_{4u}^{\beta(1)} - 2\sqrt{105}X_{4u}^{\beta(2)} + 5\sqrt{21}X_{40}^{(1)} & 7\sqrt{15}X_{4u}^{\beta(1)} + 4\sqrt{105}X_{4u}^{\beta(2)} + 7\sqrt{21}X_{40}^{(1)} & -2\sqrt{105}X_{4u}^{\beta(2)} - 12\sqrt{21}X_{40}^{(1)} \\ 7\sqrt{15}X_{4v}^{\beta(1)} - \sqrt{105}X_{4v}^{\beta(2)} & -7\sqrt{15}X_{4v}^{\beta(1)} - \sqrt{105}X_{4v}^{\beta(2)} & 2\sqrt{105}X_{4v}^{\beta(2)} \\ \frac{3\sqrt{210}X_{4u}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & \frac{\sqrt{210}X_{4u}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & -2\sqrt{210}X_{4u}^{\alpha(1)} \\ -3\sqrt{210}X_{4u}^{\alpha(1)} + 7\sqrt{30}X_{4b}^{(1)} & -\sqrt{210}X_{4u}^{\alpha(1)} - 7\sqrt{30}X_{4b}^{(1)} & 4\sqrt{210}X_{4u}^{\alpha(1)} \\ 0 & 0 & 0 \\ 3\sqrt{210}X_{4u}^{\alpha(1)} - 7\sqrt{30}X_{4b}^{(1)} & \sqrt{210}X_{4u}^{\alpha(1)} + 7\sqrt{30}X_{4b}^{(1)} & -4\sqrt{210}X_{4u}^{\alpha(1)} \\ -7\sqrt{15}X_{4v}^{\beta(1)} + \sqrt{105}X_{4v}^{\beta(2)} & 7\sqrt{15}X_{4v}^{\beta(1)} + \sqrt{105}X_{4v}^{\beta(2)} & -2\sqrt{105}X_{4v}^{\beta(2)} \\ -7\sqrt{15}X_{4u}^{\beta(1)} + 4\sqrt{105}X_{4u}^{\beta(2)} - 7\sqrt{21}X_{40}^{(1)} & 7\sqrt{15}X_{4u}^{\beta(1)} - 2\sqrt{105}X_{4u}^{\beta(2)} - 5\sqrt{21}X_{40}^{(1)} & -2\sqrt{105}X_{4u}^{\beta(2)} + 12\sqrt{21}X_{40}^{(1)} \\ -\frac{\sqrt{210}X_{4v}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & -\frac{3\sqrt{210}X_{4v}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & 2\sqrt{210}X_{4v}^{\alpha(1)} \\ -14\sqrt{15}X_{4v}^{\beta(1)} + 2\sqrt{105}X_{4v}^{\beta(2)} & 14\sqrt{15}X_{4v}^{\beta(1)} + 2\sqrt{105}X_{4v}^{\beta(2)} & -4\sqrt{105}X_{4v}^{\beta(2)} \\ 14\sqrt{15}X_{4v}^{\beta(1)} - 2\sqrt{105}X_{4v}^{\beta(2)} & -14\sqrt{15}X_{4v}^{\beta(1)} - 2\sqrt{105}X_{4v}^{\beta(2)} & 4\sqrt{105}X_{4v}^{\beta(2)} \\ 0 & 0 & 0 \\ -\frac{3\sqrt{210}X_{4u}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & -\frac{\sqrt{210}X_{4u}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & 2\sqrt{210}X_{4u}^{\alpha(1)} \\ \frac{\sqrt{210}X_{4v}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & \frac{3\sqrt{210}X_{4v}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & -2\sqrt{210}X_{4v}^{\alpha(1)} \\ 14\sqrt{15}X_{4u}^{\beta(1)} - 2\sqrt{105}X_{4u}^{\beta(2)} + 2\sqrt{21}X_{40}^{(1)} & -14\sqrt{15}X_{4u}^{\beta(1)} - 2\sqrt{105}X_{4u}^{\beta(2)} - 2\sqrt{21}X_{40}^{(1)} & 4\sqrt{105}X_{4u}^{\beta(2)} \end{pmatrix}, \quad (\text{B48})$$

$$\sigma_2^{(\text{O/L})} = \begin{pmatrix} 0 & 6\sqrt{30}X_{4v}^{\alpha(2)} - 6\sqrt{210}X_{4a}^{(2)} & -6\sqrt{30}X_{4v}^{\alpha(2)} + 6\sqrt{210}X_{4a}^{(2)} \\ -4\sqrt{30}X_{4v}^{\alpha(2)} + 4\sqrt{210}X_{4a}^{(2)} & 6\sqrt{30}X_{4v}^{\alpha(2)} + 2\sqrt{210}X_{4a}^{(2)} & -2\sqrt{30}X_{4v}^{\alpha(2)} - 6\sqrt{210}X_{4a}^{(2)} \\ 4\sqrt{30}X_{4v}^{\alpha(2)} - 4\sqrt{210}X_{4a}^{(2)} & -12\sqrt{30}X_{4v}^{\alpha(2)} + 4\sqrt{210}X_{4a}^{(2)} & 8\sqrt{30}X_{4v}^{\alpha(2)} \\ 4\sqrt{105}X_{4u}^{\beta(1)} + 8\sqrt{15}X_{4u}^{\beta(2)} - 20\sqrt{3}X_{40}^{(2)} & -4\sqrt{105}X_{4u}^{\beta(1)} + 8\sqrt{15}X_{4u}^{\beta(2)} + 20\sqrt{3}X_{40}^{(2)} & -16\sqrt{15}X_{4u}^{\beta(2)} \\ -4\sqrt{105}X_{4v}^{\beta(1)} + 4\sqrt{15}X_{4v}^{\beta(2)} & 4\sqrt{105}X_{4v}^{\beta(1)} - 20\sqrt{15}X_{4v}^{\beta(2)} & 16\sqrt{15}X_{4v}^{\beta(2)} \\ -6\sqrt{30}X_{4u}^{\alpha(2)} + 2\sqrt{210}X_{4b}^{(2)} & 4\sqrt{30}X_{4u}^{\alpha(2)} + 4\sqrt{210}X_{4b}^{(2)} & 2\sqrt{30}X_{4u}^{\alpha(2)} - 6\sqrt{210}X_{4b}^{(2)} \\ -6\sqrt{30}X_{4u}^{\alpha(2)} + 2\sqrt{210}X_{4b}^{(2)} & 4\sqrt{30}X_{4u}^{\alpha(2)} + 4\sqrt{210}X_{4b}^{(2)} & 2\sqrt{30}X_{4u}^{\alpha(2)} - 6\sqrt{210}X_{4b}^{(2)} \\ -6\sqrt{30}X_{4u}^{\alpha(2)} - 6\sqrt{210}X_{4b}^{(2)} & 0 & 6\sqrt{30}X_{4u}^{\alpha(2)} + 6\sqrt{210}X_{4b}^{(2)} \\ 12\sqrt{30}X_{4u}^{\alpha(2)} + 4\sqrt{210}X_{4b}^{(2)} & -4\sqrt{30}X_{4u}^{\alpha(2)} - 4\sqrt{210}X_{4b}^{(2)} & -8\sqrt{30}X_{4u}^{\alpha(2)} \\ 4\sqrt{105}X_{4v}^{\beta(1)} + 20\sqrt{15}X_{4v}^{\beta(2)} & -4\sqrt{105}X_{4v}^{\beta(1)} - 4\sqrt{15}X_{4v}^{\beta(2)} & -16\sqrt{15}X_{4v}^{\beta(2)} \\ 4\sqrt{105}X_{4u}^{\beta(1)} + 8\sqrt{15}X_{4u}^{\beta(2)} - 20\sqrt{3}X_{40}^{(2)} & -4\sqrt{105}X_{4u}^{\beta(1)} + 8\sqrt{15}X_{4u}^{\beta(2)} + 20\sqrt{3}X_{40}^{(2)} & -16\sqrt{15}X_{4u}^{\beta(2)} \\ -4\sqrt{30}X_{4v}^{\alpha(2)} + 4\sqrt{210}X_{4a}^{(2)} & 6\sqrt{30}X_{4v}^{\alpha(2)} + 2\sqrt{210}X_{4a}^{(2)} & -2\sqrt{30}X_{4v}^{\alpha(2)} - 6\sqrt{210}X_{4a}^{(2)} \\ -4\sqrt{105}X_{4v}^{\beta(1)} + 4\sqrt{15}X_{4v}^{\beta(2)} & 4\sqrt{105}X_{4v}^{\beta(1)} - 20\sqrt{15}X_{4v}^{\beta(2)} & 16\sqrt{15}X_{4v}^{\beta(2)} \\ 4\sqrt{105}X_{4v}^{\beta(1)} + 20\sqrt{15}X_{4v}^{\beta(2)} & -4\sqrt{105}X_{4v}^{\beta(1)} - 4\sqrt{15}X_{4v}^{\beta(2)} & -16\sqrt{15}X_{4v}^{\beta(2)} \\ -24\sqrt{15}X_{4v}^{\beta(2)} & 24\sqrt{15}X_{4v}^{\beta(2)} & 0 \\ 12\sqrt{30}X_{4u}^{\alpha(2)} + 4\sqrt{210}X_{4b}^{(2)} & -4\sqrt{30}X_{4u}^{\alpha(2)} - 4\sqrt{210}X_{4b}^{(2)} & -8\sqrt{30}X_{4u}^{\alpha(2)} \\ 4\sqrt{30}X_{4v}^{\alpha(2)} - 4\sqrt{210}X_{4a}^{(2)} & -12\sqrt{30}X_{4v}^{\alpha(2)} + 4\sqrt{210}X_{4a}^{(2)} & 8\sqrt{30}X_{4v}^{\alpha(2)} \\ 4\sqrt{105}X_{4u}^{\beta(1)} + 8\sqrt{15}X_{4u}^{\beta(2)} - 20\sqrt{3}X_{40}^{(2)} & -4\sqrt{105}X_{4u}^{\beta(1)} + 8\sqrt{15}X_{4u}^{\beta(2)} + 20\sqrt{3}X_{40}^{(2)} & -16\sqrt{15}X_{4u}^{\beta(2)} \end{pmatrix}. \quad (\text{B49})$$

$$\sigma_{ij}^{mn\alpha(O/T)} = \sigma_1^{(O/T)} + \sigma_2^{(O/T)}, \quad (B50)$$

$$\sigma_1^{(O/T)} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 4\sqrt{105}X_{4u}^{\beta 2(1)} + 8\sqrt{21}X_{40}^{(1)} & -4\sqrt{105}X_{4v}^{\beta 2(1)} & \sqrt{210}X_{4u}^{\alpha(1)} + 7\sqrt{30}X_{4b}^{(1)} \\ -4\sqrt{105}X_{4u}^{\beta 2(1)} - 8\sqrt{21}X_{40}^{(1)} & 4\sqrt{105}X_{4v}^{\beta 2(1)} & -\sqrt{210}X_{4u}^{\alpha(1)} - 7\sqrt{30}X_{4b}^{(1)} \\ -\frac{7\sqrt{210}X_{4v}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & -\frac{5\sqrt{210}X_{4u}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & -7\sqrt{15}X_{4v}^{\beta 1(1)} - 3\sqrt{105}X_{4v}^{\beta 2(1)} \\ -\frac{\sqrt{210}X_{4u}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & -\frac{\sqrt{210}X_{4v}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & -7\sqrt{15}X_{4u}^{\beta 1(1)} + \sqrt{21}X_{40}^{(1)} \\ -2\sqrt{105}X_{4v}^{\beta 2(1)} & -2\sqrt{105}X_{4u}^{\beta 2(1)} + 4\sqrt{21}X_{40}^{(1)} & \frac{\sqrt{210}X_{4v}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4a}^{(1)}}{2} \\ 4\sqrt{105}X_{4v}^{\beta 2(1)} & 4\sqrt{105}X_{4u}^{\beta 2(1)} - 8\sqrt{21}X_{40}^{(1)} & -\sqrt{210}X_{4v}^{\alpha(1)} + 7\sqrt{30}X_{4a}^{(1)} \\ 0 & 0 & 0 \\ -4\sqrt{105}X_{4v}^{\beta 2(1)} & -4\sqrt{105}X_{4u}^{\beta 2(1)} + 8\sqrt{21}X_{40}^{(1)} & \sqrt{210}X_{4v}^{\alpha(1)} - 7\sqrt{30}X_{4a}^{(1)} \\ \frac{\sqrt{210}X_{4u}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & \frac{\sqrt{210}X_{4v}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & 7\sqrt{15}X_{4u}^{\beta 1(1)} - \sqrt{21}X_{40}^{(1)} \\ \frac{5\sqrt{210}X_{4v}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4a}^{(1)}}{2} & \frac{7\sqrt{210}X_{4u}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4b}^{(1)}}{2} & -7\sqrt{15}X_{4v}^{\beta 1(1)} + 3\sqrt{105}X_{4v}^{\beta 2(1)} \\ -2\sqrt{105}X_{4u}^{\beta 2(1)} - 4\sqrt{21}X_{40}^{(1)} & 2\sqrt{105}X_{4v}^{\beta 2(1)} & -\frac{\sqrt{210}X_{4u}^{\alpha(1)}}{2} - \frac{7\sqrt{30}X_{4b}^{(1)}}{2} \\ \sqrt{210}X_{4u}^{\alpha(1)} + 7\sqrt{30}X_{4b}^{(1)} & \sqrt{210}X_{4v}^{\alpha(1)} - 7\sqrt{30}X_{4a}^{(1)} & 14\sqrt{15}X_{4u}^{\beta 1(1)} - 2\sqrt{21}X_{40}^{(1)} \\ -\sqrt{210}X_{4u}^{\alpha(1)} - 7\sqrt{30}X_{4b}^{(1)} & -\sqrt{210}X_{4v}^{\alpha(1)} + 7\sqrt{30}X_{4a}^{(1)} & -14\sqrt{15}X_{4u}^{\beta 1(1)} + 2\sqrt{21}X_{40}^{(1)} \\ 0 & 0 & 0 \\ 2\sqrt{105}X_{4v}^{\beta 2(1)} & 2\sqrt{105}X_{4u}^{\beta 2(1)} - 4\sqrt{21}X_{40}^{(1)} & -\frac{\sqrt{210}X_{4v}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4a}^{(1)}}{2} \\ 2\sqrt{105}X_{4u}^{\beta 2(1)} + 4\sqrt{21}X_{40}^{(1)} & -2\sqrt{105}X_{4v}^{\beta 2(1)} & \frac{\sqrt{210}X_{4u}^{\alpha(1)}}{2} + \frac{7\sqrt{30}X_{4b}^{(1)}}{2} \\ \sqrt{210}X_{4v}^{\alpha(1)} + 7\sqrt{30}X_{4a}^{(1)} & -\sqrt{210}X_{4u}^{\alpha(1)} + 7\sqrt{30}X_{4b}^{(1)} & 14\sqrt{15}X_{4v}^{\beta 1(1)} \end{pmatrix}, \quad (B51)$$

$$\sigma_2^{(O/T)} =$$

$$\begin{pmatrix} -6\sqrt{105}X_{4u}^{\beta 1(2)} - 12\sqrt{15}X_{4u}^{\beta 2(2)} + 30\sqrt{3}X_{40}^{(2)} & 6\sqrt{105}X_{4v}^{\beta 1(2)} - 6\sqrt{15}X_{4v}^{\beta 2(2)} & 9\sqrt{30}X_{4u}^{\alpha(2)} - 3\sqrt{210}X_{4b}^{(2)} \\ 6\sqrt{105}X_{4u}^{\beta 1(2)} + 8\sqrt{15}X_{4u}^{\beta 2(2)} + 10\sqrt{3}X_{40}^{(2)} & -6\sqrt{105}X_{4v}^{\beta 1(2)} + 10\sqrt{15}X_{4v}^{\beta 2(2)} & -\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} \\ 4\sqrt{15}X_{4u}^{\beta 2(2)} - 40\sqrt{3}X_{40}^{(2)} & -4\sqrt{15}X_{4v}^{\beta 2(2)} & -8\sqrt{30}X_{4u}^{\alpha(2)} + 4\sqrt{210}X_{4b}^{(2)} \\ -7\sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} & 7\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} & 4\sqrt{105}X_{4v}^{\beta 1(2)} \\ -13\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} & -\sqrt{30}X_{4v}^{\alpha(2)} + \sqrt{210}X_{4a}^{(2)} & 4\sqrt{105}X_{4u}^{\beta 1(2)} - 12\sqrt{15}X_{4u}^{\beta 2(2)} + 20\sqrt{3}X_{40}^{(2)} \\ -6\sqrt{105}X_{4v}^{\beta 1(2)} - 10\sqrt{15}X_{4v}^{\beta 2(2)} & -6\sqrt{105}X_{4u}^{\beta 1(2)} + 8\sqrt{15}X_{4u}^{\beta 2(2)} - 10\sqrt{3}X_{40}^{(2)} & \sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} \\ -6\sqrt{105}X_{4v}^{\beta 1(2)} - 10\sqrt{15}X_{4v}^{\beta 2(2)} & -6\sqrt{105}X_{4u}^{\beta 1(2)} + 8\sqrt{15}X_{4u}^{\beta 2(2)} - 10\sqrt{3}X_{40}^{(2)} & \sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} \\ 6\sqrt{105}X_{4v}^{\beta 1(2)} + 6\sqrt{15}X_{4v}^{\beta 2(2)} & 6\sqrt{105}X_{4u}^{\beta 1(2)} - 12\sqrt{15}X_{4u}^{\beta 2(2)} - 30\sqrt{3}X_{40}^{(2)} & -9\sqrt{30}X_{4u}^{\alpha(2)} - 3\sqrt{210}X_{4a}^{(2)} \\ 4\sqrt{15}X_{4v}^{\beta 2(2)} & 4\sqrt{15}X_{4u}^{\beta 2(2)} + 40\sqrt{3}X_{40}^{(2)} & 8\sqrt{30}X_{4v}^{\alpha(2)} + 4\sqrt{210}X_{4a}^{(2)} \\ \sqrt{30}X_{4u}^{\alpha(2)} + \sqrt{210}X_{4b}^{(2)} & 13\sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} & -4\sqrt{105}X_{4u}^{\beta 1(2)} - 12\sqrt{15}X_{4u}^{\beta 2(2)} - 20\sqrt{3}X_{40}^{(2)} \\ -7\sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} & 7\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} & 4\sqrt{105}X_{4v}^{\beta 1(2)} \\ 6\sqrt{105}X_{4u}^{\beta 1(2)} + 8\sqrt{15}X_{4u}^{\beta 2(2)} + 10\sqrt{3}X_{40}^{(2)} & -6\sqrt{105}X_{4v}^{\beta 1(2)} + 10\sqrt{15}X_{4v}^{\beta 2(2)} & -\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} \\ -13\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} & -\sqrt{30}X_{4v}^{\alpha(2)} + \sqrt{210}X_{4a}^{(2)} & 4\sqrt{105}X_{4u}^{\beta 1(2)} - 12\sqrt{15}X_{4u}^{\beta 2(2)} + 20\sqrt{3}X_{40}^{(2)} \\ \sqrt{30}X_{4u}^{\alpha(2)} + \sqrt{210}X_{4b}^{(2)} & 13\sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} & -4\sqrt{105}X_{4u}^{\beta 1(2)} - 12\sqrt{15}X_{4u}^{\beta 2(2)} - 20\sqrt{3}X_{40}^{(2)} \\ 12\sqrt{30}X_{4u}^{\alpha(2)} & -12\sqrt{30}X_{4v}^{\alpha(2)} & 24\sqrt{15}X_{4u}^{\beta 2(2)} \\ 4\sqrt{15}X_{4v}^{\beta 2(2)} & 4\sqrt{15}X_{4u}^{\beta 2(2)} + 40\sqrt{3}X_{40}^{(2)} & 8\sqrt{30}X_{4v}^{\alpha(2)} + 4\sqrt{210}X_{4a}^{(2)} \\ 4\sqrt{15}X_{4u}^{\beta 2(2)} - 40\sqrt{3}X_{40}^{(2)} & -4\sqrt{15}X_{4v}^{\beta 2(2)} & -8\sqrt{30}X_{4u}^{\alpha(2)} + 4\sqrt{210}X_{4b}^{(2)} \\ -7\sqrt{30}X_{4v}^{\alpha(2)} - \sqrt{210}X_{4a}^{(2)} & 7\sqrt{30}X_{4u}^{\alpha(2)} - \sqrt{210}X_{4b}^{(2)} & 4\sqrt{105}X_{4v}^{\beta 1(2)} \end{pmatrix}. \quad (B52)$$

$$\sigma_{ij}^{mn\alpha(H)} = \begin{pmatrix} 10\sqrt{7}X_{4u}^{\beta 1} - 20X_{4u}^{\beta 2} + 6\sqrt{5}X_{40} & 10\sqrt{7}X_{4v}^{\beta 1} - 10X_{4v}^{\beta 2} & -15\sqrt{2}X_{4u}^{\alpha} + 5\sqrt{14}X_{4b} \\ -10\sqrt{7}X_{4u}^{\beta 1} + 2\sqrt{5}X_{40} & -10\sqrt{7}X_{4v}^{\beta 1} - 10X_{4v}^{\beta 2} & -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} \\ 20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} & 20X_{4v}^{\beta 2} & 20\sqrt{2}X_{4u}^{\alpha} \\ -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} & -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} & 20X_{4v}^{\beta 2} \\ -15\sqrt{2}X_{4u}^{\alpha} + 5\sqrt{14}X_{4b} & -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} & 20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} \\ 10\sqrt{7}X_{4v}^{\beta 1} - 10X_{4v}^{\beta 2} & -10\sqrt{7}X_{4u}^{\beta 1} + 2\sqrt{5}X_{40} & -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} \\ 10\sqrt{7}X_{4v}^{\beta 1} - 10X_{4v}^{\beta 2} & -10\sqrt{7}X_{4u}^{\beta 1} + 2\sqrt{5}X_{40} & -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} \\ -10\sqrt{7}X_{4v}^{\beta 1} - 10X_{4v}^{\beta 2} & 10\sqrt{7}X_{4u}^{\beta 1} + 20X_{4u}^{\beta 2} + 6\sqrt{5}X_{40} & -15\sqrt{2}X_{4v}^{\alpha} - 5\sqrt{14}X_{4a} \\ 20X_{4v}^{\beta 2} & -20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} & 20\sqrt{2}X_{4v}^{\alpha} \\ -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} & -15\sqrt{2}X_{4v}^{\alpha} - 5\sqrt{14}X_{4a} & -20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} \\ -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} & -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} & 20X_{4v}^{\beta 2} \\ -10\sqrt{7}X_{4u}^{\beta 1} + 2\sqrt{5}X_{40} & -10\sqrt{7}X_{4v}^{\beta 1} - 10X_{4v}^{\beta 2} & -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} \\ -15\sqrt{2}X_{4u}^{\alpha} + 5\sqrt{14}X_{4b} & -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} & 20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} \\ -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} & -15\sqrt{2}X_{4v}^{\alpha} - 5\sqrt{14}X_{4a} & -20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} \\ 20\sqrt{2}X_{4u}^{\alpha} & 20\sqrt{2}X_{4v}^{\alpha} & 16\sqrt{5}X_{40} \\ 20X_{4v}^{\beta 2} & -20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} & 20\sqrt{2}X_{4v}^{\alpha} \\ 20X_{4u}^{\beta 2} - 8\sqrt{5}X_{40} & 20X_{4v}^{\beta 2} & 20\sqrt{2}X_{4u}^{\alpha} \\ -5\sqrt{2}X_{4v}^{\alpha} + 5\sqrt{14}X_{4a} & -5\sqrt{2}X_{4u}^{\alpha} - 5\sqrt{14}X_{4b} & 20X_{4v}^{\beta 2} \end{pmatrix}. \quad (B53)$$

9. rank-5 t -multipoles

$$\sigma_{ij}^{mn\alpha(O/L)} = \begin{pmatrix} 15\sqrt{7}Y_{5u}^{\alpha 1} + 5\sqrt{30}Y_{5u}^{\alpha 2} - 5\sqrt{35}Y_{5a} & -15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + \sqrt{35}Y_{5a} & -6\sqrt{30}Y_{5u}^{\alpha 2} + 4\sqrt{35}Y_{5a} \\ -15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + \sqrt{35}Y_{5a} & 15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + 3\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} \\ -6\sqrt{30}Y_{5u}^{\alpha 2} + 4\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & 8\sqrt{30}Y_{5u}^{\alpha 2} \\ 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & 2\sqrt{210}Y_{5v}^{\beta 2} \\ 3\sqrt{70}Y_{5u}^{\beta 1} - 2\sqrt{210}Y_{5u}^{\beta 2} + 15\sqrt{2}Y_{50} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & 2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} \\ 15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - 3\sqrt{35}Y_{5b} & -15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - \sqrt{35}Y_{5b} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} \\ 15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - 3\sqrt{35}Y_{5b} & -15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - \sqrt{35}Y_{5b} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} \\ -15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - \sqrt{35}Y_{5b} & 15\sqrt{7}Y_{5v}^{\alpha 1} + 5\sqrt{30}Y_{5v}^{\alpha 2} + 5\sqrt{35}Y_{5b} & -6\sqrt{30}Y_{5v}^{\alpha 2} - 4\sqrt{35}Y_{5b} \\ -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -6\sqrt{30}Y_{5v}^{\alpha 2} - 4\sqrt{35}Y_{5b} & 8\sqrt{30}Y_{5v}^{\alpha 2} \\ -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & 3\sqrt{70}Y_{5u}^{\beta 1} + 2\sqrt{210}Y_{5u}^{\beta 2} + 15\sqrt{2}Y_{50} & -2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} \\ 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & 2\sqrt{210}Y_{5v}^{\beta 2} \\ -15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + \sqrt{35}Y_{5a} & 15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + 3\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} \\ 3\sqrt{70}Y_{5u}^{\beta 1} - 2\sqrt{210}Y_{5u}^{\beta 2} + 15\sqrt{2}Y_{50} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & 2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} \\ -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & 3\sqrt{70}Y_{5u}^{\beta 1} + 2\sqrt{210}Y_{5u}^{\beta 2} + 15\sqrt{2}Y_{50} & -2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} \\ 2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} & -2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} & 40\sqrt{2}Y_{50} \\ -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -6\sqrt{30}Y_{5v}^{\alpha 2} - 4\sqrt{35}Y_{5b} & 8\sqrt{30}Y_{5v}^{\alpha 2} \\ -6\sqrt{30}Y_{5u}^{\alpha 2} + 4\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & 8\sqrt{30}Y_{5u}^{\alpha 2} \\ 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & 2\sqrt{210}Y_{5v}^{\beta 2} \end{pmatrix}. \quad (B54)$$

$$\sigma_{ij}^{mn\alpha(O/T)} = \begin{pmatrix} 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & 3\sqrt{70}Y_{5u}^{\beta 1} - 2\sqrt{210}Y_{5u}^{\beta 2} + 15\sqrt{2}Y_{50} & 15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - 3\sqrt{35}Y_{5b} \\ -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & -15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - \sqrt{35}Y_{5b} \\ 2\sqrt{210}Y_{5v}^{\beta 2} & 2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} \\ -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} \\ -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -6\sqrt{30}Y_{5u}^{\alpha 2} + 4\sqrt{35}Y_{5a} & 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} \\ -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + \sqrt{35}Y_{5a} \\ -3\sqrt{70}Y_{5v}^{\beta 1} + 5\sqrt{2}Y_{50} & 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + \sqrt{35}Y_{5a} \\ 3\sqrt{70}Y_{5u}^{\beta 1} + 2\sqrt{210}Y_{5u}^{\beta 2} + 15\sqrt{2}Y_{50} & -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & 15\sqrt{7}Y_{5u}^{\alpha 1} + \sqrt{30}Y_{5u}^{\alpha 2} + 3\sqrt{35}Y_{5a} \\ -2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} & 2\sqrt{210}Y_{5v}^{\beta 2} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} \\ -6\sqrt{30}Y_{5v}^{\alpha 2} - 4\sqrt{35}Y_{5b} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} \\ -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} \\ -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} & -15\sqrt{7}Y_{5v}^{\alpha 1} + \sqrt{30}Y_{5v}^{\alpha 2} - \sqrt{35}Y_{5b} \\ -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -6\sqrt{30}Y_{5u}^{\alpha 2} + 4\sqrt{35}Y_{5a} & 3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} \\ -6\sqrt{30}Y_{5v}^{\alpha 2} - 4\sqrt{35}Y_{5b} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & -3\sqrt{70}Y_{5v}^{\beta 1} - \sqrt{210}Y_{5v}^{\beta 2} \\ 8\sqrt{30}Y_{5v}^{\alpha 2} & 8\sqrt{30}Y_{5u}^{\alpha 2} & 2\sqrt{210}Y_{5v}^{\beta 2} \\ -2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} & 2\sqrt{210}Y_{5v}^{\beta 2} & -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} \\ 2\sqrt{210}Y_{5v}^{\beta 2} & 2\sqrt{210}Y_{5u}^{\beta 2} - 20\sqrt{2}Y_{50} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} \\ -2\sqrt{30}Y_{5u}^{\alpha 2} - 4\sqrt{35}Y_{5a} & -2\sqrt{30}Y_{5v}^{\alpha 2} + 4\sqrt{35}Y_{5b} & -3\sqrt{70}Y_{5u}^{\beta 1} + 5\sqrt{2}Y_{50} \end{pmatrix}. \quad (B55)$$

10. rank-0 c -multipoles with normalization

$$\left(\tilde{\sigma}_{ij}^{mn\alpha(O/L)}\right)^\top = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 2X_0 \\ 0 & 0 & 0 & 2X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2X_0 \\ 0 & 0 & 0 & -2X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 2X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (B56)$$

$$\left(\tilde{\sigma}_{ij}^{mn\alpha(O/T)}\right)^\top = \begin{pmatrix} 0 & -2X_0 & 2X_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_0 & 0 & 0 & 0 & 0 & -X_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -X_0 & 2X_0 & 0 & -2X_0 & 0 & 0 & 0 & 0 & 0 & 0 & X_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_0 & 0 & 0 & 0 & 0 & -X_0 & 0 & 0 & -2X_0 & 2X_0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (B57)$$

$$\begin{aligned}
\tilde{\sigma}_{ij}^{mn\alpha(H)} = & \\
& \begin{pmatrix}
2X_0^{(1)} + 8X_0^{(2)} & 0 & 0 \\
6X_0^{(2)} & 0 & 0 \\
6X_0^{(2)} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & X_0^{(1)} + X_0^{(2)} \\
0 & X_0^{(1)} + X_0^{(2)} & 0 \\
0 & 6X_0^{(2)} & 0 \\
0 & 2X_0^{(1)} + 8X_0^{(2)} & 0 \\
0 & 6X_0^{(2)} & 0 \\
0 & 0 & X_0^{(1)} + X_0^{(2)} \\
0 & 0 & 0 \\
X_0^{(1)} + X_0^{(2)} & 0 & 0 \\
0 & 0 & 6X_0^{(2)} \\
0 & 0 & 6X_0^{(2)} \\
0 & 0 & 2X_0^{(1)} + 8X_0^{(2)} \\
0 & X_0^{(1)} + X_0^{(2)} & 0 \\
X_0^{(1)} + X_0^{(2)} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\end{aligned}
\tag{B58}$$

11. rank-1 c -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mn\alpha(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)}, \quad (B59)$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} 12Y_x^{(1)} + 8Y_x^{(3)} & -6Y_x^{(1)} + 8Y_x^{(3)} & -6Y_x^{(1)} + 8Y_x^{(3)} \\ -6Y_x^{(1)} + 6Y_x^{(3)} & 8Y_x^{(1)} - 4Y_x^{(2)} + 6Y_x^{(3)} & -2Y_x^{(1)} + 4Y_x^{(2)} + 6Y_x^{(3)} \\ -6Y_x^{(1)} + 6Y_x^{(3)} & -2Y_x^{(1)} + 4Y_x^{(2)} + 6Y_x^{(3)} & 8Y_x^{(1)} - 4Y_x^{(2)} + 6Y_x^{(3)} \\ 0 & 0 & 0 \\ 8Y_z^{(1)} + 2Y_z^{(2)} + Y_z^{(3)} & -2Y_z^{(1)} - 2Y_z^{(2)} + Y_z^{(3)} & -6Y_z^{(1)} + Y_z^{(3)} \\ 8Y_y^{(1)} + 2Y_y^{(2)} + Y_y^{(3)} & -6Y_y^{(1)} + Y_y^{(3)} & -2Y_y^{(1)} - 2Y_y^{(2)} + Y_y^{(3)} \\ 8Y_y^{(1)} - 4Y_y^{(2)} + 6Y_y^{(3)} & -6Y_y^{(1)} + 6Y_y^{(3)} & -2Y_y^{(1)} + 4Y_y^{(2)} + 6Y_y^{(3)} \\ -6Y_y^{(1)} + 8Y_y^{(3)} & 12Y_y^{(1)} + 8Y_y^{(3)} & -6Y_y^{(1)} + 8Y_y^{(3)} \\ -2Y_y^{(1)} + 4Y_y^{(2)} + 6Y_y^{(3)} & -6Y_y^{(1)} + 6Y_y^{(3)} & 8Y_y^{(1)} - 4Y_y^{(2)} + 6Y_y^{(3)} \\ -2Y_z^{(1)} - 2Y_z^{(2)} + Y_z^{(3)} & 8Y_z^{(1)} + 2Y_z^{(2)} + Y_z^{(3)} & -6Y_z^{(1)} + Y_z^{(3)} \\ 0 & 0 & 0 \\ -6Y_x^{(1)} + Y_x^{(3)} & 8Y_x^{(1)} + 2Y_x^{(2)} + Y_x^{(3)} & -2Y_x^{(1)} - 2Y_x^{(2)} + Y_x^{(3)} \\ 8Y_z^{(1)} - 4Y_z^{(2)} + 6Y_z^{(3)} & -2Y_z^{(1)} + 4Y_z^{(2)} + 6Y_z^{(3)} & -6Y_z^{(1)} + 6Y_z^{(3)} \\ -2Y_z^{(1)} + 4Y_z^{(2)} + 6Y_z^{(3)} & 8Y_z^{(1)} - 4Y_z^{(2)} + 6Y_z^{(3)} & -6Y_z^{(1)} + 6Y_z^{(3)} \\ -6Y_z^{(1)} + 8Y_z^{(3)} & -6Y_z^{(1)} + 8Y_z^{(3)} & 12Y_z^{(1)} + 8Y_z^{(3)} \\ -2Y_y^{(1)} - 2Y_y^{(2)} + Y_y^{(3)} & -6Y_y^{(1)} + Y_y^{(3)} & 8Y_y^{(1)} + 2Y_y^{(2)} + Y_y^{(3)} \\ -6Y_x^{(1)} + Y_x^{(3)} & -2Y_x^{(1)} - 2Y_x^{(2)} + Y_x^{(3)} & 8Y_x^{(1)} + 2Y_x^{(2)} + Y_x^{(3)} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B60)$$

$$\tilde{\sigma}_2^{(O/L)} = \begin{pmatrix} 8Y_x^{(4)} + 32Y_x^{(5)} + 2Y_x^{(6)} & -4Y_x^{(4)} - 16Y_x^{(5)} + 2Y_x^{(6)} & -4Y_x^{(4)} - 16Y_x^{(5)} + 2Y_x^{(6)} \\ 24Y_x^{(5)} & -12Y_x^{(5)} & -12Y_x^{(5)} \\ 24Y_x^{(5)} & -12Y_x^{(5)} & -12Y_x^{(5)} \\ 0 & 0 & 0 \\ -2Y_z^{(4)} - 2Y_z^{(5)} + Y_z^{(6)} & -2Y_z^{(4)} - 2Y_z^{(5)} + Y_z^{(6)} & 4Y_z^{(4)} + 4Y_z^{(5)} + Y_z^{(6)} \\ -2Y_y^{(4)} - 2Y_y^{(5)} + Y_y^{(6)} & 4Y_y^{(4)} + 4Y_y^{(5)} + Y_y^{(6)} & -2Y_y^{(4)} - 2Y_y^{(5)} + Y_y^{(6)} \\ -12Y_y^{(5)} & 24Y_y^{(5)} & -12Y_y^{(5)} \\ -4Y_y^{(4)} - 16Y_y^{(5)} + 2Y_y^{(6)} & 8Y_y^{(4)} + 32Y_y^{(5)} + 2Y_y^{(6)} & -4Y_y^{(4)} - 16Y_y^{(5)} + 2Y_y^{(6)} \\ -12Y_y^{(5)} & 24Y_y^{(5)} & -12Y_y^{(5)} \\ -2Y_z^{(4)} - 2Y_z^{(5)} + Y_z^{(6)} & -2Y_z^{(4)} - 2Y_z^{(5)} + Y_z^{(6)} & 4Y_z^{(4)} + 4Y_z^{(5)} + Y_z^{(6)} \\ 0 & 0 & 0 \\ 4Y_x^{(4)} + 4Y_x^{(5)} + Y_x^{(6)} & -2Y_x^{(4)} - 2Y_x^{(5)} + Y_x^{(6)} & -2Y_x^{(4)} - 2Y_x^{(5)} + Y_x^{(6)} \\ -12Y_z^{(5)} & -12Y_z^{(5)} & 24Y_z^{(5)} \\ -12Y_z^{(5)} & -12Y_z^{(5)} & 24Y_z^{(5)} \\ -4Y_z^{(4)} - 16Y_z^{(5)} + 2Y_z^{(6)} & -4Y_z^{(4)} - 16Y_z^{(5)} + 2Y_z^{(6)} & 8Y_z^{(4)} + 32Y_z^{(5)} + 2Y_z^{(6)} \\ -2Y_y^{(4)} - 2Y_y^{(5)} + Y_y^{(6)} & 4Y_y^{(4)} + 4Y_y^{(5)} + Y_y^{(6)} & -2Y_y^{(4)} - 2Y_y^{(5)} + Y_y^{(6)} \\ 4Y_x^{(4)} + 4Y_x^{(5)} + Y_x^{(6)} & -2Y_x^{(4)} - 2Y_x^{(5)} + Y_x^{(6)} & -2Y_x^{(4)} - 2Y_x^{(5)} + Y_x^{(6)} \\ 0 & 0 & 0 \end{pmatrix}. \quad (B61)$$

$$\tilde{\sigma}_{ij}^{mna(O/T)} = \tilde{\sigma}_1^{(O/T)} + \tilde{\sigma}_2^{(O/T)}, \quad (\text{B62})$$

$$\tilde{\sigma}_1^{(O/T)} = \begin{pmatrix} 0 & -6Y_z^{(1)} & -6Y_y^{(1)} \\ 0 & -2Y_z^{(1)} - 2Y_z^{(2)} & 8Y_y^{(1)} + 2Y_y^{(2)} \\ 0 & 8Y_z^{(1)} + 2Y_z^{(2)} & -2Y_y^{(1)} - 2Y_y^{(2)} \\ 5Y_x^{(1)} - 4Y_x^{(2)} & 5Y_y^{(1)} + 2Y_y^{(2)} & 5Y_z^{(1)} + 2Y_z^{(2)} \\ -2Y_y^{(1)} + Y_y^{(2)} & 8Y_x^{(1)} - Y_x^{(2)} & 0 \\ -2Y_z^{(1)} + Y_z^{(2)} & 0 & 8Y_x^{(1)} - Y_x^{(2)} \\ -2Y_z^{(1)} - 2Y_z^{(2)} & 0 & 8Y_x^{(1)} + 2Y_x^{(2)} \\ -6Y_z^{(1)} & 0 & -6Y_x^{(1)} \\ 8Y_z^{(1)} + 2Y_z^{(2)} & 0 & -2Y_x^{(1)} - 2Y_x^{(2)} \\ 8Y_y^{(1)} - Y_y^{(2)} & -2Y_x^{(1)} + Y_x^{(2)} & 0 \\ 5Y_x^{(1)} + 2Y_x^{(2)} & 5Y_y^{(1)} - 4Y_y^{(2)} & 5Y_z^{(1)} + 2Y_z^{(2)} \\ 0 & -2Y_z^{(1)} + Y_z^{(2)} & 8Y_y^{(1)} - Y_y^{(2)} \\ -2Y_y^{(1)} - 2Y_y^{(2)} & 8Y_x^{(1)} + 2Y_x^{(2)} & 0 \\ 8Y_y^{(1)} + 2Y_y^{(2)} & -2Y_x^{(1)} - 2Y_x^{(2)} & 0 \\ -6Y_y^{(1)} & -6Y_x^{(1)} & 0 \\ 8Y_z^{(1)} - Y_z^{(2)} & 0 & -2Y_x^{(1)} + Y_x^{(2)} \\ 0 & 8Y_z^{(1)} - Y_z^{(2)} & -2Y_y^{(1)} + Y_y^{(2)} \\ 5Y_x^{(1)} + 2Y_x^{(2)} & 5Y_y^{(1)} + 2Y_y^{(2)} & 5Y_z^{(1)} - 4Y_z^{(2)} \end{pmatrix}, \quad (\text{B63})$$

$$\tilde{\sigma}_2^{(O/T)} = \begin{pmatrix} 0 & 6Y_z^{(4)} + 24Y_z^{(5)} & 6Y_y^{(4)} + 24Y_y^{(5)} \\ 0 & 18Y_z^{(5)} & 18Y_y^{(5)} \\ 0 & 18Y_z^{(5)} & 18Y_y^{(5)} \\ 0 & 0 & 0 \\ 3Y_y^{(4)} + 3Y_y^{(5)} & 3Y_x^{(4)} + 3Y_x^{(5)} & 0 \\ 3Y_z^{(4)} + 3Y_z^{(5)} & 0 & 3Y_x^{(4)} + 3Y_x^{(5)} \\ 18Y_z^{(5)} & 0 & 18Y_x^{(5)} \\ 6Y_z^{(4)} + 24Y_z^{(5)} & 0 & 6Y_x^{(4)} + 24Y_x^{(5)} \\ 18Y_z^{(5)} & 0 & 18Y_x^{(5)} \\ 3Y_y^{(4)} + 3Y_y^{(5)} & 3Y_x^{(4)} + 3Y_x^{(5)} & 0 \\ 0 & 0 & 0 \\ 0 & 3Y_z^{(4)} + 3Y_z^{(5)} & 3Y_y^{(4)} + 3Y_y^{(5)} \\ 18Y_y^{(5)} & 18Y_x^{(5)} & 0 \\ 18Y_y^{(5)} & 18Y_x^{(5)} & 0 \\ 6Y_y^{(4)} + 24Y_y^{(5)} & 6Y_x^{(4)} + 24Y_x^{(5)} & 0 \\ 3Y_z^{(4)} + 3Y_z^{(5)} & 0 & 3Y_x^{(4)} + 3Y_x^{(5)} \\ 0 & 3Y_z^{(4)} + 3Y_z^{(5)} & 3Y_y^{(4)} + 3Y_y^{(5)} \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{B64})$$

where the multipole components with superscript (3) and (6) of $\tilde{\sigma}_{ij}^{mn\alpha(O/T)}$ do not appear.

$$\tilde{\sigma}_{ij}^{mn\alpha(H)} = \begin{pmatrix} 0 & -2Y_z^{(1)} - 8Y_z^{(2)} & 2Y_y^{(1)} + 8Y_y^{(2)} \\ 0 & -6Y_z^{(2)} - 2Y_z^{(3)} & 6Y_y^{(2)} - 2Y_y^{(3)} \\ 0 & -6Y_z^{(2)} + 2Y_z^{(3)} & 6Y_y^{(2)} + 2Y_y^{(3)} \\ 0 & 2Y_y^{(3)} & -2Y_z^{(3)} \\ -Y_y^{(1)} - Y_y^{(2)} + Y_y^{(3)} & Y_x^{(1)} + Y_x^{(2)} + Y_x^{(3)} & 0 \\ Y_z^{(1)} + Y_z^{(2)} - Y_z^{(3)} & 0 & -Y_x^{(1)} - Y_x^{(2)} - Y_x^{(3)} \\ 6Y_z^{(2)} + 2Y_z^{(3)} & 0 & -6Y_x^{(2)} + 2Y_x^{(3)} \\ 2Y_z^{(1)} + 8Y_z^{(2)} & 0 & -2Y_x^{(1)} - 8Y_x^{(2)} \\ 6Y_z^{(2)} - 2Y_z^{(3)} & 0 & -6Y_x^{(2)} - 2Y_x^{(3)} \\ -Y_y^{(1)} - Y_y^{(2)} - Y_y^{(3)} & Y_x^{(1)} + Y_x^{(2)} - Y_x^{(3)} & 0 \\ -2Y_x^{(3)} & 0 & 2Y_z^{(3)} \\ 0 & -Y_z^{(1)} - Y_z^{(2)} + Y_z^{(3)} & Y_y^{(1)} + Y_y^{(2)} + Y_y^{(3)} \\ -6Y_y^{(2)} - 2Y_y^{(3)} & 6Y_x^{(2)} - 2Y_x^{(3)} & 0 \\ -6Y_y^{(2)} + 2Y_y^{(3)} & 6Y_x^{(2)} + 2Y_x^{(3)} & 0 \\ -2Y_y^{(1)} - 8Y_y^{(2)} & 2Y_x^{(1)} + 8Y_x^{(2)} & 0 \\ Y_z^{(1)} + Y_z^{(2)} + Y_z^{(3)} & 0 & -Y_x^{(1)} - Y_x^{(2)} + Y_x^{(3)} \\ 0 & -Y_z^{(1)} - Y_z^{(2)} - Y_z^{(3)} & Y_y^{(1)} + Y_y^{(2)} - Y_y^{(3)} \\ 2Y_x^{(3)} & -2Y_y^{(3)} & 0 \end{pmatrix}. \quad (B65)$$

12. rank-2 c -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mna(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)}, \quad (B66)$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} 0 & 4X_{yz}^{(1)} + 16X_{yz}^{(2)} & -4X_{yz}^{(1)} - 16X_{yz}^{(2)} \\ 2X_{yz}^{(3)} & 12X_{yz}^{(2)} + 2X_{yz}^{(3)} & -12X_{yz}^{(2)} + 2X_{yz}^{(3)} \\ -2X_{yz}^{(3)} & 12X_{yz}^{(2)} - 2X_{yz}^{(3)} & -12X_{yz}^{(2)} - 2X_{yz}^{(3)} \\ X_u^{(3)} + X_v^{(3)} & X_u^{(3)} + X_v^{(3)} & X_u^{(3)} + X_v^{(3)} \\ 2X_{xy}^{(1)} + 2X_{xy}^{(2)} - X_{xy}^{(3)} & -2X_{xy}^{(1)} - 2X_{xy}^{(2)} - X_{xy}^{(3)} & -X_{xy}^{(3)} \\ -2X_{zx}^{(1)} - 2X_{zx}^{(2)} + X_{zx}^{(3)} & X_{zx}^{(3)} & 2X_{zx}^{(1)} + 2X_{zx}^{(2)} + X_{zx}^{(3)} \\ -12X_{zx}^{(2)} - 2X_{zx}^{(3)} & -2X_{zx}^{(3)} & 12X_{zx}^{(2)} - 2X_{zx}^{(3)} \\ -4X_{zx}^{(1)} - 16X_{zx}^{(2)} & 0 & 4X_{zx}^{(1)} + 16X_{zx}^{(2)} \\ -12X_{zx}^{(2)} + 2X_{zx}^{(3)} & 2X_{zx}^{(3)} & 12X_{zx}^{(2)} + 2X_{zx}^{(3)} \\ 2X_{xy}^{(1)} + 2X_{xy}^{(2)} + X_{xy}^{(3)} & -2X_{xy}^{(1)} - 2X_{xy}^{(2)} + X_{xy}^{(3)} & X_{xy}^{(3)} \\ -X_u^{(3)} + X_v^{(3)} & -X_u^{(3)} + X_v^{(3)} & -X_u^{(3)} + X_v^{(3)} \\ -X_{yz}^{(3)} & 2X_{yz}^{(1)} + 2X_{yz}^{(2)} - X_{yz}^{(3)} & -2X_{yz}^{(1)} - 2X_{yz}^{(2)} - X_{yz}^{(3)} \\ 12X_{xy}^{(2)} + 2X_{xy}^{(3)} & -12X_{xy}^{(2)} + 2X_{xy}^{(3)} & 2X_{xy}^{(3)} \\ 12X_{xy}^{(2)} - 2X_{xy}^{(3)} & -12X_{xy}^{(2)} - 2X_{xy}^{(3)} & -2X_{xy}^{(3)} \\ 4X_{xy}^{(1)} + 16X_{xy}^{(2)} & -4X_{xy}^{(1)} - 16X_{xy}^{(2)} & 0 \\ -2X_{zx}^{(1)} - 2X_{zx}^{(2)} - X_{zx}^{(3)} & -X_{zx}^{(3)} & 2X_{zx}^{(1)} + 2X_{zx}^{(2)} - X_{zx}^{(3)} \\ X_{yz}^{(3)} & 2X_{yz}^{(1)} + 2X_{yz}^{(2)} + X_{yz}^{(3)} & -2X_{yz}^{(1)} - 2X_{yz}^{(2)} + X_{yz}^{(3)} \\ -2X_v^{(3)} & -2X_v^{(3)} & -2X_v^{(3)} \end{pmatrix}, \quad (B67)$$

$$\tilde{\sigma}_2^{(O/L)} = \begin{pmatrix} 0 & -3X_{yz}^{(5)} & 3X_{yz}^{(5)} \\ -8X_{yz}^{(4)} - 5X_{yz}^{(5)} & 4X_{yz}^{(4)} + 4X_{yz}^{(5)} & 4X_{yz}^{(4)} + X_{yz}^{(5)} \\ 8X_{yz}^{(4)} + 5X_{yz}^{(5)} & -4X_{yz}^{(4)} - X_{yz}^{(5)} & -4X_{yz}^{(4)} - 4X_{yz}^{(5)} \\ -4X_u^{(4)} - 5X_u^{(5)} - 4X_v^{(4)} - 5X_v^{(5)} & 5X_u^{(5)} + 8X_v^{(4)} - 5X_v^{(5)} & 4X_u^{(4)} - 4X_v^{(4)} + 10X_v^{(5)} \\ -2X_{xy}^{(4)} + 4X_{xy}^{(5)} & -2X_{xy}^{(4)} + X_{xy}^{(5)} & 4X_{xy}^{(4)} - 5X_{xy}^{(5)} \\ 2X_{zx}^{(4)} - 4X_{zx}^{(5)} & -4X_{zx}^{(4)} + 5X_{zx}^{(5)} & 2X_{zx}^{(4)} - X_{zx}^{(5)} \\ -4X_{zx}^{(4)} - 4X_{zx}^{(5)} & 8X_{zx}^{(4)} + 5X_{zx}^{(5)} & -4X_{zx}^{(4)} - X_{zx}^{(5)} \\ 3X_{zx}^{(5)} & 0 & -3X_{zx}^{(5)} \\ 4X_{zx}^{(4)} + X_{zx}^{(5)} & -8X_{zx}^{(4)} - 5X_{zx}^{(5)} & 4X_{zx}^{(4)} + 4X_{zx}^{(5)} \\ 2X_{xy}^{(4)} - X_{xy}^{(5)} & 2X_{xy}^{(4)} - 4X_{xy}^{(5)} & -4X_{xy}^{(4)} + 5X_{xy}^{(5)} \\ -5X_u^{(5)} + 8X_v^{(4)} - 5X_v^{(5)} & 4X_u^{(4)} + 5X_u^{(5)} - 4X_v^{(4)} - 5X_v^{(5)} & -4X_u^{(4)} - 4X_v^{(4)} + 10X_v^{(5)} \\ 4X_{yz}^{(4)} - 5X_{yz}^{(5)} & -2X_{yz}^{(4)} + 4X_{yz}^{(5)} & -2X_{yz}^{(4)} + X_{yz}^{(5)} \\ 4X_{xy}^{(4)} + 4X_{xy}^{(5)} & 4X_{xy}^{(4)} + X_{xy}^{(5)} & -8X_{xy}^{(4)} - 5X_{xy}^{(5)} \\ -4X_{xy}^{(4)} - X_{xy}^{(5)} & -4X_{xy}^{(4)} - 4X_{xy}^{(5)} & 8X_{xy}^{(4)} + 5X_{xy}^{(5)} \\ -3X_{xy}^{(5)} & 3X_{xy}^{(5)} & 0 \\ -2X_{zx}^{(4)} + X_{zx}^{(5)} & 4X_{zx}^{(4)} - 5X_{zx}^{(5)} & -2X_{zx}^{(4)} + 4X_{zx}^{(5)} \\ -4X_{yz}^{(4)} + 5X_{yz}^{(5)} & 2X_{yz}^{(4)} - X_{yz}^{(5)} & 2X_{yz}^{(4)} - 4X_{yz}^{(5)} \\ 4X_u^{(4)} - 5X_u^{(5)} - 4X_v^{(4)} - 5X_v^{(5)} & -4X_u^{(4)} + 5X_u^{(5)} - 4X_v^{(4)} - 5X_v^{(5)} & 8X_v^{(4)} + 10X_v^{(5)} \end{pmatrix}. \quad (B68)$$

$$\tilde{\sigma}_{ij}^{mn\alpha(O/T)} = \tilde{\sigma}_1^{(O/T)} + \tilde{\sigma}_2^{(O/T)}, \quad (B69)$$

$$\tilde{\sigma}_1^{(O/T)} = \begin{pmatrix} 2X_u^{(1)} + 8X_u^{(2)} + 2X_v^{(1)} + 8X_v^{(2)} & -2X_{xy}^{(1)} - 8X_{xy}^{(2)} & 2X_{zx}^{(1)} + 8X_{zx}^{(2)} \\ 6X_u^{(2)} + 6X_v^{(2)} & -6X_{xy}^{(2)} & 6X_{zx}^{(2)} \\ 6X_u^{(2)} + 6X_v^{(2)} & -6X_{xy}^{(2)} & 6X_{zx}^{(2)} \\ 0 & 0 & 0 \\ -X_{zx}^{(1)} - X_{zx}^{(2)} & X_{yz}^{(1)} + X_{yz}^{(2)} & -2X_v^{(1)} - 2X_v^{(2)} \\ X_{xy}^{(1)} + X_{xy}^{(2)} & -X_u^{(1)} - X_u^{(2)} + X_v^{(1)} + X_v^{(2)} & -X_{yz}^{(1)} - X_{yz}^{(2)} \\ 6X_{xy}^{(2)} & -6X_u^{(2)} + 6X_v^{(2)} & -6X_{yz}^{(2)} \\ 2X_{xy}^{(1)} + 8X_{xy}^{(2)} & -2X_u^{(1)} - 8X_u^{(2)} + 2X_v^{(1)} + 8X_v^{(2)} & -2X_{yz}^{(1)} - 8X_{yz}^{(2)} \\ 6X_{xy}^{(2)} & -6X_u^{(2)} + 6X_v^{(2)} & -6X_{yz}^{(2)} \\ -X_{zx}^{(1)} - X_{zx}^{(2)} & X_{yz}^{(1)} + X_{yz}^{(2)} & -2X_v^{(1)} - 2X_v^{(2)} \\ 0 & 0 & 0 \\ X_u^{(1)} + X_u^{(2)} + X_v^{(1)} + X_v^{(2)} & -X_{xy}^{(1)} - X_{xy}^{(2)} & X_{zx}^{(1)} + X_{zx}^{(2)} \\ -6X_{zx}^{(2)} & 6X_{yz}^{(2)} & -12X_v^{(2)} \\ -6X_{zx}^{(2)} & 6X_{yz}^{(2)} & -12X_v^{(2)} \\ -2X_{zx}^{(1)} - 8X_{zx}^{(2)} & 2X_{yz}^{(1)} + 8X_{yz}^{(2)} & -4X_v^{(1)} - 16X_v^{(2)} \\ X_{xy}^{(1)} + X_{xy}^{(2)} & -X_u^{(1)} - X_u^{(2)} + X_v^{(1)} + X_v^{(2)} & -X_{yz}^{(1)} - X_{yz}^{(2)} \\ X_u^{(1)} + X_u^{(2)} + X_v^{(1)} + X_v^{(2)} & -X_{xy}^{(1)} - X_{xy}^{(2)} & X_{zx}^{(1)} + X_{zx}^{(2)} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B70)$$

$$\tilde{\sigma}_2^{(O/T)} = \begin{pmatrix} -3X_u^{(5)} - 3X_v^{(5)} & -6X_{xy}^{(5)} & 6X_{zx}^{(5)} \\ 2X_u^{(4)} - X_u^{(5)} - 6X_v^{(4)} + 9X_v^{(5)} & 6X_{xy}^{(4)} + 3X_{xy}^{(5)} & 6X_{zx}^{(4)} - 3X_{zx}^{(5)} \\ -2X_u^{(4)} + 4X_u^{(5)} + 6X_v^{(4)} - 6X_v^{(5)} & -6X_{xy}^{(4)} + 3X_{xy}^{(5)} & -6X_{zx}^{(4)} - 3X_{zx}^{(5)} \\ 0 & 6X_{zx}^{(4)} & -6X_{xy}^{(4)} \\ -3X_{zx}^{(4)} + 3X_{zx}^{(5)} & -3X_{yz}^{(4)} - 3X_{yz}^{(5)} & 2X_u^{(4)} + 5X_u^{(5)} - 3X_v^{(5)} \\ 3X_{xy}^{(4)} - 3X_{xy}^{(5)} & X_u^{(4)} + X_u^{(5)} + 3X_v^{(4)} + 9X_v^{(5)} & 3X_{yz}^{(4)} + 3X_{yz}^{(5)} \\ -6X_{xy}^{(4)} - 3X_{xy}^{(5)} & -2X_u^{(4)} + X_u^{(5)} - 6X_v^{(4)} + 9X_v^{(5)} & -6X_{yz}^{(4)} + 3X_{yz}^{(5)} \\ 6X_{xy}^{(5)} & 3X_u^{(5)} - 3X_v^{(5)} & -6X_{yz}^{(5)} \\ 6X_{xy}^{(4)} - 3X_{xy}^{(5)} & 2X_u^{(4)} - 4X_u^{(5)} + 6X_v^{(4)} - 6X_v^{(5)} & 6X_{yz}^{(4)} + 3X_{yz}^{(5)} \\ 3X_{zx}^{(4)} + 3X_{zx}^{(5)} & 3X_{yz}^{(4)} - 3X_{yz}^{(5)} & -2X_u^{(4)} - 5X_u^{(5)} - 3X_v^{(5)} \\ -6X_{yz}^{(4)} & 0 & 6X_{xy}^{(4)} \\ -X_u^{(4)} - X_u^{(5)} + 3X_v^{(4)} + 9X_v^{(5)} & -3X_{xy}^{(4)} + 3X_{xy}^{(5)} & -3X_{zx}^{(4)} - 3X_{zx}^{(5)} \\ 6X_{zx}^{(4)} + 3X_{zx}^{(5)} & 6X_{yz}^{(4)} - 3X_{yz}^{(5)} & -4X_u^{(4)} + 5X_u^{(5)} - 3X_v^{(5)} \\ -6X_{zx}^{(4)} + 3X_{zx}^{(5)} & -6X_{yz}^{(4)} - 3X_{yz}^{(5)} & 4X_u^{(4)} - 5X_u^{(5)} - 3X_v^{(5)} \\ -6X_{zx}^{(5)} & 6X_{yz}^{(5)} & 6X_v^{(5)} \\ -3X_{xy}^{(4)} - 3X_{xy}^{(5)} & -X_u^{(4)} - 4X_u^{(5)} - 3X_v^{(4)} - 6X_v^{(5)} & -3X_{yz}^{(4)} + 3X_{yz}^{(5)} \\ X_u^{(4)} + 4X_u^{(5)} - 3X_v^{(4)} - 6X_v^{(5)} & 3X_{xy}^{(4)} + 3X_{xy}^{(5)} & 3X_{zx}^{(4)} - 3X_{zx}^{(5)} \\ 6X_{yz}^{(4)} & -6X_{zx}^{(4)} & 0 \end{pmatrix}, \quad (B71)$$

where the multipole component with superscript (3) of $\tilde{\sigma}_{ij}^{mna(O/T)}$ does not appear.

$$\begin{aligned} \tilde{\sigma}_{ij}^{mna(H)} &= \tilde{\sigma}_1^{(H)} + \tilde{\sigma}_2^{(H)}, \\ \tilde{\sigma}_1^{(H)} &= \begin{pmatrix} 2X_u^{(1)} + 8X_u^{(2)} - 2X_v^{(1)} - 8X_v^{(2)} & -2X_{xy}^{(1)} - 8X_{xy}^{(2)} & -2X_{zx}^{(1)} - 8X_{zx}^{(2)} \\ 6X_u^{(2)} - 6X_v^{(2)} & -6X_{xy}^{(2)} & -6X_{zx}^{(2)} \\ 6X_u^{(2)} - 6X_v^{(2)} & -6X_{xy}^{(2)} & -6X_{zx}^{(2)} \\ 0 & 0 & 0 \\ -X_{zx}^{(1)} - X_{zx}^{(2)} & -X_{yz}^{(1)} - X_{yz}^{(2)} & -2X_u^{(1)} - 2X_u^{(2)} \\ -X_{xy}^{(1)} - X_{xy}^{(2)} & X_u^{(1)} + X_u^{(2)} + X_v^{(1)} + X_v^{(2)} & -X_{yz}^{(1)} - X_{yz}^{(2)} \\ -6X_{xy}^{(2)} & 6X_u^{(2)} + 6X_v^{(2)} & -6X_{yz}^{(2)} \\ -2X_{xy}^{(1)} - 8X_{xy}^{(2)} & 2X_u^{(1)} + 8X_u^{(2)} + 2X_v^{(1)} + 8X_v^{(2)} & -2X_{yz}^{(1)} - 8X_{yz}^{(2)} \\ -6X_{xy}^{(2)} & 6X_u^{(2)} + 6X_v^{(2)} & -6X_{yz}^{(2)} \\ -X_{zx}^{(1)} - X_{zx}^{(2)} & -X_{yz}^{(1)} - X_{yz}^{(2)} & -2X_u^{(1)} - 2X_u^{(2)} \\ 0 & 0 & 0 \\ X_u^{(1)} + X_u^{(2)} - X_v^{(1)} - X_v^{(2)} & -X_{xy}^{(1)} - X_{xy}^{(2)} & -X_{zx}^{(1)} - X_{zx}^{(2)} \\ -6X_{zx}^{(2)} & -6X_{yz}^{(2)} & -12X_u^{(2)} \\ -6X_{zx}^{(2)} & -6X_{yz}^{(2)} & -12X_u^{(2)} \\ -2X_{zx}^{(1)} - 8X_{zx}^{(2)} & -2X_{yz}^{(1)} - 8X_{yz}^{(2)} & -4X_u^{(1)} - 16X_u^{(2)} \\ -X_{xy}^{(1)} - X_{xy}^{(2)} & X_u^{(1)} + X_u^{(2)} + X_v^{(1)} + X_v^{(2)} & -X_{yz}^{(1)} - X_{yz}^{(2)} \\ X_u^{(1)} + X_u^{(2)} - X_v^{(1)} - X_v^{(2)} & -X_{xy}^{(1)} - X_{xy}^{(2)} & -X_{zx}^{(1)} - X_{zx}^{(2)} \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (B72)$$

$$\begin{aligned} \tilde{\sigma}_2^{(H)} &= \begin{pmatrix} 3X_u^{(4)} - 9X_v^{(4)} & 6X_{xy}^{(4)} & 6X_{zx}^{(4)} \\ 2X_u^{(3)} + X_u^{(4)} + 2X_v^{(3)} + 7X_v^{(4)} & 2X_{xy}^{(3)} - 8X_{xy}^{(4)} & -2X_{zx}^{(3)} + 2X_{zx}^{(4)} \\ -2X_u^{(3)} - 4X_u^{(4)} - 2X_v^{(3)} + 2X_v^{(4)} & -2X_{xy}^{(3)} + 2X_{xy}^{(4)} & 2X_{zx}^{(3)} - 8X_{zx}^{(4)} \\ -4X_{yz}^{(3)} - 5X_{yz}^{(4)} & 2X_{zx}^{(3)} - 5X_{zx}^{(4)} & 2X_{xy}^{(3)} - 5X_{xy}^{(4)} \\ -X_{zx}^{(3)} - 8X_{zx}^{(4)} & X_{yz}^{(3)} + 2X_{yz}^{(4)} & 3X_u^{(4)} + 2X_v^{(3)} - 5X_v^{(4)} \\ -X_{xy}^{(3)} - 8X_{xy}^{(4)} & -X_u^{(3)} + X_u^{(4)} + X_v^{(3)} - 7X_v^{(4)} & X_{yz}^{(3)} + 2X_{yz}^{(4)} \\ 2X_{xy}^{(3)} - 8X_{xy}^{(4)} & 2X_u^{(3)} + X_u^{(4)} - 2X_v^{(3)} - 7X_v^{(4)} & -2X_{yz}^{(3)} + 2X_{yz}^{(4)} \\ 6X_{xy}^{(4)} & 3X_u^{(4)} + 9X_v^{(4)} & 6X_{yz}^{(4)} \\ -2X_{xy}^{(3)} + 2X_{xy}^{(4)} & -2X_u^{(3)} - 4X_u^{(4)} + 2X_v^{(3)} - 2X_v^{(4)} & 2X_{yz}^{(3)} - 8X_{yz}^{(4)} \\ X_{zx}^{(3)} + 2X_{zx}^{(4)} & -X_{yz}^{(3)} - 8X_{yz}^{(4)} & 3X_u^{(4)} - 2X_v^{(3)} + 5X_v^{(4)} \\ 2X_{yz}^{(3)} - 5X_{yz}^{(4)} & -4X_{zx}^{(3)} - 5X_{zx}^{(4)} & 2X_{xy}^{(3)} - 5X_{xy}^{(4)} \\ -X_u^{(3)} + X_u^{(4)} - X_v^{(3)} + 7X_v^{(4)} & -X_{xy}^{(3)} - 8X_{xy}^{(4)} & X_{zx}^{(3)} + 2X_{zx}^{(4)} \\ 2X_{zx}^{(3)} - 8X_{zx}^{(4)} & -2X_{yz}^{(3)} + 2X_{yz}^{(4)} & 3X_u^{(4)} - 4X_v^{(3)} - 5X_v^{(4)} \\ -2X_{zx}^{(3)} + 2X_{zx}^{(4)} & 2X_{yz}^{(3)} - 8X_{yz}^{(4)} & 3X_u^{(4)} + 4X_v^{(3)} + 5X_v^{(4)} \\ 6X_{zx}^{(4)} & 6X_{yz}^{(4)} & -6X_u^{(4)} \\ X_{xy}^{(3)} + 2X_{xy}^{(4)} & X_u^{(3)} - 4X_u^{(4)} - X_v^{(3)} - 2X_v^{(4)} & -X_{yz}^{(3)} - 8X_{yz}^{(4)} \\ X_u^{(3)} - 4X_u^{(4)} + X_v^{(3)} + 2X_v^{(4)} & X_{xy}^{(3)} + 2X_{xy}^{(4)} & -X_{zx}^{(3)} - 8X_{zx}^{(4)} \\ 2X_{yz}^{(3)} - 5X_{yz}^{(4)} & 2X_{zx}^{(3)} - 5X_{zx}^{(4)} & -4X_{xy}^{(3)} - 5X_{xy}^{(4)} \end{pmatrix}. \end{aligned} \quad (B73)$$

13. rank-3 c -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mn\alpha(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)} + \tilde{\sigma}_3^{(O/L)}, \quad (B74)$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} -4Y_x^{\alpha(1)} - 16Y_x^{\alpha(2)} & 2Y_x^{\alpha(1)} + 8Y_x^{\alpha(2)} - 2Y_x^{\beta(1)} + 4Y_x^{\beta(2)} & 2Y_x^{\alpha(1)} + 8Y_x^{\alpha(2)} + 2Y_x^{\beta(1)} - 4Y_x^{\beta(2)} \\ 8Y_x^{\alpha(2)} & 6Y_x^{\alpha(2)} - 2Y_x^{\beta(2)} & -14Y_x^{\alpha(2)} + 2Y_x^{\beta(2)} \\ 8Y_x^{\alpha(2)} & -14Y_x^{\alpha(2)} - 2Y_x^{\beta(2)} & 6Y_x^{\alpha(2)} + 2Y_x^{\beta(2)} \\ 0 & 0 & 0 \\ Y_z^{\alpha(1)} + 6Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & Y_z^{\alpha(1)} - 14Y_z^{\alpha(2)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} & -2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} \\ Y_y^{\alpha(1)} + 6Y_y^{\alpha(2)} + Y_y^{\beta(1)} + 2Y_y^{\beta(2)} & -2Y_y^{\alpha(1)} + 8Y_y^{\alpha(2)} & Y_y^{\alpha(1)} - 14Y_y^{\alpha(2)} - Y_y^{\beta(1)} - 2Y_y^{\beta(2)} \\ 6Y_y^{\alpha(2)} + 2Y_y^{\beta(2)} & 8Y_y^{\alpha(2)} & -14Y_y^{\alpha(2)} - 2Y_y^{\beta(2)} \\ 2Y_y^{\alpha(1)} + 8Y_y^{\alpha(2)} + 2Y_y^{\beta(1)} - 4Y_y^{\beta(2)} & -4Y_y^{\alpha(1)} - 16Y_y^{\alpha(2)} & 2Y_y^{\alpha(1)} + 8Y_y^{\alpha(2)} - 2Y_y^{\beta(1)} + 4Y_y^{\beta(2)} \\ -14Y_y^{\alpha(2)} + 2Y_y^{\beta(2)} & 8Y_y^{\alpha(2)} & 6Y_y^{\alpha(2)} - 2Y_y^{\beta(2)} \\ Y_z^{\alpha(1)} - 14Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & Y_z^{\alpha(1)} + 6Y_z^{\alpha(2)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} & -2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} \\ 0 & 0 & 0 \\ -2Y_x^{\alpha(1)} + 8Y_x^{\alpha(2)} & Y_x^{\alpha(1)} + 6Y_x^{\alpha(2)} - Y_x^{\beta(1)} - 2Y_x^{\beta(2)} & Y_x^{\alpha(1)} - 14Y_x^{\alpha(2)} + Y_x^{\beta(1)} + 2Y_x^{\beta(2)} \\ 6Y_z^{\alpha(2)} - 2Y_z^{\beta(2)} & -14Y_z^{\alpha(2)} + 2Y_z^{\beta(2)} & 8Y_z^{\alpha(2)} \\ -14Y_z^{\alpha(2)} - 2Y_z^{\beta(2)} & 6Y_z^{\alpha(2)} + 2Y_z^{\beta(2)} & 8Y_z^{\alpha(2)} \\ 2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} + 4Y_z^{\beta(2)} & 2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} - 4Y_z^{\beta(2)} & -4Y_z^{\alpha(1)} - 16Y_z^{\alpha(2)} \\ Y_y^{\alpha(1)} - 14Y_y^{\alpha(2)} + Y_y^{\beta(1)} + 2Y_y^{\beta(2)} & -2Y_y^{\alpha(1)} + 8Y_y^{\alpha(2)} & Y_y^{\alpha(1)} + 6Y_y^{\alpha(2)} - Y_y^{\beta(1)} - 2Y_y^{\beta(2)} \\ -2Y_x^{\alpha(1)} + 8Y_x^{\alpha(2)} & Y_x^{\alpha(1)} - 14Y_x^{\alpha(2)} - Y_x^{\beta(1)} - 2Y_x^{\beta(2)} & Y_x^{\alpha(1)} + 6Y_x^{\alpha(2)} + Y_x^{\beta(1)} + 2Y_x^{\beta(2)} \\ 0 & 0 & 0 \end{pmatrix}, \quad (B75)$$

$$\tilde{\sigma}_2^{(O/L)} = \begin{pmatrix} -16Y_x^{\alpha(4)} & 8Y_x^{\alpha(4)} - 8Y_x^{\beta(4)} & 8Y_x^{\alpha(4)} + 8Y_x^{\beta(4)} \\ -12Y_x^{\alpha(4)} - 4Y_x^{\beta(3)} & -2Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} + 2Y_x^{\beta(3)} - 6Y_x^{\beta(4)} & 2Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} + 2Y_x^{\beta(3)} + 6Y_x^{\beta(4)} \\ -12Y_x^{\alpha(4)} + 4Y_x^{\beta(3)} & 2Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} - 2Y_x^{\beta(3)} - 6Y_x^{\beta(4)} & -2Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} - 2Y_x^{\beta(3)} + 6Y_x^{\beta(4)} \\ -2Y_{xyz}^{(3)} & Y_{xyz}^{(3)} & Y_{xyz}^{(3)} \\ Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(3)} - Y_z^{\beta(4)} & -Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(3)} + Y_z^{\beta(4)} & -2Y_z^{\alpha(4)} + 2Y_z^{\beta(3)} \\ Y_y^{\alpha(3)} + Y_y^{\alpha(4)} + Y_y^{\beta(3)} + Y_y^{\beta(4)} & -2Y_y^{\alpha(4)} - 2Y_y^{\beta(3)} & -Y_y^{\alpha(3)} + Y_y^{\alpha(4)} + Y_y^{\beta(3)} - Y_y^{\beta(4)} \\ -2Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} - 2Y_y^{\beta(3)} + 6Y_y^{\beta(4)} & -12Y_y^{\alpha(4)} + 4Y_y^{\beta(3)} & 2Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} - 2Y_y^{\beta(3)} - 6Y_y^{\beta(4)} \\ 8Y_y^{\alpha(4)} + 8Y_y^{\beta(4)} & -16Y_y^{\alpha(4)} & 8Y_y^{\alpha(4)} - 8Y_y^{\beta(4)} \\ 2Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} + 2Y_y^{\beta(3)} + 6Y_y^{\beta(4)} & -12Y_y^{\alpha(4)} - 4Y_y^{\beta(3)} & -2Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} + 2Y_y^{\beta(3)} - 6Y_y^{\beta(4)} \\ -Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(3)} - Y_z^{\beta(4)} & Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(3)} + Y_z^{\beta(4)} & -2Y_z^{\alpha(4)} - 2Y_z^{\beta(3)} \\ Y_{xyz}^{(3)} & -2Y_{xyz}^{(3)} & Y_{xyz}^{(3)} \\ -2Y_x^{\alpha(4)} + 2Y_x^{\beta(3)} & Y_x^{\alpha(3)} + Y_x^{\alpha(4)} - Y_x^{\beta(3)} - Y_x^{\beta(4)} & -Y_x^{\alpha(3)} + Y_x^{\alpha(4)} - Y_x^{\beta(3)} + Y_x^{\beta(4)} \\ -2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 2Y_z^{\beta(3)} - 6Y_z^{\beta(4)} & 2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 2Y_z^{\beta(3)} + 6Y_z^{\beta(4)} & -12Y_z^{\alpha(4)} - 4Y_z^{\beta(3)} \\ 2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 2Y_z^{\beta(3)} - 6Y_z^{\beta(4)} & -2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 2Y_z^{\beta(3)} + 6Y_z^{\beta(4)} & -12Y_z^{\alpha(4)} + 4Y_z^{\beta(3)} \\ 8Y_z^{\alpha(4)} - 8Y_z^{\beta(4)} & 8Y_z^{\alpha(4)} + 8Y_z^{\beta(4)} & -16Y_z^{\alpha(4)} \\ -Y_y^{\alpha(3)} + Y_y^{\alpha(4)} - Y_y^{\beta(3)} + Y_y^{\beta(4)} & -2Y_y^{\alpha(4)} + 2Y_y^{\beta(3)} & Y_y^{\alpha(3)} + Y_y^{\alpha(4)} - Y_y^{\beta(3)} - Y_y^{\beta(4)} \\ -2Y_x^{\alpha(4)} - 2Y_x^{\beta(3)} & -Y_x^{\alpha(3)} + Y_x^{\alpha(4)} + Y_x^{\beta(3)} - Y_x^{\beta(4)} & Y_x^{\alpha(3)} + Y_x^{\alpha(4)} + Y_x^{\beta(3)} + Y_x^{\beta(4)} \\ Y_{xyz}^{(3)} & Y_{xyz}^{(3)} & -2Y_{xyz}^{(3)} \end{pmatrix}, \quad (B76)$$

$$\tilde{\sigma}_3^{(O/L)} = \begin{pmatrix} -2Y_x^{\alpha(5)} & -2Y_x^{\alpha(5)} & -2Y_x^{\alpha(5)} \\ Y_x^{\alpha(5)} - Y_x^{\beta(5)} & Y_x^{\alpha(5)} - Y_x^{\beta(5)} & Y_x^{\alpha(5)} - Y_x^{\beta(5)} \\ Y_x^{\alpha(5)} + Y_x^{\beta(5)} & Y_x^{\alpha(5)} + Y_x^{\beta(5)} & Y_x^{\alpha(5)} + Y_x^{\beta(5)} \\ -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} \\ Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} \\ Y_y^{\alpha(5)} + Y_y^{\beta(5)} & Y_y^{\alpha(5)} + Y_y^{\beta(5)} & Y_y^{\alpha(5)} + Y_y^{\beta(5)} \\ Y_y^{\alpha(5)} + Y_y^{\beta(5)} & Y_y^{\alpha(5)} + Y_y^{\beta(5)} & Y_y^{\alpha(5)} + Y_y^{\beta(5)} \\ -2Y_y^{\alpha(5)} & -2Y_y^{\alpha(5)} & -2Y_y^{\alpha(5)} \\ Y_y^{\alpha(5)} - Y_y^{\beta(5)} & Y_y^{\alpha(5)} - Y_y^{\beta(5)} & Y_y^{\alpha(5)} - Y_y^{\beta(5)} \\ Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} \\ -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} \\ Y_x^{\alpha(5)} - Y_x^{\beta(5)} & Y_x^{\alpha(5)} - Y_x^{\beta(5)} & Y_x^{\alpha(5)} - Y_x^{\beta(5)} \\ Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} \\ Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} \\ -2Y_z^{\alpha(5)} & -2Y_z^{\alpha(5)} & -2Y_z^{\alpha(5)} \\ Y_y^{\alpha(5)} - Y_y^{\beta(5)} & Y_y^{\alpha(5)} - Y_y^{\beta(5)} & Y_y^{\alpha(5)} - Y_y^{\beta(5)} \\ Y_x^{\alpha(5)} + Y_x^{\beta(5)} & Y_x^{\alpha(5)} + Y_x^{\beta(5)} & Y_x^{\alpha(5)} + Y_x^{\beta(5)} \\ -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} \end{pmatrix}. \quad (B77)$$

$$\tilde{\sigma}_{ij}^{mna(O/T)} = \tilde{\sigma}_1^{(O/T)} + \tilde{\sigma}_2^{(O/T)} + \tilde{\sigma}_3^{(O/T)}, \quad (B78)$$

$$\tilde{\sigma}_1^{(O/T)} = \begin{pmatrix} -2Y_{xyz}^{(1)} + 2Y_{xyz}^{(2)} & 2Y_z^{\alpha(1)} + 3Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} - Y_z^{\beta(2)} & 2Y_y^{\alpha(1)} + 3Y_y^{\alpha(2)} + 2Y_y^{\beta(1)} + Y_y^{\beta(2)} \\ -Y_{xyz}^{(2)} & Y_z^{\alpha(2)} + 3Y_z^{\beta(2)} & -4Y_y^{\alpha(2)} + 2Y_y^{\beta(2)} \\ -Y_{xyz}^{(2)} & -4Y_z^{\alpha(2)} - 2Y_z^{\beta(2)} & Y_y^{\alpha(2)} - 3Y_y^{\beta(2)} \\ 10Y_x^{\alpha(2)} & 10Y_y^{\alpha(2)} & 10Y_z^{\alpha(2)} \\ Y_y^{\alpha(1)} + Y_y^{\alpha(2)} - Y_y^{\beta(1)} + 3Y_y^{\beta(2)} & Y_x^{\alpha(1)} - 4Y_x^{\alpha(2)} + Y_x^{\beta(1)} + 2Y_x^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ Y_z^{\alpha(1)} + Y_z^{\alpha(2)} + Y_z^{\beta(1)} - 3Y_z^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_x^{\alpha(1)} - 4Y_x^{\alpha(2)} - Y_x^{\beta(1)} - 2Y_x^{\beta(2)} \\ Y_z^{\alpha(2)} - 3Y_z^{\beta(2)} & -Y_{xyz}^{(2)} & -4Y_x^{\alpha(2)} - 2Y_x^{\beta(2)} \\ 2Y_z^{\alpha(1)} + 3Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} + Y_z^{\beta(2)} & -2Y_{xyz}^{(1)} + 2Y_{xyz}^{(2)} & 2Y_x^{\alpha(1)} + 3Y_x^{\alpha(2)} - 2Y_x^{\beta(1)} - Y_x^{\beta(2)} \\ -4Y_z^{\alpha(2)} + 2Y_z^{\beta(2)} & -Y_{xyz}^{(2)} & Y_x^{\alpha(2)} + 3Y_x^{\beta(2)} \\ Y_y^{\alpha(1)} - 4Y_y^{\alpha(2)} - Y_y^{\beta(1)} - 2Y_y^{\beta(2)} & Y_x^{\alpha(1)} + Y_x^{\alpha(2)} + Y_x^{\beta(1)} - 3Y_x^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ 10Y_x^{\alpha(2)} & 10Y_y^{\alpha(2)} & 10Y_z^{\alpha(2)} \\ -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_z^{\alpha(1)} + Y_z^{\alpha(2)} - Y_z^{\beta(1)} + 3Y_z^{\beta(2)} & Y_y^{\alpha(1)} - 4Y_y^{\alpha(2)} + Y_y^{\beta(1)} + 2Y_y^{\beta(2)} \\ Y_y^{\alpha(2)} + 3Y_y^{\beta(2)} & -4Y_x^{\alpha(2)} + 2Y_x^{\beta(2)} & -Y_{xyz}^{(2)} \\ -4Y_y^{\alpha(2)} - 2Y_y^{\beta(2)} & Y_x^{\alpha(2)} - 3Y_x^{\beta(2)} & -Y_{xyz}^{(2)} \\ 2Y_y^{\alpha(1)} + 3Y_y^{\alpha(2)} - 2Y_y^{\beta(1)} - Y_y^{\beta(2)} & 2Y_x^{\alpha(1)} + 3Y_x^{\alpha(2)} + 2Y_x^{\beta(1)} + Y_x^{\beta(2)} & -2Y_{xyz}^{(1)} + 2Y_{xyz}^{(2)} \\ Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_x^{\alpha(1)} + Y_x^{\alpha(2)} - Y_x^{\beta(1)} + 3Y_x^{\beta(2)} \\ -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & Y_y^{\alpha(1)} + Y_y^{\alpha(2)} + Y_y^{\beta(1)} - 3Y_y^{\beta(2)} \\ 10Y_x^{\alpha(2)} & 10Y_y^{\alpha(2)} & 10Y_z^{\alpha(2)} \end{pmatrix}, \quad (B79)$$

$$\tilde{\sigma}_2^{(O/T)} = \begin{pmatrix} -8Y_{xyz}^{(4)} & 8Y_z^{\alpha(4)} - 8Y_z^{\beta(4)} & 8Y_y^{\alpha(4)} + 8Y_y^{\beta(4)} \\ -6Y_{xyz}^{(4)} & 4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 6Y_z^{\beta(4)} & -4Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} + 6Y_y^{\beta(4)} \\ -6Y_{xyz}^{(4)} & -4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 6Y_z^{\beta(4)} & 4Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} + 6Y_y^{\beta(4)} \\ -2Y_x^{\alpha(3)} & Y_y^{\alpha(3)} - 3Y_y^{\beta(3)} & Y_z^{\alpha(3)} + 3Y_z^{\beta(3)} \\ -2Y_y^{\alpha(3)} + Y_y^{\alpha(4)} - Y_y^{\beta(4)} & 2Y_x^{\alpha(3)} + Y_x^{\alpha(4)} + Y_x^{\beta(4)} & -Y_{xyz}^{(4)} \\ -2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(4)} & -Y_{xyz}^{(4)} & 2Y_x^{\alpha(3)} + Y_x^{\alpha(4)} - Y_x^{\beta(4)} \\ 4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 6Y_z^{\beta(4)} & -6Y_{xyz}^{(4)} & -4Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} - 6Y_x^{\beta(4)} \\ 8Y_z^{\alpha(4)} + 8Y_z^{\beta(4)} & -8Y_{xyz}^{(4)} & 8Y_x^{\alpha(4)} - 8Y_x^{\beta(4)} \\ -4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 6Y_z^{\beta(4)} & -6Y_{xyz}^{(4)} & 4Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} - 6Y_x^{\beta(4)} \\ 2Y_y^{\alpha(3)} + Y_y^{\alpha(4)} - Y_y^{\beta(4)} & -2Y_x^{\alpha(3)} + Y_x^{\alpha(4)} + Y_x^{\beta(4)} & -Y_{xyz}^{(4)} \\ Y_x^{\alpha(3)} + 3Y_x^{\beta(3)} & -2Y_y^{\alpha(3)} & Y_z^{\alpha(3)} - 3Y_z^{\beta(3)} \\ -Y_{xyz}^{(4)} & -2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(4)} & 2Y_y^{\alpha(3)} + Y_y^{\alpha(4)} + Y_y^{\beta(4)} \\ 4Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} - 6Y_y^{\beta(4)} & -4Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} + 6Y_x^{\beta(4)} & -6Y_{xyz}^{(4)} \\ -4Y_y^{\alpha(3)} + 6Y_y^{\alpha(4)} - 6Y_y^{\beta(4)} & 4Y_x^{\alpha(3)} + 6Y_x^{\alpha(4)} + 6Y_x^{\beta(4)} & -6Y_{xyz}^{(4)} \\ 8Y_y^{\alpha(4)} - 8Y_y^{\beta(4)} & 8Y_x^{\alpha(4)} + 8Y_x^{\beta(4)} & -8Y_{xyz}^{(4)} \\ 2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(4)} & -Y_{xyz}^{(4)} & -2Y_x^{\alpha(3)} + Y_x^{\alpha(4)} - Y_x^{\beta(4)} \\ -Y_{xyz}^{(4)} & 2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(4)} & -2Y_y^{\alpha(3)} + Y_y^{\alpha(4)} + Y_y^{\beta(4)} \\ Y_x^{\alpha(3)} - 3Y_x^{\beta(3)} & Y_y^{\alpha(3)} + 3Y_y^{\beta(3)} & -2Y_z^{\alpha(3)} \end{pmatrix}, \quad (B80)$$

where the multipole component with superscript (5) of $\tilde{\sigma}_{ij}^{mn\alpha(\text{O/T})}$ does not appear.

$$\tilde{\sigma}_{ij}^{mn\alpha(\text{H})} = \begin{pmatrix} 0 & -3Y_z^{\alpha(2)} + 3Y_z^{\beta(2)} & 3Y_y^{\alpha(2)} + 3Y_y^{\beta(2)} \\ 2Y_{xy}^{(1)} - Y_{xyz}^{(2)} & -2Y_z^{\alpha(1)} - Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} - Y_z^{\beta(2)} & -2Y_y^{\alpha(1)} - 4Y_y^{\alpha(2)} + 2Y_y^{\beta(1)} - 2Y_y^{\beta(2)} \\ -2Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & 2Y_z^{\alpha(1)} + 4Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & 2Y_y^{\alpha(1)} + Y_y^{\alpha(2)} - 2Y_y^{\beta(1)} - Y_y^{\beta(2)} \\ -2Y_x^{\beta(1)} + 2Y_x^{\beta(2)} & -3Y_y^{\alpha(1)} + Y_y^{\beta(1)} + 2Y_y^{\beta(2)} & 3Y_z^{\alpha(1)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} \\ Y_y^{\alpha(1)} - Y_y^{\alpha(2)} + Y_y^{\beta(1)} - Y_y^{\beta(2)} & Y_x^{\alpha(1)} - 4Y_x^{\alpha(2)} - Y_x^{\beta(1)} - 2Y_x^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ -Y_z^{\alpha(1)} + Y_z^{\alpha(2)} + Y_z^{\beta(1)} - Y_z^{\beta(2)} & Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & -Y_x^{\alpha(1)} + 4Y_x^{\alpha(2)} - Y_x^{\beta(1)} - 2Y_x^{\beta(2)} \\ 2Y_z^{\alpha(1)} + Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} - Y_z^{\beta(2)} & -2Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & 2Y_x^{\alpha(1)} + 4Y_x^{\alpha(2)} + 2Y_x^{\beta(1)} - 2Y_x^{\beta(2)} \\ 3Y_z^{\alpha(2)} + 3Y_z^{\beta(2)} & 0 & -3Y_x^{\alpha(2)} + 3Y_x^{\beta(2)} \\ -2Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & 2Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & -2Y_x^{\alpha(1)} - Y_x^{\alpha(2)} - 2Y_x^{\beta(1)} - Y_x^{\beta(2)} \\ -Y_y^{\alpha(1)} + 4Y_y^{\alpha(2)} - Y_y^{\beta(1)} - 2Y_y^{\beta(2)} & -Y_x^{\alpha(1)} + Y_x^{\alpha(2)} + Y_x^{\beta(1)} - Y_x^{\beta(2)} & Y_{xyz}^{(1)} + Y_{xyz}^{(2)} \\ 3Y_x^{\alpha(1)} + Y_x^{\beta(1)} + 2Y_x^{\beta(2)} & -2Y_y^{\beta(1)} + 2Y_y^{\beta(2)} & -3Y_z^{\alpha(1)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} \\ -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_z^{\alpha(1)} - Y_z^{\alpha(2)} + Y_z^{\beta(1)} - Y_z^{\beta(2)} & Y_y^{\alpha(1)} - 4Y_y^{\alpha(2)} - Y_y^{\beta(1)} - 2Y_y^{\beta(2)} \\ -2Y_y^{\alpha(1)} - Y_y^{\alpha(2)} - 2Y_y^{\beta(1)} - Y_y^{\beta(2)} & -2Y_x^{\alpha(1)} - 4Y_x^{\alpha(2)} + 2Y_x^{\beta(1)} - 2Y_x^{\beta(2)} & 2Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ 2Y_y^{\alpha(1)} + 4Y_y^{\alpha(2)} + 2Y_y^{\beta(1)} - 2Y_y^{\beta(2)} & 2Y_x^{\alpha(1)} + Y_x^{\alpha(2)} - 2Y_x^{\beta(1)} - Y_x^{\beta(2)} & -2Y_{xyz}^{(1)} + Y_{xyz}^{(2)} \\ -3Y_y^{\alpha(2)} + 3Y_y^{\beta(2)} & 3Y_x^{\alpha(2)} + 3Y_x^{\beta(2)} & 0 \\ Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_x^{\alpha(1)} - Y_x^{\alpha(2)} + Y_x^{\beta(1)} - Y_x^{\beta(2)} \\ Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & -Y_z^{\alpha(1)} + 4Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & -Y_y^{\alpha(1)} + Y_y^{\alpha(2)} + Y_y^{\beta(1)} - Y_y^{\beta(2)} \\ -3Y_x^{\alpha(1)} + Y_x^{\beta(1)} + 2Y_x^{\beta(2)} & 3Y_y^{\alpha(1)} + Y_y^{\beta(1)} + 2Y_y^{\beta(2)} & -2Y_z^{\beta(1)} + 2Y_z^{\beta(2)} \end{pmatrix}. \quad (\text{B81})$$

14. rank-4 *c*-multipoles with normalization

$$\tilde{\sigma}_{ij}^{mn\alpha(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)}, \quad (B82)$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} 0 & 0 & 0 \\ -4X_{4x}^{\beta(1)} & -2X_{4x}^{\alpha(1)} + 2X_{4x}^{\beta(1)} & 2X_{4x}^{\alpha(1)} + 2X_{4x}^{\beta(1)} \\ 4X_{4x}^{\beta(1)} & 2X_{4x}^{\alpha(1)} - 2X_{4x}^{\beta(1)} & -2X_{4x}^{\alpha(1)} - 2X_{4x}^{\beta(1)} \\ -3X_{4u}^{(1)} - X_{4v}^{(1)} & 2X_{4v}^{(1)} + 3X_{4v}^{(1)} & 3X_{4u}^{(1)} - X_{4v}^{(1)} - 3X_{4v}^{(1)} \\ X_{4z}^{\alpha(1)} - X_{4z}^{\beta(1)} & -X_{4z}^{\alpha(1)} - X_{4z}^{\beta(1)} & 2X_{4z}^{\beta(1)} \\ X_{4y}^{\alpha(1)} + X_{4y}^{\beta(1)} & -2X_{4y}^{\beta(1)} & -X_{4y}^{\alpha(1)} + X_{4y}^{\beta(1)} \\ -2X_{4y}^{\alpha(1)} - 2X_{4y}^{\beta(1)} & 4X_{4y}^{\beta(1)} & 2X_{4y}^{\alpha(1)} - 2X_{4y}^{\beta(1)} \\ 0 & 0 & 0 \\ 2X_{4y}^{\alpha(1)} + 2X_{4y}^{\beta(1)} & -4X_{4y}^{\beta(1)} & -2X_{4y}^{\alpha(1)} + 2X_{4y}^{\beta(1)} \\ -X_{4z}^{\alpha(1)} + X_{4z}^{\beta(1)} & X_{4z}^{\alpha(1)} + X_{4z}^{\beta(1)} & -2X_{4z}^{\beta(1)} \\ 2X_{4v}^{(1)} - 3X_{4v}^{(1)} & 3X_{4u}^{(1)} - X_{4v}^{(1)} & -3X_{4u}^{(1)} - X_{4v}^{(1)} + 3X_{4v}^{(1)} \\ 2X_{4x}^{\beta(1)} & X_{4x}^{\alpha(1)} - X_{4x}^{\beta(1)} & -X_{4x}^{\alpha(1)} - X_{4x}^{\beta(1)} \\ -2X_{4z}^{\alpha(1)} + 2X_{4z}^{\beta(1)} & 2X_{4z}^{\alpha(1)} + 2X_{4z}^{\beta(1)} & -4X_{4z}^{\beta(1)} \\ 2X_{4z}^{\alpha(1)} - 2X_{4z}^{\beta(1)} & -2X_{4z}^{\alpha(1)} - 2X_{4z}^{\beta(1)} & 4X_{4z}^{\beta(1)} \\ 0 & 0 & 0 \\ -X_{4y}^{\alpha(1)} - X_{4y}^{\beta(1)} & 2X_{4y}^{\beta(1)} & X_{4y}^{\alpha(1)} - X_{4y}^{\beta(1)} \\ -2X_{4x}^{\beta(1)} & -X_{4x}^{\alpha(1)} + X_{4x}^{\beta(1)} & X_{4x}^{\alpha(1)} + X_{4x}^{\beta(1)} \\ 3X_{4u}^{(1)} - X_{4v}^{(1)} + 3X_{4v}^{(1)} & -3X_{4u}^{(1)} - X_{4v}^{(1)} - 3X_{4v}^{(1)} & 2X_{4v}^{(1)} \end{pmatrix}, \quad (B83)$$

$$\tilde{\sigma}_2^{(O/L)} = \begin{pmatrix} 0 & -12X_{4x}^{\beta(2)} & 12X_{4x}^{\beta(2)} \\ 8X_{4x}^{\beta(2)} & -2X_{4x}^{\alpha(2)} + 2X_{4x}^{\beta(2)} & 2X_{4x}^{\alpha(2)} - 10X_{4x}^{\beta(2)} \\ -8X_{4x}^{\beta(2)} & 2X_{4x}^{\alpha(2)} + 10X_{4x}^{\beta(2)} & -2X_{4x}^{\alpha(2)} - 2X_{4x}^{\beta(2)} \\ 6X_{4u}^{(2)} + 2X_{4v}^{(2)} & -6X_{4u}^{(2)} + 2X_{4v}^{(2)} & -4X_{4v}^{(2)} \\ -2X_{4z}^{\alpha(2)} + 2X_{4z}^{\beta(2)} & 2X_{4z}^{\alpha(2)} - 10X_{4z}^{\beta(2)} & 8X_{4z}^{\beta(2)} \\ -2X_{4y}^{\alpha(2)} - 2X_{4y}^{\beta(2)} & -8X_{4y}^{\beta(2)} & 2X_{4y}^{\alpha(2)} + 10X_{4y}^{\beta(2)} \\ -2X_{4y}^{\alpha(2)} - 2X_{4y}^{\beta(2)} & -8X_{4y}^{\beta(2)} & 2X_{4y}^{\alpha(2)} + 10X_{4y}^{\beta(2)} \\ 12X_{4y}^{\beta(2)} & 0 & -12X_{4y}^{\beta(2)} \\ 2X_{4y}^{\alpha(2)} - 10X_{4y}^{\beta(2)} & 8X_{4y}^{\beta(2)} & -2X_{4y}^{\alpha(2)} + 2X_{4y}^{\beta(2)} \\ 2X_{4z}^{\alpha(2)} + 10X_{4z}^{\beta(2)} & -2X_{4z}^{\alpha(2)} - 2X_{4z}^{\beta(2)} & -8X_{4z}^{\beta(2)} \\ 6X_{4u}^{(2)} + 2X_{4v}^{(2)} & -6X_{4u}^{(2)} + 2X_{4v}^{(2)} & -4X_{4v}^{(2)} \\ 8X_{4x}^{\beta(2)} & -2X_{4x}^{\alpha(2)} + 2X_{4x}^{\beta(2)} & 2X_{4x}^{\alpha(2)} - 10X_{4x}^{\beta(2)} \\ -2X_{4z}^{\alpha(2)} + 2X_{4z}^{\beta(2)} & 2X_{4z}^{\alpha(2)} - 10X_{4z}^{\beta(2)} & 8X_{4z}^{\beta(2)} \\ 2X_{4z}^{\alpha(2)} + 10X_{4z}^{\beta(2)} & -2X_{4z}^{\alpha(2)} - 2X_{4z}^{\beta(2)} & -8X_{4z}^{\beta(2)} \\ -12X_{4z}^{\beta(2)} & 12X_{4z}^{\beta(2)} & 0 \\ 2X_{4y}^{\alpha(2)} - 10X_{4y}^{\beta(2)} & 8X_{4y}^{\beta(2)} & -2X_{4y}^{\alpha(2)} + 2X_{4y}^{\beta(2)} \\ -8X_{4x}^{\beta(2)} & 2X_{4x}^{\alpha(2)} + 10X_{4x}^{\beta(2)} & -2X_{4x}^{\alpha(2)} - 2X_{4x}^{\beta(2)} \\ 6X_{4u}^{(2)} + 2X_{4v}^{(2)} & -6X_{4u}^{(2)} + 2X_{4v}^{(2)} & -4X_{4v}^{(2)} \end{pmatrix}. \quad (B84)$$

$$\tilde{\sigma}_{ij}^{mn\alpha(O/T)} = \tilde{\sigma}_1^{(O/T)} + \tilde{\sigma}_2^{(O/T)}, \quad (B85)$$

$$\tilde{\sigma}_1^{(O/T)} = \begin{pmatrix} 0 & 0 & 0 \\ -2X_{4u}^{(1)} + 2X_{4v}^{(1)} + 2X_4^{(1)} & -4X_{4z}^{\beta(1)} & -4X_{4y}^{\beta(1)} \\ 2X_{4u}^{(1)} - 2X_{4v}^{(1)} - 2X_4^{(1)} & 4X_{4z}^{\beta(1)} & 4X_{4y}^{\beta(1)} \\ 2X_{4x}^{\alpha(1)} & -X_{4y}^{\alpha(1)} + 3X_{4y}^{\beta(1)} & -X_{4z}^{\alpha(1)} - 3X_{4z}^{\beta(1)} \\ 2X_{4y}^{\beta(1)} & 2X_{4x}^{\beta(1)} & -2X_{4u}^{(1)} - X_4^{(1)} \\ -2X_{4z}^{\beta(1)} & -X_{4u}^{(1)} - X_{4v}^{(1)} + X_4^{(1)} & -2X_{4x}^{\beta(1)} \\ 4X_{4z}^{\beta(1)} & 2X_{4u}^{(1)} + 2X_{4v}^{(1)} - 2X_4^{(1)} & 4X_{4x}^{\beta(1)} \\ 0 & 0 & 0 \\ -4X_{4z}^{\beta(1)} & -2X_{4u}^{(1)} - 2X_{4v}^{(1)} + 2X_4^{(1)} & -4X_{4x}^{\beta(1)} \\ -2X_{4y}^{\beta(1)} & -2X_{4x}^{\beta(1)} & 2X_{4u}^{(1)} + X_4^{(1)} \\ -X_{4x}^{\alpha(1)} - 3X_{4x}^{\beta(1)} & 2X_{4y}^{\alpha(1)} & -X_{4z}^{\alpha(1)} + 3X_{4z}^{\beta(1)} \\ X_{4u}^{(1)} - X_{4v}^{(1)} - X_4^{(1)} & 2X_{4z}^{\beta(1)} & 2X_{4y}^{\beta(1)} \\ -4X_{4y}^{\beta(1)} & -4X_{4x}^{\beta(1)} & 4X_{4u}^{(1)} + 2X_4^{(1)} \\ 4X_{4y}^{\beta(1)} & 4X_{4x}^{\beta(1)} & -4X_{4u}^{(1)} - 2X_4^{(1)} \\ 0 & 0 & 0 \\ 2X_{4z}^{\beta(1)} & X_{4u}^{(1)} + X_{4v}^{(1)} - X_4^{(1)} & 2X_{4x}^{\beta(1)} \\ -X_{4u}^{(1)} + X_{4v}^{(1)} + X_4^{(1)} & -2X_{4z}^{\beta(1)} & -2X_{4y}^{\beta(1)} \\ -X_{4x}^{\alpha(1)} + 3X_{4x}^{\beta(1)} & -X_{4y}^{\alpha(1)} - 3X_{4y}^{\beta(1)} & 2X_{4z}^{\alpha(1)} \end{pmatrix}, \quad (B86)$$

$$\tilde{\sigma}_2^{(O/T)} = \begin{pmatrix} -9X_{4u}^{(2)} - 3X_{4v}^{(2)} & 3X_{4z}^{\alpha(2)} - 3X_{4z}^{\beta(2)} & 3X_{4y}^{\alpha(2)} + 3X_{4y}^{\beta(2)} \\ 4X_{4u}^{(2)} + 2X_{4v}^{(2)} + X_4^{(2)} & -3X_{4z}^{\alpha(2)} + 5X_{4z}^{\beta(2)} & 2X_{4y}^{\beta(2)} \\ 5X_{4u}^{(2)} + X_{4v}^{(2)} - X_4^{(2)} & -2X_{4z}^{\beta(2)} & -3X_{4y}^{\alpha(2)} - 5X_{4y}^{\beta(2)} \\ 2X_{4x}^{\alpha(2)} & 2X_{4y}^{\alpha(2)} & 2X_{4z}^{\alpha(2)} \\ -3X_{4y}^{\alpha(2)} + 5X_{4y}^{\beta(2)} & 2X_{4x}^{\beta(2)} & X_{4u}^{(2)} - 3X_{4v}^{(2)} + X_4^{(2)} \\ -3X_{4z}^{\alpha(2)} - 5X_{4z}^{\beta(2)} & -4X_{4u}^{(2)} + 2X_{4v}^{(2)} - X_4^{(2)} & -2X_{4x}^{\beta(2)} \\ -3X_{4z}^{\alpha(2)} - 5X_{4z}^{\beta(2)} & -4X_{4u}^{(2)} + 2X_{4v}^{(2)} - X_4^{(2)} & -2X_{4x}^{\beta(2)} \\ 3X_{4z}^{\alpha(2)} + 3X_{4z}^{\beta(2)} & 9X_{4u}^{(2)} - 3X_{4v}^{(2)} & 3X_{4x}^{\alpha(2)} - 3X_{4x}^{\beta(2)} \\ 2X_{4z}^{\beta(2)} & -5X_{4u}^{(2)} + X_{4v}^{(2)} + X_4^{(2)} & -3X_{4x}^{\alpha(2)} + 5X_{4x}^{\beta(2)} \\ -2X_{4y}^{\beta(2)} & -3X_{4x}^{\alpha(2)} - 5X_{4x}^{\beta(2)} & -X_{4u}^{(2)} - 3X_{4v}^{(2)} - X_4^{(2)} \\ 2X_{4x}^{\alpha(2)} & 2X_{4y}^{\alpha(2)} & 2X_{4z}^{\alpha(2)} \\ 4X_{4u}^{(2)} + 2X_{4v}^{(2)} + X_4^{(2)} & -3X_{4z}^{\alpha(2)} + 5X_{4z}^{\beta(2)} & 2X_{4y}^{\beta(2)} \\ -3X_{4y}^{\alpha(2)} + 5X_{4y}^{\beta(2)} & 2X_{4x}^{\beta(2)} & X_{4u}^{(2)} - 3X_{4v}^{(2)} + X_4^{(2)} \\ -2X_{4y}^{\beta(2)} & -3X_{4x}^{\alpha(2)} - 5X_{4x}^{\beta(2)} & -X_{4u}^{(2)} - 3X_{4v}^{(2)} - X_4^{(2)} \\ 3X_{4y}^{\alpha(2)} - 3X_{4y}^{\beta(2)} & 3X_{4x}^{\alpha(2)} + 3X_{4x}^{\beta(2)} & 6X_{4v}^{(2)} \\ 2X_{4z}^{\beta(2)} & -5X_{4u}^{(2)} + X_{4v}^{(2)} + X_4^{(2)} & -3X_{4x}^{\alpha(2)} + 5X_{4x}^{\beta(2)} \\ 5X_{4u}^{(2)} + X_{4v}^{(2)} - X_4^{(2)} & -2X_{4z}^{\beta(2)} & -3X_{4y}^{\alpha(2)} - 5X_{4y}^{\beta(2)} \\ 2X_{4x}^{\alpha(2)} & 2X_{4y}^{\alpha(2)} & 2X_{4z}^{\alpha(2)} \end{pmatrix}. \quad (B87)$$

$$\tilde{\sigma}_{ij}^{mn\alpha(H)} = \begin{pmatrix} X_{4u} - X_{4v} + 2X_4 & X_{4z}^\alpha - X_{4z}^\beta & -X_{4y}^\alpha - X_{4y}^\beta \\ -2X_{4u} - X_4 & -X_{4z}^\alpha - X_{4z}^\beta & 2X_{4y}^\beta \\ X_{4u} + X_{4v} - X_4 & 2X_{4z}^\beta & X_{4y}^\alpha - X_{4y}^\beta \\ 2X_{4x}^\beta & 2X_{4y}^\beta & 2X_{4z}^\beta \\ -X_{4y}^\alpha - X_{4y}^\beta & 2X_{4x}^\beta & X_{4u} + X_{4v} - X_4 \\ X_{4z}^\alpha - X_{4z}^\beta & -2X_{4u} - X_4 & 2X_{4x}^\beta \\ X_{4z}^\alpha - X_{4z}^\beta & -2X_{4u} - X_4 & 2X_{4x}^\beta \\ -X_{4z}^\alpha - X_{4z}^\beta & X_{4u} + X_{4v} + 2X_4 & X_{4x}^\alpha - X_{4x}^\beta \\ 2X_{4z}^\beta & X_{4u} - X_{4v} - X_4 & -X_{4x}^\alpha - X_{4x}^\beta \\ 2X_{4y}^\beta & X_{4x}^\alpha - X_{4x}^\beta & X_{4u} - X_{4v} - X_4 \\ 2X_{4x}^\beta & 2X_{4y}^\beta & 2X_{4z}^\beta \\ -2X_{4u} - X_4 & -X_{4z}^\alpha - X_{4z}^\beta & 2X_{4y}^\beta \\ -X_{4y}^\alpha - X_{4y}^\beta & 2X_{4x}^\beta & X_{4u} + X_{4v} - X_4 \\ 2X_{4y}^\beta & X_{4x}^\alpha - X_{4x}^\beta & X_{4u} - X_{4v} - X_4 \\ X_{4y}^\alpha - X_{4y}^\beta & -X_{4x}^\alpha - X_{4x}^\beta & -2X_{4u} + 2X_4 \\ 2X_{4z}^\beta & X_{4u} - X_{4v} - X_4 & -X_{4x}^\alpha - X_{4x}^\beta \\ X_{4u} + X_{4v} - X_4 & 2X_{4z}^\beta & X_{4y}^\alpha - X_{4y}^\beta \\ 2X_{4x}^\beta & 2X_{4y}^\beta & 2X_{4z}^\beta \end{pmatrix}. \quad (\text{B88})$$

15. rank-5 c -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mn\alpha(O/L)} = \begin{pmatrix} 8Y_{5x}^{\alpha 1} & -4Y_{5x}^{\alpha 1} + Y_{5x}^\beta & -4Y_{5x}^{\alpha 1} - Y_{5x}^\beta \\ -4Y_{5x}^{\alpha 1} + Y_{5x}^\beta & 3Y_{5x}^{\alpha 1} + Y_{5x}^{\alpha 2} - Y_{5x}^\beta & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} \\ -4Y_{5x}^{\alpha 1} - Y_{5x}^\beta & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & 3Y_{5x}^{\alpha 1} + Y_{5x}^{\alpha 2} + Y_{5x}^\beta \\ Y_{5u} - Y_{5v} & -Y_{5u} - Y_{5v} & 2Y_{5v} \\ 3Y_{5z}^{\alpha 1} + Y_{5z}^{\alpha 2} - Y_{5z}^\beta & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & -4Y_{5z}^{\alpha 1} + Y_{5z}^\beta \\ 3Y_{5y}^{\alpha 1} + Y_{5y}^{\alpha 2} + Y_{5y}^\beta & -4Y_{5y}^{\alpha 1} - Y_{5y}^\beta & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} \\ 3Y_{5y}^{\alpha 1} + Y_{5y}^{\alpha 2} + Y_{5y}^\beta & -4Y_{5y}^{\alpha 1} - Y_{5y}^\beta & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} \\ -4Y_{5y}^{\alpha 1} - Y_{5y}^\beta & 8Y_{5y}^{\alpha 1} & -4Y_{5y}^{\alpha 1} + Y_{5y}^\beta \\ Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & -4Y_{5y}^{\alpha 1} + Y_{5y}^\beta & 3Y_{5y}^{\alpha 1} + Y_{5y}^{\alpha 2} - Y_{5y}^\beta \\ Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & 3Y_{5z}^{\alpha 1} + Y_{5z}^{\alpha 2} + Y_{5z}^\beta & -4Y_{5z}^{\alpha 1} - Y_{5z}^\beta \\ Y_{5u} - Y_{5v} & -Y_{5u} - Y_{5v} & 2Y_{5v} \\ -4Y_{5x}^{\alpha 1} + Y_{5x}^\beta & 3Y_{5x}^{\alpha 1} + Y_{5x}^{\alpha 2} - Y_{5x}^\beta & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} \\ 3Y_{5z}^{\alpha 1} + Y_{5z}^{\alpha 2} - Y_{5z}^\beta & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & -4Y_{5z}^{\alpha 1} + Y_{5z}^\beta \\ Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & 3Y_{5z}^{\alpha 1} + Y_{5z}^{\alpha 2} + Y_{5z}^\beta & -4Y_{5z}^{\alpha 1} - Y_{5z}^\beta \\ -4Y_{5z}^{\alpha 1} + Y_{5z}^\beta & -4Y_{5z}^{\alpha 1} - Y_{5z}^\beta & 8Y_{5z}^{\alpha 1} \\ Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & -4Y_{5y}^{\alpha 1} + Y_{5y}^\beta & 3Y_{5y}^{\alpha 1} + Y_{5y}^{\alpha 2} - Y_{5y}^\beta \\ -4Y_{5x}^{\alpha 1} - Y_{5x}^\beta & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & 3Y_{5x}^{\alpha 1} + Y_{5x}^{\alpha 2} + Y_{5x}^\beta \\ Y_{5u} - Y_{5v} & -Y_{5u} - Y_{5v} & 2Y_{5v} \end{pmatrix}. \quad (\text{B89})$$

$$\tilde{\sigma}_{ij}^{mna(O/T)} \begin{pmatrix} Y_{5u} - Y_{5v} & 3Y_{5z}^{\alpha 1} + Y_{5z}^{\alpha 2} - Y_{5z}^{\beta} & 3Y_{5y}^{\alpha 1} + Y_{5y}^{\alpha 2} + Y_{5y}^{\beta} \\ -Y_{5u} - Y_{5v} & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & -4Y_{5y}^{\alpha 1} - Y_{5y}^{\beta} \\ 2Y_{5v} & -4Y_{5z}^{\alpha 1} + Y_{5z}^{\beta} & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} \\ Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} \\ Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & -4Y_{5x}^{\alpha 1} - Y_{5x}^{\beta} & Y_{5u} - Y_{5v} \\ Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & Y_{5u} - Y_{5v} & -4Y_{5x}^{\alpha 1} + Y_{5x}^{\beta} \\ Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & Y_{5u} - Y_{5v} & -4Y_{5x}^{\alpha 1} + Y_{5x}^{\beta} \\ 3Y_{5z}^{\alpha 1} + Y_{5z}^{\alpha 2} + Y_{5z}^{\beta} & -Y_{5u} - Y_{5v} & 3Y_{5x}^{\alpha 1} + Y_{5x}^{\alpha 2} - Y_{5x}^{\beta} \\ -4Y_{5z}^{\alpha 1} - Y_{5z}^{\beta} & 2Y_{5v} & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} \\ -4Y_{5y}^{\alpha 1} + Y_{5y}^{\beta} & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & -Y_{5u} - Y_{5v} \\ Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} \\ -Y_{5u} - Y_{5v} & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} & -4Y_{5y}^{\alpha 1} - Y_{5y}^{\beta} \\ Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & -4Y_{5x}^{\alpha 1} - Y_{5x}^{\beta} & Y_{5u} - Y_{5v} \\ -4Y_{5y}^{\alpha 1} + Y_{5y}^{\beta} & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & -Y_{5u} - Y_{5v} \\ 3Y_{5y}^{\alpha 1} + Y_{5y}^{\alpha 2} - Y_{5y}^{\beta} & 3Y_{5x}^{\alpha 1} + Y_{5x}^{\alpha 2} + Y_{5x}^{\beta} & 2Y_{5v} \\ -4Y_{5z}^{\alpha 1} - Y_{5z}^{\beta} & 2Y_{5v} & Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} \\ 2Y_{5v} & -4Y_{5z}^{\alpha 1} + Y_{5z}^{\beta} & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} \\ Y_{5x}^{\alpha 1} - Y_{5x}^{\alpha 2} & Y_{5y}^{\alpha 1} - Y_{5y}^{\alpha 2} & Y_{5z}^{\alpha 1} - Y_{5z}^{\alpha 2} \end{pmatrix}. \quad (\text{B90})$$

16. rank-3 t -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mna(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)} + \tilde{\sigma}_3^{(O/L)}, \quad (\text{B91})$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} -2Y_{3a}^{(1)} - 8Y_{3a}^{(2)} + 6Y_{3u}^{(1)} + 8Y_{3u}^{(2)} & 2Y_{3a}^{(1)} - 2Y_{3a}^{(2)} + 2Y_{3u}^{(1)} - 14Y_{3u}^{(2)} & 10Y_{3a}^{(2)} - 8Y_{3u}^{(1)} + 6Y_{3u}^{(2)} \\ 4Y_{3a}^{(2)} - 4Y_{3u}^{(2)} & 6Y_{3a}^{(2)} + 2Y_{3u}^{(2)} & -10Y_{3a}^{(2)} + 2Y_{3u}^{(2)} \\ 4Y_{3a}^{(2)} - 4Y_{3u}^{(2)} & -4Y_{3a}^{(2)} + 12Y_{3u}^{(2)} & -8Y_{3u}^{(2)} \\ 0 & 0 & 0 \\ Y_z^{\alpha(1)} + 6Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & Y_z^{\alpha(1)} - 14Y_z^{\alpha(2)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} & -2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} \\ -Y_{3b}^{(1)} - 6Y_{3b}^{(2)} + Y_{3v}^{(1)} + 2Y_{3v}^{(2)} & Y_{3b}^{(1)} - 4Y_{3b}^{(2)} + 3Y_{3v}^{(1)} - 4Y_{3v}^{(2)} & 10Y_{3b}^{(2)} - 4Y_{3v}^{(1)} + 2Y_{3v}^{(2)} \\ -6Y_{3b}^{(2)} + 2Y_{3v}^{(2)} & -4Y_{3b}^{(2)} - 4Y_{3v}^{(2)} & 10Y_{3b}^{(2)} + 2Y_{3v}^{(2)} \\ -2Y_{3b}^{(1)} + 2Y_{3b}^{(2)} + 2Y_{3v}^{(1)} - 14Y_{3v}^{(2)} & 2Y_{3b}^{(1)} + 8Y_{3b}^{(2)} + 6Y_{3v}^{(1)} + 8Y_{3v}^{(2)} & -10Y_{3b}^{(2)} - 8Y_{3v}^{(1)} + 6Y_{3v}^{(2)} \\ 4Y_{3b}^{(2)} + 12Y_{3v}^{(2)} & -4Y_{3b}^{(2)} - 4Y_{3v}^{(2)} & -8Y_{3v}^{(2)} \\ Y_z^{\alpha(1)} - 14Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & Y_z^{\alpha(1)} + 6Y_z^{\alpha(2)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} & -2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} \\ 0 & 0 & 0 \\ -Y_{3a}^{(1)} + 4Y_{3a}^{(2)} + 3Y_{3u}^{(1)} - 4Y_{3u}^{(2)} & Y_{3a}^{(1)} + 6Y_{3a}^{(2)} + Y_{3u}^{(1)} + 2Y_{3u}^{(2)} & -10Y_{3a}^{(2)} - 4Y_{3u}^{(1)} + 2Y_{3u}^{(2)} \\ 6Y_z^{\alpha(2)} - 2Y_z^{\beta(2)} & -14Y_z^{\alpha(2)} + 2Y_z^{\beta(2)} & 8Y_z^{\alpha(2)} \\ -14Y_z^{\alpha(2)} - 2Y_z^{\beta(2)} & 6Y_z^{\alpha(2)} + 2Y_z^{\beta(2)} & 8Y_z^{\alpha(2)} \\ 2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} + 4Y_z^{\beta(2)} & 2Y_z^{\alpha(1)} + 8Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} - 4Y_z^{\beta(2)} & -4Y_z^{\alpha(1)} - 16Y_z^{\alpha(2)} \\ -Y_{3b}^{(1)} + 4Y_{3b}^{(2)} + Y_{3v}^{(1)} + 12Y_{3v}^{(2)} & Y_{3b}^{(1)} - 4Y_{3b}^{(2)} + 3Y_{3v}^{(1)} - 4Y_{3v}^{(2)} & -4Y_{3b}^{(1)} - 8Y_{3v}^{(2)} \\ -Y_{3a}^{(1)} + 4Y_{3a}^{(2)} + 3Y_{3u}^{(1)} - 4Y_{3u}^{(2)} & Y_{3a}^{(1)} - 4Y_{3a}^{(2)} + Y_{3u}^{(1)} + 12Y_{3u}^{(2)} & -4Y_{3a}^{(1)} - 8Y_{3u}^{(2)} \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{B92})$$

$$\tilde{\sigma}_2^{(O/L)} = \begin{pmatrix} -8Y_{3a}^{(4)} + 24Y_{3u}^{(4)} & 8Y_{3a}^{(4)} + 8Y_{3u}^{(4)} & -32Y_{3u}^{(4)} \\ 2Y_{3a}^{(3)} - 6Y_{3a}^{(4)} + 10Y_{3u}^{(3)} + 18Y_{3u}^{(4)} & -2Y_{3a}^{(3)} + 6Y_{3a}^{(4)} - 2Y_{3u}^{(3)} + 6Y_{3u}^{(4)} & -8Y_{3u}^{(3)} - 24Y_{3u}^{(4)} \\ -2Y_{3a}^{(3)} - 6Y_{3a}^{(4)} - 10Y_{3u}^{(3)} + 18Y_{3u}^{(4)} & 2Y_{3a}^{(3)} + 6Y_{3a}^{(4)} + 2Y_{3u}^{(3)} + 6Y_{3u}^{(4)} & 8Y_{3u}^{(3)} - 24Y_{3u}^{(4)} \\ -2Y_{xy}^{(3)} & Y_{xy}^{(3)} & Y_{xy}^{(3)} \\ Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(3)} - Y_z^{\beta(4)} & -Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(3)} + Y_z^{\beta(4)} & -2Y_z^{\alpha(4)} + 2Y_z^{\beta(3)} \\ -Y_{3b}^{(3)} - Y_{3b}^{(4)} + Y_{3v}^{(3)} + Y_{3v}^{(4)} & Y_{3b}^{(3)} + Y_{3b}^{(4)} - 5Y_{3v}^{(3)} + 3Y_{3v}^{(4)} & 4Y_{3v}^{(3)} - 4Y_{3v}^{(4)} \\ 2Y_{3b}^{(3)} - 6Y_{3b}^{(4)} - 2Y_{3v}^{(3)} + 6Y_{3v}^{(4)} & -2Y_{3b}^{(3)} + 6Y_{3b}^{(4)} + 10Y_{3v}^{(3)} + 18Y_{3v}^{(4)} & -8Y_{3v}^{(3)} - 24Y_{3v}^{(4)} \\ -8Y_{3b}^{(4)} + 8Y_{3v}^{(4)} & 8Y_{3b}^{(4)} + 24Y_{3v}^{(4)} & -32Y_{3v}^{(4)} \\ -2Y_{3b}^{(3)} - 6Y_{3b}^{(4)} + 2Y_{3v}^{(3)} + 6Y_{3v}^{(4)} & 2Y_{3b}^{(3)} + 6Y_{3b}^{(4)} - 10Y_{3v}^{(3)} + 18Y_{3v}^{(4)} & 8Y_{3v}^{(3)} - 24Y_{3v}^{(4)} \\ -Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(3)} - Y_z^{\beta(4)} & Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(3)} + Y_z^{\beta(4)} & -2Y_z^{\alpha(4)} - 2Y_z^{\beta(3)} \\ Y_{xyz}^{(3)} & -2Y_{xyz}^{(3)} & Y_{xyz}^{(3)} \\ -Y_{3a}^{(3)} - Y_{3a}^{(4)} - 5Y_{3u}^{(3)} + 3Y_{3u}^{(4)} & Y_{3a}^{(3)} + Y_{3a}^{(4)} + Y_{3u}^{(3)} + Y_{3u}^{(4)} & 4Y_{3u}^{(3)} - 4Y_{3u}^{(4)} \\ -2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 2Y_z^{\beta(3)} - 6Y_z^{\beta(4)} & 2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 2Y_z^{\beta(3)} + 6Y_z^{\beta(4)} & -12Y_z^{\alpha(4)} - 4Y_z^{\beta(3)} \\ 2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 2Y_z^{\beta(3)} - 6Y_z^{\beta(4)} & -2Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 2Y_z^{\beta(3)} + 6Y_z^{\beta(4)} & -12Y_z^{\alpha(4)} + 4Y_z^{\beta(3)} \\ 8Y_z^{\alpha(4)} - 8Y_z^{\beta(4)} & 8Y_z^{\alpha(4)} + 8Y_z^{\beta(4)} & -16Y_z^{\alpha(4)} \\ Y_{3b}^{(3)} - Y_{3b}^{(4)} - Y_{3v}^{(3)} + Y_{3v}^{(4)} & -Y_{3b}^{(3)} + Y_{3b}^{(4)} + 5Y_{3v}^{(3)} + 3Y_{3v}^{(4)} & -4Y_{3v}^{(3)} - 4Y_{3v}^{(4)} \\ Y_{3a}^{(3)} - Y_{3a}^{(4)} + 5Y_{3u}^{(3)} + 3Y_{3u}^{(4)} & -Y_{3a}^{(3)} + Y_{3a}^{(4)} - Y_{3u}^{(3)} + Y_{3u}^{(4)} & -4Y_{3u}^{(3)} - 4Y_{3u}^{(4)} \\ Y_{xyz}^{(3)} & Y_{xyz}^{(3)} & -2Y_{xyz}^{(3)} \end{pmatrix}, \quad (\text{B93})$$

$$\tilde{\sigma}_3^{(O/L)} = \begin{pmatrix} -Y_{3a}^{(5)} + 3Y_{3u}^{(5)} & -Y_{3a}^{(5)} + 3Y_{3u}^{(5)} & -Y_{3a}^{(5)} + 3Y_{3u}^{(5)} \\ Y_{3a}^{(5)} + Y_{3u}^{(5)} & Y_{3a}^{(5)} + Y_{3u}^{(5)} & Y_{3a}^{(5)} + Y_{3u}^{(5)} \\ -4Y_{3u}^{(5)} & -4Y_{3u}^{(5)} & -4Y_{3u}^{(5)} \\ -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} \\ Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} \\ -Y_{3b}^{(5)} + Y_{3v}^{(5)} & -Y_{3b}^{(5)} + Y_{3v}^{(5)} & -Y_{3b}^{(5)} + Y_{3v}^{(5)} \\ -Y_{3b}^{(5)} + Y_{3v}^{(5)} & -Y_{3b}^{(5)} + Y_{3v}^{(5)} & -Y_{3b}^{(5)} + Y_{3v}^{(5)} \\ Y_{3b}^{(5)} + 3Y_{3v}^{(5)} & Y_{3b}^{(5)} + 3Y_{3v}^{(5)} & Y_{3b}^{(5)} + 3Y_{3v}^{(5)} \\ -4Y_{3v}^{(5)} & -4Y_{3v}^{(5)} & -4Y_{3v}^{(5)} \\ Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} \\ -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} \\ Y_{3a}^{(5)} + Y_{3u}^{(5)} & Y_{3a}^{(5)} + Y_{3u}^{(5)} & Y_{3a}^{(5)} + Y_{3u}^{(5)} \\ Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} & Y_z^{\alpha(5)} - Y_z^{\beta(5)} \\ Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} & Y_z^{\alpha(5)} + Y_z^{\beta(5)} \\ -2Y_z^{\alpha(5)} & -2Y_z^{\alpha(5)} & -2Y_z^{\alpha(5)} \\ -4Y_{3v}^{(5)} & -4Y_{3v}^{(5)} & -4Y_{3v}^{(5)} \\ -4Y_{3u}^{(5)} & -4Y_{3u}^{(5)} & -4Y_{3u}^{(5)} \\ -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} & -Y_{xyz}^{(5)} \end{pmatrix}. \quad (\text{B94})$$

$$\tilde{\sigma}_{ij}^{mn\alpha(O/T)} = \tilde{\sigma}_1^{(O/T)} + \tilde{\sigma}_2^{(O/T)} + \tilde{\sigma}_3^{(O/T)}, \quad (B95)$$

$$\tilde{\sigma}_1^{(O/T)} = \begin{pmatrix} -2Y_{xyz}^{(1)} + 2Y_{xyz}^{(2)} & 2Y_z^{\alpha(1)} + 3Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} - Y_z^{\beta(2)} & -2Y_{3b}^{(1)} - 3Y_{3b}^{(2)} + 2Y_{3v}^{(1)} + Y_{3v}^{(2)} \\ -Y_{xyz}^{(2)} & Y_z^{\alpha(2)} + 3Y_z^{\beta(2)} & -Y_{3b}^{(2)} + 7Y_{3v}^{(2)} \\ -Y_{xyz}^{(2)} & -4Y_z^{\alpha(2)} - 2Y_z^{\beta(2)} & 4Y_{3b}^{(2)} - 8Y_{3v}^{(2)} \\ 5Y_{3a}^{(2)} - 5Y_{3u}^{(2)} & -5Y_{3b}^{(2)} - 5Y_{3v}^{(2)} & 10Y_z^{\alpha(2)} \\ -5Y_{3b}^{(2)} - 4Y_{3v}^{(1)} + 7Y_{3v}^{(2)} & -5Y_{3a}^{(2)} - 4Y_{3u}^{(1)} - 3Y_{3u}^{(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ Y_z^{\alpha(1)} + Y_z^{\alpha(2)} + Y_z^{\beta(1)} - 3Y_z^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_{3a}^{(1)} + Y_{3a}^{(2)} + Y_{3u}^{(1)} + 7Y_{3u}^{(2)} \\ Y_z^{\alpha(2)} - 3Y_z^{\beta(2)} & -Y_{xyz}^{(2)} & Y_{3a}^{(2)} + 7Y_{3u}^{(2)} \\ 2Y_z^{\alpha(1)} + 3Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} + Y_z^{\beta(2)} & -2Y_{xyz}^{(1)} + 2Y_{xyz}^{(2)} & 2Y_{3a}^{(1)} + 3Y_{3a}^{(2)} + 2Y_{3u}^{(1)} + Y_{3u}^{(2)} \\ -4Y_z^{\alpha(2)} + 2Y_z^{\beta(2)} & -Y_{xyz}^{(2)} & -4Y_{3a}^{(2)} - 8Y_{3u}^{(2)} \\ 5Y_{3b}^{(2)} - 4Y_{3v}^{(1)} - 3Y_{3v}^{(2)} & 5Y_{3a}^{(2)} - 4Y_{3u}^{(1)} + 7Y_{3u}^{(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ 5Y_{3a}^{(2)} - 5Y_{3u}^{(2)} & -5Y_{3b}^{(2)} - 5Y_{3v}^{(2)} & 10Y_z^{\alpha(2)} \\ -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_z^{\alpha(1)} + Y_z^{\alpha(2)} - Y_z^{\beta(1)} + 3Y_z^{\beta(2)} & -Y_{3b}^{(1)} - Y_{3b}^{(2)} + Y_{3v}^{(1)} + 7Y_{3v}^{(2)} \\ -5Y_{3b}^{(2)} + 7Y_{3v}^{(2)} & -5Y_{3a}^{(2)} - 3Y_{3u}^{(2)} & -Y_{xyz}^{(2)} \\ 5Y_{3b}^{(2)} - 3Y_{3v}^{(2)} & 5Y_{3a}^{(2)} + 7Y_{3u}^{(2)} & -Y_{xyz}^{(2)} \\ -8Y_{3v}^{(1)} - 4Y_{3v}^{(2)} & -8Y_{3u}^{(1)} - 4Y_{3u}^{(2)} & -2Y_{xyz}^{(1)} + 2Y_{xyz}^{(2)} \\ Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_{3a}^{(1)} - 4Y_{3a}^{(2)} + Y_{3u}^{(1)} - 8Y_{3u}^{(2)} \\ -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & -Y_{3b}^{(1)} + 4Y_{3b}^{(2)} + Y_{3v}^{(1)} - 8Y_{3v}^{(2)} \\ 5Y_{3a}^{(2)} - 5Y_{3u}^{(2)} & -5Y_{3b}^{(2)} - 5Y_{3v}^{(2)} & 10Y_z^{\alpha(2)} \end{pmatrix}, \quad (B96)$$

$$\tilde{\sigma}_2^{(O/T)} = \begin{pmatrix} -8Y_{xyz}^{(4)} & 8Y_z^{\alpha(4)} - 8Y_z^{\beta(4)} & -8Y_{3b}^{(4)} + 8Y_{3v}^{(4)} \\ -6Y_{xyz}^{(4)} & 4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 6Y_z^{\beta(4)} & 2Y_{3b}^{(3)} - 6Y_{3b}^{(4)} + 6Y_{3v}^{(3)} + 6Y_{3v}^{(4)} \\ -6Y_{xyz}^{(4)} & -4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} - 6Y_z^{\beta(4)} & -2Y_{3b}^{(3)} - 6Y_{3b}^{(4)} - 6Y_{3v}^{(3)} + 6Y_{3v}^{(4)} \\ -Y_{3a}^{(3)} + 3Y_{3u}^{(3)} & Y_{3b}^{(3)} - 9Y_{3v}^{(3)} & Y_z^{\alpha(3)} + 3Y_z^{\beta(3)} \\ Y_{3b}^{(3)} + 3Y_{3v}^{(3)} - 4Y_{3v}^{(4)} & Y_{3a}^{(3)} - 3Y_{3u}^{(3)} - 4Y_{3u}^{(4)} & -Y_{xyz}^{(4)} \\ -2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(4)} & -Y_{xyz}^{(4)} & Y_{3a}^{(3)} + Y_{3a}^{(4)} - 3Y_{3u}^{(3)} + Y_{3u}^{(4)} \\ 4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 6Y_z^{\beta(4)} & -6Y_{xyz}^{(4)} & -2Y_{3a}^{(3)} + 6Y_{3a}^{(4)} + 6Y_{3u}^{(3)} + 6Y_{3u}^{(4)} \\ 8Y_z^{\alpha(4)} + 8Y_z^{\beta(4)} & -8Y_{xyz}^{(4)} & 8Y_{3a}^{(4)} + 8Y_{3u}^{(4)} \\ -4Y_z^{\alpha(3)} + 6Y_z^{\alpha(4)} + 6Y_z^{\beta(4)} & -6Y_{xyz}^{(4)} & 2Y_{3a}^{(3)} + 6Y_{3a}^{(4)} - 6Y_{3u}^{(3)} + 6Y_{3u}^{(4)} \\ -Y_{3b}^{(3)} - 3Y_{3v}^{(3)} - 4Y_{3v}^{(4)} & -Y_{3a}^{(3)} + 3Y_{3u}^{(3)} - 4Y_{3u}^{(4)} & -Y_{xyz}^{(4)} \\ -Y_{3a}^{(3)} - 9Y_{3u}^{(3)} & Y_{3b}^{(3)} + 3Y_{3v}^{(3)} & Y_z^{\alpha(3)} - 3Y_z^{\beta(3)} \\ -Y_{xyz}^{(4)} & -2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(4)} & -Y_{3b}^{(3)} - Y_{3b}^{(4)} - 3Y_{3v}^{(3)} + Y_{3v}^{(4)} \\ -2Y_{3b}^{(3)} - 6Y_{3v}^{(3)} - 24Y_{3v}^{(4)} & -2Y_{3a}^{(3)} + 6Y_{3u}^{(3)} - 24Y_{3u}^{(4)} & -6Y_{xyz}^{(4)} \\ 2Y_{3b}^{(3)} + 6Y_{3v}^{(3)} - 24Y_{3v}^{(4)} & 2Y_{3a}^{(3)} - 6Y_{3u}^{(3)} - 24Y_{3u}^{(4)} & -6Y_{xyz}^{(4)} \\ -32Y_{3v}^{(4)} & -32Y_{3u}^{(4)} & -8Y_{xyz}^{(4)} \\ 2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} + Y_z^{\beta(4)} & -Y_{xyz}^{(4)} & -Y_{3a}^{(3)} + Y_{3a}^{(4)} + 3Y_{3u}^{(3)} + Y_{3u}^{(4)} \\ -Y_{xyz}^{(4)} & 2Y_z^{\alpha(3)} + Y_z^{\alpha(4)} - Y_z^{\beta(4)} & Y_{3b}^{(3)} - Y_{3b}^{(4)} + 3Y_{3v}^{(3)} + Y_{3v}^{(4)} \\ 2Y_{3a}^{(3)} + 6Y_{3u}^{(3)} & -2Y_{3b}^{(3)} + 6Y_{3v}^{(3)} & -2Y_z^{\alpha(3)} \end{pmatrix}, \quad (B97)$$

where the multipole component with superscript (5) of $\tilde{\sigma}_{ij}^{mn\alpha(\text{O/T})}$ does not appear.

$$\tilde{\sigma}_{ij}^{mn\alpha(\text{H})} = \begin{pmatrix} 0 & -3Y_z^{\alpha(2)} + 3Y_z^{\beta(2)} & -3Y_{3b}^{(2)} + 3Y_{3v}^{(2)} \\ 2Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & -2Y_z^{\alpha(1)} - Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} - Y_z^{\beta(2)} & 3Y_{3b}^{(2)} + 8Y_{3v}^{(1)} + Y_{3v}^{(2)} \\ -2Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & 2Y_z^{\alpha(1)} + 4Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & -8Y_{3v}^{(1)} - 4Y_{3v}^{(2)} \\ Y_{3a}^{(1)} - Y_{3a}^{(2)} + 5Y_{3u}^{(1)} - 5Y_{3u}^{(2)} & Y_{3b}^{(1)} - Y_{3b}^{(2)} + 7Y_{3v}^{(1)} + 5Y_{3v}^{(2)} & 3Y_z^{\alpha(1)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} \\ -Y_{3b}^{(1)} + Y_{3b}^{(2)} + Y_{3v}^{(1)} - Y_{3v}^{(2)} & Y_{3a}^{(1)} - Y_{3a}^{(2)} + Y_{3u}^{(1)} + 11Y_{3u}^{(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ -Y_z^{\alpha(1)} + Y_z^{\alpha(2)} + Y_z^{\beta(1)} - Y_z^{\beta(2)} & Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & 3Y_{3a}^{(2)} + 4Y_{3u}^{(1)} - Y_{3u}^{(2)} \\ 2Y_z^{\alpha(1)} + Y_z^{\alpha(2)} - 2Y_z^{\beta(1)} - Y_z^{\beta(2)} & -2Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & 3Y_{3a}^{(2)} - 8Y_{3u}^{(1)} - Y_{3u}^{(2)} \\ 3Y_z^{\alpha(2)} + 3Y_z^{\beta(2)} & 0 & -3Y_{3a}^{(2)} - 3Y_{3u}^{(2)} \\ -2Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} + 2Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & 2Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & 8Y_{3u}^{(1)} + 4Y_{3u}^{(2)} \\ Y_{3b}^{(1)} - Y_{3b}^{(2)} - Y_{3v}^{(1)} - 11Y_{3v}^{(2)} & -Y_{3a}^{(1)} + Y_{3a}^{(2)} - Y_{3u}^{(1)} + Y_{3u}^{(2)} & Y_{xyz}^{(1)} + Y_{xyz}^{(2)} \\ Y_{3a}^{(1)} - Y_{3a}^{(2)} - 7Y_{3u}^{(1)} - 5Y_{3u}^{(2)} & Y_{3b}^{(1)} - Y_{3b}^{(2)} - 5Y_{3v}^{(1)} + 5Y_{3v}^{(2)} & -3Y_z^{\alpha(1)} + Y_z^{\beta(1)} + 2Y_z^{\beta(2)} \\ -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & Y_z^{\alpha(1)} - Y_z^{\alpha(2)} + Y_z^{\beta(1)} - Y_z^{\beta(2)} & 3Y_{3b}^{(2)} - 4Y_{3v}^{(1)} + Y_{3v}^{(2)} \\ 2Y_{3b}^{(1)} + Y_{3b}^{(2)} - 2Y_{3v}^{(1)} - Y_{3v}^{(2)} & -2Y_{3a}^{(1)} - Y_{3a}^{(2)} - 2Y_{3u}^{(1)} + 11Y_{3u}^{(2)} & 2Y_{xyz}^{(1)} - Y_{xyz}^{(2)} \\ -2Y_{3b}^{(1)} - Y_{3b}^{(2)} + 2Y_{3v}^{(1)} - 11Y_{3v}^{(2)} & 2Y_{3a}^{(1)} + Y_{3a}^{(2)} + 2Y_{3u}^{(1)} + Y_{3u}^{(2)} & -2Y_{xyz}^{(1)} + Y_{xyz}^{(2)} \\ 12Y_{3v}^{(2)} & -12Y_{3u}^{(2)} & 0 \\ Y_z^{\alpha(1)} - 4Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & -Y_{xyz}^{(1)} - Y_{xyz}^{(2)} & -4Y_{3u}^{(1)} + 4Y_{3u}^{(2)} \\ Y_{xyz}^{(1)} + Y_{xyz}^{(2)} & -Y_z^{\alpha(1)} + 4Y_z^{\alpha(2)} - Y_z^{\beta(1)} - 2Y_z^{\beta(2)} & 4Y_{3v}^{(1)} - 4Y_{3v}^{(2)} \\ -2Y_{3a}^{(1)} - Y_{3a}^{(2)} + 2Y_{3u}^{(1)} - 5Y_{3u}^{(2)} & -2Y_{3b}^{(1)} - Y_{3b}^{(2)} - 2Y_{3v}^{(1)} + 5Y_{3v}^{(2)} & -2Y_z^{\beta(1)} + 2Y_z^{\beta(2)} \end{pmatrix}. \quad (\text{B98})$$

17. rank-4 t -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mn\alpha(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)}, \quad (B99)$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} 0 & 0 & 0 \\ 2X_{4v}^{\alpha(1)} - 2X_{4a}^{(1)} & 6X_{4v}^{\alpha(1)} + 2X_{4a}^{(1)} & -8X_{4v}^{\alpha(1)} \\ -2X_{4v}^{\alpha(1)} + 2X_{4a}^{(1)} & -6X_{4v}^{\alpha(1)} - 2X_{4a}^{(1)} & 8X_{4v}^{\alpha(1)} \\ -X_{4u}^{\beta 1(1)} - X_{4u}^{\beta 2(1)} + 5X_{40}^{(1)} & X_{4u}^{\beta 1(1)} + 2X_{4u}^{\beta 2(1)} + 7X_{40}^{(1)} & -X_{4u}^{\beta 2(1)} - 12X_{40}^{(1)} \\ X_{4v}^{\beta 1(1)} - X_{4v}^{\beta 2(1)} & -X_{4v}^{\beta 1(1)} - X_{4v}^{\beta 2(1)} & 2X_{4v}^{\beta 2(1)} \\ 3X_{4u}^{\alpha(1)} - X_{4b}^{(1)} & X_{4u}^{\alpha(1)} + X_{4b}^{(1)} & -4X_{4u}^{\alpha(1)} \\ -6X_{4u}^{\alpha(1)} + 2X_{4b}^{(1)} & -2X_{4u}^{\alpha(1)} - 2X_{4b}^{(1)} & 8X_{4u}^{\alpha(1)} \\ 0 & 0 & 0 \\ 6X_{4u}^{\alpha(1)} - 2X_{4b}^{(1)} & 2X_{4u}^{\alpha(1)} + 2X_{4b}^{(1)} & -8X_{4u}^{\alpha(1)} \\ -X_{4v}^{\beta 1(1)} + X_{4v}^{\beta 2(1)} & X_{4v}^{\beta 1(1)} + X_{4v}^{\beta 2(1)} & -2X_{4v}^{\beta 2(1)} \\ -X_{4u}^{\beta 1(1)} + 2X_{4u}^{\beta 2(1)} - 7X_{40}^{(1)} & X_{4u}^{\beta 1(1)} - X_{4u}^{\beta 2(1)} - 5X_{40}^{(1)} & -X_{4u}^{\beta 2(1)} + 12X_{40}^{(1)} \\ -X_{4v}^{\alpha(1)} + X_{4a}^{(1)} & -3X_{4v}^{\alpha(1)} - X_{4a}^{(1)} & 4X_{4v}^{\alpha(1)} \\ -2X_{4v}^{\beta 1(1)} + 2X_{4v}^{\beta 2(1)} & 2X_{4v}^{\beta 1(1)} + 2X_{4v}^{\beta 2(1)} & -4X_{4v}^{\beta 2(1)} \\ 2X_{4v}^{\beta 1(1)} - 2X_{4v}^{\beta 2(1)} & -2X_{4v}^{\beta 1(1)} - 2X_{4v}^{\beta 2(1)} & 4X_{4v}^{\beta 2(1)} \\ 0 & 0 & 0 \\ -3X_{4u}^{\alpha(1)} + X_{4b}^{(1)} & -X_{4u}^{\alpha(1)} - X_{4b}^{(1)} & 4X_{4u}^{\alpha(1)} \\ X_{4v}^{\alpha(1)} - X_{4a}^{(1)} & 3X_{4v}^{\alpha(1)} + X_{4a}^{(1)} & -4X_{4v}^{\alpha(1)} \\ 2X_{4u}^{\beta 1(1)} - X_{4u}^{\beta 2(1)} + 2X_{40}^{(1)} & -2X_{4u}^{\beta 1(1)} - X_{4u}^{\beta 2(1)} - 2X_{40}^{(1)} & 2X_{4u}^{\beta 2(1)} \end{pmatrix}, \quad (B100)$$

$$\tilde{\sigma}_2^{(O/L)} = \begin{pmatrix} 0 & 6X_{4v}^{\alpha(2)} - 6X_{4a}^{(2)} & -6X_{4v}^{\alpha(2)} + 6X_{4a}^{(2)} \\ -4X_{4v}^{\alpha(2)} + 4X_{4a}^{(2)} & 6X_{4v}^{\alpha(2)} + 2X_{4a}^{(2)} & -2X_{4v}^{\alpha(2)} - 6X_{4a}^{(2)} \\ 4X_{4v}^{\alpha(2)} - 4X_{4a}^{(2)} & -12X_{4v}^{\alpha(2)} + 4X_{4a}^{(2)} & 8X_{4v}^{\alpha(2)} \\ 2X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} - 2X_{40}^{(2)} & -2X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} + 2X_{40}^{(2)} & -4X_{4u}^{\beta 2(2)} \\ -2X_{4v}^{\beta 1(2)} + 2X_{4v}^{\beta 2(2)} & 2X_{4v}^{\beta 1(2)} - 10X_{4v}^{\beta 2(2)} & 8X_{4v}^{\beta 2(2)} \\ -6X_{4u}^{\alpha(2)} + 2X_{4b}^{(2)} & 4X_{4u}^{\alpha(2)} + 4X_{4b}^{(2)} & 2X_{4u}^{\alpha(2)} - 6X_{4b}^{(2)} \\ -6X_{4u}^{\alpha(2)} + 2X_{4b}^{(2)} & 4X_{4u}^{\alpha(2)} + 4X_{4b}^{(2)} & 2X_{4u}^{\alpha(2)} - 6X_{4b}^{(2)} \\ -6X_{4u}^{\alpha(2)} - 6X_{4b}^{(2)} & 0 & 6X_{4u}^{\alpha(2)} + 6X_{4b}^{(2)} \\ 12X_{4u}^{\alpha(2)} + 4X_{4b}^{(2)} & -4X_{4u}^{\alpha(2)} - 4X_{4b}^{(2)} & -8X_{4u}^{\alpha(2)} \\ 2X_{4v}^{\beta 1(2)} + 10X_{4v}^{\beta 2(2)} & -2X_{4v}^{\beta 1(2)} - 2X_{4v}^{\beta 2(2)} & -8X_{4v}^{\beta 2(2)} \\ 2X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} - 2X_{40}^{(2)} & -2X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} + 2X_{40}^{(2)} & -4X_{4u}^{\beta 2(2)} \\ -4X_{4v}^{\alpha(2)} + 4X_{4a}^{(2)} & 6X_{4v}^{\alpha(2)} + 2X_{4a}^{(2)} & -2X_{4v}^{\alpha(2)} - 6X_{4a}^{(2)} \\ -2X_{4v}^{\beta 1(2)} + 2X_{4v}^{\beta 2(2)} & 2X_{4v}^{\beta 1(2)} - 10X_{4v}^{\beta 2(2)} & 8X_{4v}^{\beta 2(2)} \\ 2X_{4v}^{\beta 1(2)} + 10X_{4v}^{\beta 2(2)} & -2X_{4v}^{\beta 1(2)} - 2X_{4v}^{\beta 2(2)} & -8X_{4v}^{\beta 2(2)} \\ -12X_{4v}^{\beta 2(2)} & 12X_{4v}^{\beta 2(2)} & 0 \\ 12X_{4u}^{\alpha(2)} + 4X_{4b}^{(2)} & -4X_{4u}^{\alpha(2)} - 4X_{4b}^{(2)} & -8X_{4u}^{\alpha(2)} \\ 4X_{4v}^{\alpha(2)} - 4X_{4a}^{(2)} & -12X_{4v}^{\alpha(2)} + 4X_{4a}^{(2)} & 8X_{4v}^{\alpha(2)} \\ 2X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} - 2X_{40}^{(2)} & -2X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} + 2X_{40}^{(2)} & -4X_{4u}^{\beta 2(2)} \end{pmatrix}. \quad (B102)$$

$$\tilde{\sigma}_{ij}^{mna(O/L)} = \tilde{\sigma}_1^{(O/L)} + \tilde{\sigma}_2^{(O/L)}, \quad (B103)$$

$$\tilde{\sigma}_1^{(O/L)} = \begin{pmatrix} 0 & 0 & 0 \\ 2X_{4u}^{\beta 2(1)} + 8X_{40}^{(1)} & -4X_{4v}^{\beta 2(1)} & 2X_{4u}^{\alpha(1)} + 2X_{4b}^{(1)} \\ -2X_{4u}^{\beta 2(1)} - 8X_{40}^{(1)} & 4X_{4v}^{\beta 2(1)} & -2X_{4u}^{\alpha(1)} - 2X_{4b}^{(1)} \\ -7X_{4v}^{\alpha(1)} - X_{4a}^{(1)} & -5X_{4u}^{\alpha(1)} - X_{4b}^{(1)} & -X_{4v}^{\beta 1(1)} - 3X_{4v}^{\beta 2(1)} \\ -X_{4u}^{\alpha(1)} - X_{4b}^{(1)} & -X_{4v}^{\alpha(1)} + X_{4a}^{(1)} & -X_{4u}^{\beta 1(1)} + X_{40}^{(1)} \\ -2X_{4v}^{\beta 2(1)} & -X_{4u}^{\beta 2(1)} + 4X_{40}^{(1)} & X_{4v}^{\alpha(1)} - X_{4a}^{(1)} \\ 4X_{4v}^{\beta 2(1)} & 2X_{4u}^{\beta 2(1)} - 8X_{40}^{(1)} & -2X_{4v}^{\alpha(1)} + 2X_{4a}^{(1)} \\ 0 & 0 & 0 \\ -4X_{4v}^{\beta 2(1)} & -2X_{4u}^{\beta 2(1)} + 8X_{40}^{(1)} & 2X_{4v}^{\alpha(1)} - 2X_{4a}^{(1)} \\ X_{4u}^{\alpha(1)} + X_{4b}^{(1)} & X_{4v}^{\alpha(1)} - X_{4a}^{(1)} & X_{4u}^{\beta 1(1)} - X_{40}^{(1)} \\ 5X_{4v}^{\alpha(1)} - X_{4a}^{(1)} & 7X_{4u}^{\alpha(1)} - X_{4b}^{(1)} & -X_{4v}^{\beta 1(1)} + 3X_{4v}^{\beta 2(1)} \\ -X_{4u}^{\beta 2(1)} - 4X_{40}^{(1)} & 2X_{4v}^{\beta 2(1)} & -X_{4u}^{\alpha(1)} - X_{4b}^{(1)} \\ 2X_{4u}^{\alpha(1)} + 2X_{4b}^{(1)} & 2X_{4v}^{\alpha(1)} - 2X_{4a}^{(1)} & 2X_{4u}^{\beta 1(1)} - 2X_{40}^{(1)} \\ -2X_{4u}^{\alpha(1)} - 2X_{4b}^{(1)} & -2X_{4v}^{\alpha(1)} + 2X_{4a}^{(1)} & -2X_{4u}^{\beta 1(1)} + 2X_{40}^{(1)} \\ 0 & 0 & 0 \\ 2X_{4v}^{\beta 2(1)} & X_{4u}^{\beta 2(1)} - 4X_{40}^{(1)} & -X_{4v}^{\alpha(1)} + X_{4a}^{(1)} \\ X_{4u}^{\beta 2(1)} + 4X_{40}^{(1)} & -2X_{4v}^{\beta 2(1)} & X_{4u}^{\alpha(1)} + X_{4b}^{(1)} \\ 2X_{4v}^{\alpha(1)} + 2X_{4a}^{(1)} & -2X_{4u}^{\alpha(1)} + 2X_{4b}^{(1)} & 2X_{4v}^{\beta 1(1)} \end{pmatrix}, \quad (B104)$$

$$\tilde{\sigma}_2^{(O/L)} = \quad (B105)$$

$$\begin{pmatrix} -3X_{4u}^{\beta 1(2)} - 3X_{4u}^{\beta 2(2)} + 3X_{40}^{(2)} & 3X_{4v}^{\beta 1(2)} - 3X_{4v}^{\beta 2(2)} & 9X_{4u}^{\alpha(2)} - 3X_{4b}^{(2)} \\ 3X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} + X_{40}^{(2)} & -3X_{4v}^{\beta 1(2)} + 5X_{4v}^{\beta 2(2)} & -X_{4u}^{\alpha(2)} - X_{4b}^{(2)} \\ X_{4u}^{\beta 2(2)} - 4X_{40}^{(2)} & -2X_{4v}^{\beta 2(2)} & -8X_{4u}^{\alpha(2)} + 4X_{4b}^{(2)} \\ -7X_{4v}^{\alpha(2)} - X_{4a}^{(2)} & 7X_{4u}^{\alpha(2)} - X_{4b}^{(2)} & 2X_{4v}^{\beta 1(2)} \\ -13X_{4u}^{\alpha(2)} - X_{4b}^{(2)} & -X_{4v}^{\alpha(2)} + X_{4a}^{(2)} & 2X_{4u}^{\beta 1(2)} - 3X_{4u}^{\beta 2(2)} + 2X_{40}^{(2)} \\ -3X_{4v}^{\beta 1(2)} - 5X_{4v}^{\beta 2(2)} & -3X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} - X_{40}^{(2)} & X_{4v}^{\alpha(2)} - X_{4a}^{(2)} \\ -3X_{4v}^{\beta 1(2)} - 5X_{4v}^{\beta 2(2)} & -3X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} - X_{40}^{(2)} & X_{4v}^{\alpha(2)} - X_{4a}^{(2)} \\ 3X_{4v}^{\beta 1(2)} + 3X_{4v}^{\beta 2(2)} & 3X_{4u}^{\beta 1(2)} - 3X_{4u}^{\beta 2(2)} - 3X_{40}^{(2)} & -9X_{4v}^{\alpha(2)} - 3X_{4a}^{(2)} \\ 2X_{4v}^{\beta 2(2)} & X_{4u}^{\beta 2(2)} + 4X_{40}^{(2)} & 8X_{4v}^{\alpha(2)} + 4X_{4a}^{(2)} \\ X_{4u}^{\alpha(2)} + X_{4b}^{(2)} & 13X_{4v}^{\alpha(2)} - X_{4a}^{(2)} & -2X_{4u}^{\beta 1(2)} - 3X_{4u}^{\beta 2(2)} - 2X_{40}^{(2)} \\ -7X_{4v}^{\alpha(2)} - X_{4a}^{(2)} & 7X_{4u}^{\alpha(2)} - X_{4b}^{(2)} & 2X_{4v}^{\beta 1(2)} \\ 3X_{4u}^{\beta 1(2)} + 2X_{4u}^{\beta 2(2)} + X_{40}^{(2)} & -3X_{4v}^{\beta 1(2)} + 5X_{4v}^{\beta 2(2)} & -X_{4u}^{\alpha(2)} - X_{4b}^{(2)} \\ -13X_{4u}^{\alpha(2)} - X_{4b}^{(2)} & -X_{4v}^{\alpha(2)} + X_{4a}^{(2)} & 2X_{4u}^{\beta 1(2)} - 3X_{4u}^{\beta 2(2)} + 2X_{40}^{(2)} \\ X_{4u}^{\alpha(2)} + X_{4b}^{(2)} & 13X_{4v}^{\alpha(2)} - X_{4a}^{(2)} & -2X_{4u}^{\beta 1(2)} - 3X_{4u}^{\beta 2(2)} - 2X_{40}^{(2)} \\ 12X_{4u}^{\alpha(2)} & -12X_{4v}^{\alpha(2)} & 6X_{4u}^{\beta 2(2)} \\ 2X_{4v}^{\beta 2(2)} & X_{4u}^{\beta 2(2)} + 4X_{40}^{(2)} & 8X_{4v}^{\alpha(2)} + 4X_{4a}^{(2)} \\ X_{4u}^{\beta 2(2)} - 4X_{40}^{(2)} & -2X_{4v}^{\beta 2(2)} & -8X_{4u}^{\alpha(2)} + 4X_{4b}^{(2)} \\ -7X_{4v}^{\alpha(2)} - X_{4a}^{(2)} & 7X_{4u}^{\alpha(2)} - X_{4b}^{(2)} & 2X_{4v}^{\beta 1(2)} \end{pmatrix}. \quad (B106)$$

$$\tilde{\sigma}_{ij}^{mn\alpha(H)} = \begin{pmatrix} X_{4u}^{\beta 1} - X_{4u}^{\beta 2} + 3X_{40} & X_{4v}^{\beta 1} - X_{4v}^{\beta 2} & -3X_{4u}^{\alpha} + X_{4b} \\ -X_{4u}^{\beta 1} + X_{40} & -X_{4v}^{\beta 1} - X_{4v}^{\beta 2} & -X_{4u}^{\alpha} - X_{4b} \\ X_{4u}^{\beta 2} - 4X_{40} & 2X_{4v}^{\beta 2} & 4X_{4u}^{\alpha} \\ -X_{4v}^{\alpha} + X_{4a} & -X_{4u}^{\alpha} - X_{4b} & 2X_{4v}^{\beta 2} \\ -3X_{4u}^{\alpha} + X_{4b} & -X_{4v}^{\alpha} + X_{4a} & X_{4u}^{\beta 2} - 4X_{40} \\ X_{4v}^{\beta 1} - X_{4v}^{\beta 2} & -X_{4u}^{\beta 1} + X_{40} & -X_{4v}^{\alpha} + X_{4a} \\ X_{4v}^{\beta 1} - X_{4v}^{\beta 2} & -X_{4u}^{\beta 1} + X_{40} & -X_{4v}^{\alpha} + X_{4a} \\ -X_{4v}^{\beta 1} - X_{4v}^{\beta 2} & X_{4u}^{\beta 1} + X_{4u}^{\beta 2} + 3X_{40} & -3X_{4v}^{\alpha} - X_{4a} \\ 2X_{4v}^{\beta 2} & -X_{4u}^{\beta 2} - 4X_{40} & 4X_{4v}^{\alpha} \\ -X_{4u}^{\alpha} - X_{4b} & -3X_{4v}^{\alpha} - X_{4a} & -X_{4u}^{\beta 2} - 4X_{40} \\ -X_{4v}^{\alpha} + X_{4a} & -X_{4u}^{\alpha} - X_{4b} & 2X_{4v}^{\beta 2} \\ -X_{4u}^{\beta 1} + X_{40} & -X_{4v}^{\beta 1} - X_{4v}^{\beta 2} & -X_{4u}^{\alpha} - X_{4b} \\ -3X_{4u}^{\alpha} + X_{4b} & -X_{4v}^{\alpha} + X_{4a} & X_{4u}^{\beta 2} - 4X_{40} \\ -X_{4u}^{\alpha} - X_{4b} & -3X_{4v}^{\alpha} - X_{4a} & -X_{4u}^{\beta 2} - 4X_{40} \\ 4X_{4u}^{\alpha} & 4X_{4v}^{\alpha} & 8X_{40} \\ 2X_{4v}^{\beta 2} & -X_{4u}^{\beta 2} - 4X_{40} & 4X_{4v}^{\alpha} \\ X_{4u}^{\beta 2} - 4X_{40} & 2X_{4v}^{\beta 2} & 4X_{4u}^{\alpha} \\ -X_{4v}^{\alpha} + X_{4a} & -X_{4u}^{\alpha} - X_{4b} & 2X_{4v}^{\beta 2} \end{pmatrix}. \quad (\text{B107})$$

18. rank-5 t -multipoles with normalization

$$\tilde{\sigma}_{ij}^{mn\alpha(O/L)} = \begin{pmatrix} Y_{5u}^{\alpha 1} + 5Y_{5u}^{\alpha 2} - 5Y_{5a} & -Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + Y_{5a} & -6Y_{5u}^{\alpha 2} + 4Y_{5a} \\ -Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + Y_{5a} & Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + 3Y_{5a} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} \\ -6Y_{5u}^{\alpha 2} + 4Y_{5a} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} & 8Y_{5u}^{\alpha 2} \\ Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & 2Y_{5v}^{\beta 2} \\ Y_{5u}^{\beta 1} - Y_{5u}^{\beta 2} + 3Y_{50} & -Y_{5u}^{\beta 1} + Y_{50} & Y_{5u}^{\beta 2} - 4Y_{50} \\ Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - 3Y_{5b} & -Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - Y_{5b} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} \\ Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - 3Y_{5b} & -Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - Y_{5b} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} \\ -Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - Y_{5b} & Y_{5v}^{\alpha 1} + 5Y_{5v}^{\alpha 2} + 5Y_{5b} & -6Y_{5v}^{\alpha 2} - 4Y_{5b} \\ -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -6Y_{5v}^{\alpha 2} - 4Y_{5b} & 8Y_{5v}^{\alpha 2} \\ -Y_{5u}^{\beta 1} + Y_{50} & Y_{5u}^{\beta 1} + Y_{5u}^{\beta 2} + 3Y_{50} & -Y_{5u}^{\beta 2} - 4Y_{50} \\ Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & 2Y_{5v}^{\beta 2} \\ -Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + Y_{5a} & Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + 3Y_{5a} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} \\ Y_{5u}^{\beta 1} - Y_{5u}^{\beta 2} + 3Y_{50} & -Y_{5u}^{\beta 1} + Y_{50} & Y_{5u}^{\beta 2} - 4Y_{50} \\ -Y_{5u}^{\beta 1} + Y_{50} & Y_{5u}^{\beta 1} + Y_{5u}^{\beta 2} + 3Y_{50} & -Y_{5u}^{\beta 2} - 4Y_{50} \\ Y_{5u}^{\beta 2} - 4Y_{50} & -Y_{5u}^{\beta 2} - 4Y_{50} & 8Y_{50} \\ -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -6Y_{5v}^{\alpha 2} - 4Y_{5b} & 8Y_{5v}^{\alpha 2} \\ -6Y_{5u}^{\alpha 2} + 4Y_{5a} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} & 8Y_{5u}^{\alpha 2} \\ Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & 2Y_{5v}^{\beta 2} \end{pmatrix}. \quad (\text{B108})$$

$$\tilde{\sigma}_{ij}^{mna(O/T)} = \begin{pmatrix} Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & Y_{5u}^{\beta 1} - Y_{5u}^{\beta 2} + 3Y_{50} & Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - 3Y_{5b} \\ -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5u}^{\beta 1} + Y_{50} & -Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - Y_{5b} \\ 2Y_{5v}^{\beta 2} & Y_{5u}^{\beta 2} - 4Y_{50} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} \\ -2Y_{5u}^{\alpha 2} - 4Y_{5a} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -Y_{5u}^{\beta 1} + Y_{50} \\ -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -6Y_{5u}^{\alpha 2} + 4Y_{5a} & Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} \\ -Y_{5u}^{\beta 1} + Y_{50} & Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + Y_{5a} \\ -Y_{5u}^{\beta 1} + Y_{50} & Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + Y_{5a} \\ Y_{5u}^{\beta 1} + Y_{5u}^{\beta 2} + 3Y_{50} & -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & Y_{5u}^{\alpha 1} + Y_{5u}^{\alpha 2} + 3Y_{5a} \\ -Y_{5u}^{\beta 2} - 4Y_{50} & 2Y_{5v}^{\beta 2} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} \\ -6Y_{5v}^{\alpha 2} - 4Y_{5b} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} & -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} \\ -2Y_{5u}^{\alpha 2} - 4Y_{5a} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -Y_{5u}^{\beta 1} + Y_{50} \\ -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} & -Y_{5u}^{\beta 1} + Y_{50} & -Y_{5v}^{\alpha 1} + Y_{5v}^{\alpha 2} - Y_{5b} \\ -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -6Y_{5u}^{\alpha 2} + 4Y_{5a} & Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} \\ -6Y_{5v}^{\alpha 2} - 4Y_{5b} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} & -Y_{5v}^{\beta 1} - Y_{5v}^{\beta 2} \\ 8Y_{5v}^{\alpha 2} & 8Y_{5u}^{\alpha 2} & 2Y_{5v}^{\beta 2} \\ -Y_{5u}^{\beta 2} - 4Y_{50} & 2Y_{5v}^{\beta 2} & -2Y_{5u}^{\alpha 2} - 4Y_{5a} \\ 2Y_{5v}^{\beta 2} & Y_{5u}^{\beta 2} - 4Y_{50} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} \\ -2Y_{5u}^{\alpha 2} - 4Y_{5a} & -2Y_{5v}^{\alpha 2} + 4Y_{5b} & -Y_{5u}^{\beta 1} + Y_{50} \end{pmatrix}. \quad (\text{B109})$$

Appendix C: Comparison between the magnetic octupole conductivities and spin conductivities for t -multipoles

We give nonvanishing tensor components of MO and spin conductivities with activations of t -multipoles corresponding to Tables [XII](#), [XIII](#), and [XIV](#). As shown in Tables [XII–XIV](#), the conclusions discussed in Sec. [IIB](#)—namely,

- certain multipoles generate only the MO currents,
- the MO conductivities carry different spin components from the spin conductivities, and
- the ij -plane in which the MO or spin currents flow differs even for the same multipole component,

do not depend on whether the c - or t -multipoles are used.

Appendix D: Hopping Hamiltonian

Here, the hopping matrices with basis functions $\{p_x, p_y, p_z, f_x^\beta, f_y^\beta, f_z^\beta\}$ are given below for each parameter $\{(ppt), (ppt), (pft), (pft), (fft), (fft), (fft), (fft)\}$, presented in the following form:

$$\begin{pmatrix} H_{pp}^t & H_{pf}^t \\ (H_{pf}^t)^\dagger & H_{ff}^t \end{pmatrix}, \quad (\text{D1})$$

where $t \in \{\sigma, \pi, \delta, \phi\}$.

1. First-nearest-neighbor hopping Hamiltonian

The abbreviation as $c(k_i) = \cos k_i$, $s(k_i) = \sin k_i$ for $i = x, y, z$ is used for notational simplicity.

$$H_{pp}^\sigma = 2(pp\sigma) \begin{pmatrix} c(k_x) & 0 & 0 \\ 0 & c(k_y) & 0 \\ 0 & 0 & c(k_z) \end{pmatrix} \quad (\text{D2})$$

TABLE XII. Symmetry-allowed MO Ohmic conductivity tensors and spin Ohmic conductivity tensors with activations rank-3 t -multipoles(t -MP).

t -MP	MO conductivity	spin conductivity
Y_z^α	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^y, O_{zx}^z, \mathbf{O}^y \\ \hline & & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x \\ \hline & S_z & S_y \\ \hline & & S_z \end{bmatrix}$
Y_{3a}	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^z, O_{xx}^z, O_{yy}^y \\ \hline & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ \hline & & O_{xx}^x, O_{yy}^y, O_{xy}^y \end{bmatrix}$	$\begin{bmatrix} S_x & S_y & - \\ \hline & S_x & - \\ \hline & & - \end{bmatrix}$
Y_{3b}	$\begin{bmatrix} O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ \hline & O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^z, O_{xx}^z, O_{yy}^y \\ \hline & & O_{xy}^y, O_{xx}^x, O_{yy}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & - \\ \hline & S_y & - \\ \hline & & - \end{bmatrix}$
Y_{3u}	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^z, \mathbf{O}^z \\ \hline & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ \hline & & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \end{bmatrix}$	$\begin{bmatrix} S_x & S_y & S_z \\ \hline & S_x & - \\ \hline & & S_x \end{bmatrix}$
Y_{3v}	$\begin{bmatrix} O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ \hline & O_{xy}^y, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^z, \mathbf{O}^z \\ \hline & & O_{xy}^y, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$	$\begin{bmatrix} S_y & S_x & - \\ \hline & S_y & S_z \\ \hline & & S_y \end{bmatrix}$
Y_z^β	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{yz}^y, O_{zx}^x & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^y, O_{yz}^z, \mathbf{O}^y \\ \hline & & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^y \end{bmatrix}$	$\begin{bmatrix} S_z & - & S_x \\ \hline & S_z & S_y \\ \hline & & - \end{bmatrix}$
Y_{xyz}	$\begin{bmatrix} \mathbf{O}_{lm}^\alpha & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^y, O_{yz}^z, \mathbf{O}^y \\ \hline & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ \hline & & \mathbf{O}_{lm}^\alpha \end{bmatrix}$	$\begin{bmatrix} - & S_z & S_y \\ \hline & - & S_x \\ \hline & & - \end{bmatrix}$

$$H_{pp}^\pi = 2(pp\pi) \begin{pmatrix} c(k_y) + c(k_z) & 0 & 0 \\ 0 & c(k_x) + c(k_z) & 0 \\ 0 & 0 & c(k_x) + c(k_y) \end{pmatrix} \quad (D3)$$

$$H_{pf}^\pi = \frac{\sqrt{10}(pf\pi)}{2} \begin{pmatrix} c(k_y) - c(k_z) & 0 & 0 \\ 0 & -c(k_x) + c(k_z) & 0 \\ 0 & 0 & c(k_x) - c(k_y) \end{pmatrix} \quad (D4)$$

$$H_{ff}^\pi = \frac{5(ff\pi)}{4} \begin{pmatrix} c(k_y) + c(k_z) & 0 & 0 \\ 0 & c(k_x) + c(k_z) & 0 \\ 0 & 0 & c(k_x) + c(k_y) \end{pmatrix} \quad (D5)$$

$$H_{ff}^\delta = 2(ff\delta) \begin{pmatrix} c(k_x) & 0 & 0 \\ 0 & c(k_y) & 0 \\ 0 & 0 & c(k_z) \end{pmatrix} \quad (D6)$$

$$H_{ff}^\phi = \frac{3(ff\phi)}{4} \begin{pmatrix} c(k_y) + c(k_z) & 0 & 0 \\ 0 & c(k_x) + c(k_z) & 0 \\ 0 & 0 & c(k_x) + c(k_y) \end{pmatrix} \quad (D7)$$

2. Second-nearest-neighbor hopping Hamiltonian

$$H_{pp}^\sigma = 2(pp\sigma) \begin{pmatrix} (c(k_y) + c(k_z))c(k_x) & -2s(k_x)s(k_y) & -2s(k_x)s(k_z) \\ -2s(k_x)s(k_y) & (c(k_x) + c(k_z))c(k_y) & -2s(k_y)s(k_z) \\ -2s(k_x)s(k_z) & -2s(k_y)s(k_z) & (c(k_x) + c(k_y))c(k_z) \end{pmatrix} \quad (D8)$$

TABLE XIII. Symmetry-allowed MO Ohmic conductivity tensors with activations of rank-4 and -5 t -multipoles (t -MP). It is noted that a corresponding spin conductivity tensor does not exist.

t -MP	MO conductivity	t -MP	MO conductivity
X_{40}	$\begin{bmatrix} \mathbf{O}_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & & O_{yz}^x, O_{zx}^y \end{bmatrix}$	Y_{50}	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \end{bmatrix}$
X_{4a}	$\begin{bmatrix} O_{yy}^x, O_{zz}^x, O_{xy}^y, O_{zx}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & & O_{xx}^x, O_{yy}^y, O_{xy}^y \end{bmatrix}$	Y_{5a}	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & & O_{xx}^x, O_{yy}^y, O_{xy}^y \end{bmatrix}$
X_{4b}	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{yy}^z & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{yy}^z \end{bmatrix}$	Y_{5b}	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{yy}^z \end{bmatrix}$
X_{4u}^α	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & O_{xy}^x, O_{yz}^z, O_{xx}^z, O_{yy}^z & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ & & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$	$Y_{5u}^{\alpha 1}$	$\begin{bmatrix} O_{xx}^x, O_{yy}^y, O_{xy}^y & O_{xx}^x, O_{yy}^y, O_{xy}^y & - \\ & O_{xx}^x, O_{yy}^y, O_{xy}^y & - \\ & & - \end{bmatrix}$
X_{4v}^α	$\begin{bmatrix} O_{xy}^x, O_{zz}^x, O_{xy}^y, O_{zx}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & & O_{xy}^x, O_{yz}^z, \mathbf{O}^x \end{bmatrix}$	$Y_{5v}^{\alpha 1}$	$\begin{bmatrix} O_{xy}^x, O_{xx}^x, O_{yy}^y & O_{xx}^x, O_{xx}^x, O_{yy}^y & - \\ & O_{xy}^x, O_{xx}^x, O_{yy}^y & - \\ & & - \end{bmatrix}$
$X_{4u}^{\beta 1}$	$\begin{bmatrix} \mathbf{O}_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{xx}^z, O_{yy}^z \\ & \mathbf{O}_{lm}^\alpha & O_{xx}^x, O_{yy}^z, O_{xy}^y \\ & & - \end{bmatrix}$	$Y_{5u}^{\alpha 2}$	$\begin{bmatrix} O_{xy}^y, O_{zx}^z, \mathbf{O}^x & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \end{bmatrix}$
$X_{4v}^{\beta 1}$	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & \mathbf{O}_{lm}^\alpha & O_{xx}^x, O_{yy}^z, O_{xy}^y \\ & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{yy}^z, O_{xx}^z, O_{yy}^z \\ & & - \end{bmatrix}$	$Y_{5v}^{\alpha 2}$	$\begin{bmatrix} O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ & & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \end{bmatrix}$
$X_{4u}^{\beta 2}$	$\begin{bmatrix} \mathbf{O}_{lm}^\alpha & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & & \mathbf{O}_{lm}^\alpha \end{bmatrix}$	$Y_{5u}^{\beta 1}$	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & \mathbf{O}_{lm}^\alpha & O_{xx}^x, O_{yy}^z, O_{xy}^y \\ & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{xx}^z, O_{yy}^y \\ & & - \end{bmatrix}$
$X_{4v}^{\beta 2}$	$\begin{bmatrix} O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{yz}^z, O_{xx}^z & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \end{bmatrix}$	$Y_{5v}^{\beta 1}$	$\begin{bmatrix} \mathbf{O}_{lm}^\alpha & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^y, O_{xx}^z, O_{yy}^y \\ & \mathbf{O}_{lm}^\alpha & O_{xx}^x, O_{yy}^z, O_{xy}^y \\ & & - \end{bmatrix}$
		$Y_{5u}^{\beta 2}$	$\begin{bmatrix} O_{zx}^x, O_{xx}^z, O_{zz}^z & - & O_{xx}^x, O_{zz}^z, O_{zx}^z \\ & O_{yz}^y, O_{yy}^z, O_{zz}^z & O_{yy}^y, O_{zz}^z, O_{yz}^z \\ & & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \end{bmatrix}$
		$Y_{5v}^{\beta 2}$	$\begin{bmatrix} \mathbf{O}_{lm}^\alpha & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^y, O_{yz}^z, \mathbf{O}^y \\ & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & & \mathbf{O}_{lm}^\alpha \end{bmatrix}$

$$H_{pp}^\pi = 2(pp\pi) \begin{pmatrix} c(k_x)c(k_y) + c(k_x)c(k_z) + 2c(k_y)c(k_z) & 2s(k_x)s(k_y) & 2s(k_x)s(k_z) \\ 2s(k_x)s(k_y) & c(k_x)c(k_y) + 2c(k_x)c(k_z) + c(k_y)c(k_z) & 2s(k_y)s(k_z) \\ 2s(k_x)s(k_z) & 2s(k_y)s(k_z) & 2c(k_x)c(k_y) + c(k_x)c(k_z) + c(k_y)c(k_z) \end{pmatrix} \quad (\text{D9})$$

$$H_{pf}^\sigma = \frac{\sqrt{15}(pf\sigma)}{2} \begin{pmatrix} (c(k_y) - c(k_z))c(k_x) & s(k_x)s(k_y) & -s(k_x)s(k_z) \\ -s(k_x)s(k_y) & (-c(k_x) + c(k_z))c(k_y) & s(k_y)s(k_z) \\ s(k_x)s(k_z) & -s(k_y)s(k_z) & (c(k_x) - c(k_y))c(k_z) \end{pmatrix} \quad (\text{D10})$$

$$H_{pf}^\pi = \frac{\sqrt{10}(pf\pi)}{4} \begin{pmatrix} (-c(k_y) + c(k_z))c(k_x) & s(k_x)s(k_y) & -s(k_x)s(k_z) \\ -s(k_x)s(k_y) & (c(k_x) - c(k_z))c(k_y) & s(k_y)s(k_z) \\ s(k_x)s(k_z) & -s(k_y)s(k_z) & (-c(k_x) + c(k_y))c(k_z) \end{pmatrix} \quad (\text{D11})$$

$$H_{ff}^\sigma = \frac{15(ff\sigma)}{8} \begin{pmatrix} (c(k_y) + c(k_z))c(k_x) & 2s(k_x)s(k_y) & 2s(k_x)s(k_z) \\ 2s(k_x)s(k_y) & (c(k_x) + c(k_z))c(k_y) & 2s(k_y)s(k_z) \\ 2s(k_x)s(k_z) & 2s(k_y)s(k_z) & (c(k_x) + c(k_y))c(k_z) \end{pmatrix} \quad (\text{D12})$$

TABLE XIV. Symmetry-allowed MO Hall conductivity tensors with activations of rank-3 and -4 t -multipoles (t -MP). It is noted that a corresponding spin conductivity tensor does not exist.

t -MP	MO conductivity	t -MP	MO conductivity
Y_z^α	$\begin{bmatrix} - & O_{yz}^x, O_{zx}^y & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & & - \end{bmatrix}$	X_{40}	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, \mathbf{O}^z & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & & - \end{bmatrix}$
Y_{3a}	$\begin{bmatrix} - & O_{xy}, O_{xx}^y, O_{yy}^y & O_{zx}, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & - & \mathbf{O}_{lm}^\alpha \\ & & - \end{bmatrix}$	X_{4a}	$\begin{bmatrix} - & O_{xy}, O_{xx}^y, O_{yy}^y & O_{zx}, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & - & \mathbf{O}_{lm}^\alpha \\ & & - \end{bmatrix}$
Y_{3b}	$\begin{bmatrix} - & O_{xx}^x, O_{yy}^x, O_{xy}^y & \mathbf{O}_{lm}^\alpha \\ & - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & & - \end{bmatrix}$	X_{4b}	$\begin{bmatrix} - & O_{xx}^x, O_{yy}^x, O_{xy}^y & \mathbf{O}_{lm}^\alpha \\ & - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z \\ & & - \end{bmatrix}$
Y_{3u}	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ & - & \mathbf{O}_{lm}^\alpha \\ & & - \end{bmatrix}$	X_{4u}^α	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & - & O_{zx}^x, O_{yz}^z, \mathbf{O}^z \\ & & - \end{bmatrix}$
Y_{3v}	$\begin{bmatrix} - & O_{xy}^y, O_{zx}^z, \mathbf{O}^x & \mathbf{O}_{lm}^\alpha \\ & - & O_{zx}^x, O_{yz}^z, \mathbf{O}^z \\ & & - \end{bmatrix}$	X_{4v}^α	$\begin{bmatrix} - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y & O_{zx}^x, O_{yz}^y, \mathbf{O}^z \\ & - & \mathbf{O}_{lm}^\alpha \\ & & - \end{bmatrix}$
Y_z^β	$\begin{bmatrix} - & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & & - \end{bmatrix}$	$X_{4u}^{\beta 1}$	$\begin{bmatrix} - & - & O_{xy}^x, O_{xx}^y, O_{yy}^y \\ & - & O_{xx}^x, O_{xy}^y, O_{yy}^y \\ & & - \end{bmatrix}$
Y_{xyz}	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^x, O_{xx}^y, O_{zz}^y, O_{yz}^z \\ & - & O_{xy}^y, O_{zz}^x, O_{xy}^z, O_{zx}^z \\ & & - \end{bmatrix}$	$X_{4v}^{\beta 1}$	$\begin{bmatrix} - & - & O_{xx}^x, O_{yy}^x, O_{xy}^y \\ & - & O_{xy}^x, O_{xx}^y, O_{yy}^y \\ & & - \end{bmatrix}$
		$X_{4u}^{\beta 2}$	$\begin{bmatrix} - & O_{zx}^x, O_{yz}^y, O_{xx}^z, O_{yy}^z & O_{xy}^y, O_{zz}^y, O_{yz}^z \\ & - & O_{xx}^x, O_{zz}^x, O_{zx}^z \\ & & - \end{bmatrix}$
		$X_{4v}^{\beta 2}$	$\begin{bmatrix} - & \mathbf{O}_{lm}^\alpha & O_{xy}^y, O_{zx}^z, \mathbf{O}^x \\ & - & O_{xy}^x, O_{yz}^z, \mathbf{O}^y \\ & & - \end{bmatrix}$

$$H_{ff}^\pi = \frac{5(ff\pi)}{16} \begin{pmatrix} (c(k_y) + c(k_z))c(k_x) & -2s(k_x)s(k_y) & -2s(k_x)s(k_z) \\ -2s(k_x)s(k_y) & (c(k_x) + c(k_z))c(k_y) & -2s(k_y)s(k_z) \\ -2s(k_x)s(k_z) & -2s(k_y)s(k_z) & (c(k_x) + c(k_y))c(k_z) \end{pmatrix} \quad (D13)$$

$$H_{ff}^\delta = \frac{(ff\delta)}{8} \begin{pmatrix} c(k_x)c(k_y) + c(k_x)c(k_z) + 32c(k_y)c(k_z) & 2s(k_x)s(k_y) & 2s(k_x)s(k_z) \\ 2s(k_x)s(k_y) & c(k_x)c(k_y) + 32c(k_x)c(k_z) + c(k_y)c(k_z) & 2s(k_y)s(k_z) \\ 2s(k_x)s(k_z) & 2s(k_y)s(k_z) & 32c(k_x)c(k_y) + c(k_x)c(k_z) + c(k_y)c(k_z) \end{pmatrix} \quad (D14)$$

$$H_{ff}^\phi = \frac{27(ff\phi)}{16} \begin{pmatrix} (c(k_y) + c(k_z))c(k_x) & -2s(k_x)s(k_y) & -2s(k_x)s(k_z) \\ -2s(k_x)s(k_y) & (c(k_x) + c(k_z))c(k_y) & -2s(k_y)s(k_z) \\ -2s(k_x)s(k_z) & -2s(k_y)s(k_z) & (c(k_x) + c(k_y))c(k_z) \end{pmatrix} \quad (D15)$$

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